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# NON MONOTONICITY OF APPROXIMATION IN $L^p$

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SUNTO. — In questo articolo si dimostra che l'approssimazione di funzioni con nuclei di sommabilità non è, in generale, monotona negli spazi $L^p$ sul toro $N$-dimensionale.
La dimostrazione si basa su proprietà teoriche della misura di Lebesgue; un'analoga proprietà si potrebbe, dunque, dimostrare per gruppi compatti generali.

## 1. - Introduction

If $K_n$ is the Féjer kernel, by Plancherel formula, it is trivial that $\| f-K_n * f \|_2 \downarrow 0$. For $p \neq 2$ this fact is no longer true (see [1] and [3]). For a general approximate unity there are results for $C(T^N)$ [2].

In this paper it is proved that the approximation is not monotone in $L^p(T^N)$ for a large class of approximate unities, which includes classical kernels.

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## 2. - Results

Let $\{K_n\}$ be an approximate unity (or summability kernel) on the $N$-dimensional torus $T^N = R^N/Z^N \simeq [-1/2, 1/2]^N$, $N \geqslant 1$, i.e a family $\{K_n\}_{n \in R^+}$ where:

$$\int_{T^N} K_n(t)\, dt = 1$$

$$\operatorname*{Sup}_n \| K_n \|_1 < M$$

$$\forall\, \delta,\ 0 < \delta < 1/2,\ \text{if}\ C_\delta = \{x \in T^N : |x| > \delta\}$$

$$\text{then} \int_{C_\delta} |K_n(t)|\, dt \longrightarrow 0\,, \qquad n \longrightarrow \infty\,.$$

Let us indifferently name $B$ the Lebesgue space $L^p(T^N)$, $1 \leqslant p < \infty$, or the space of continuous functions on the torus, $C(T^N)$ and denote its norm by $\| \cdot \|_B$.

It's well known thath if $f \in B$, then $K_n * f$ is an approximation of $f$ in $B$, i.e.

$$\lim_{n \to \infty} \| f - K_n * f \|_B = 0\,.$$

In some recent papers, [1], [2], [3], L. Colzani, L. De Michele and D. Roux proved there is a good control of the oscillation of $\| f - K_n * f \|_B$; nevertheless for quite general approximate unities, theorem 1 says that this approximation is not monotone in $L^p$, $p \neq 2$.

THEOREM 1. - *Let $p \in R$, $1 < p < \infty$, $p \neq 2$. Let $K_n$ be an approximate unity, $K_n \in L^p(T^N)$ $\forall\, n$.*

*Then $\forall\, m$ there exist $n$, $n > m$ and a function $f \in L^p(T^N)$ such that*

$$\| f - K_n * f \|_p > \| f - K_m * f \|_p\,.$$

*If $p = 1$ the theorem is still true with the additional hypotesis $K_n \geqslant 0$.*

We prove theorem 1 for $1 < p < 2$, $2 < p < \infty$ and $p = 1$, separately.

PROOF. - The case $2 < p < \infty$.

We need the following.

LEMMA 2. - *Let $p, a, b, A, B, C$ be in $\mathbf{R}$, $p > 2$, $A > 0$. Let $\Phi(a, b)$ be the function*

$$\Phi(a, b) = ab \cdot A + ab^{p-1} \cdot B + (a^2 + b^p) \cdot C . \tag{1}$$

*Then there exist $a_0$ and $b_0$ such that $\Phi(a, b) > 0 \ \forall (a, b)$, $a < a_0$, $b < b_0$.*

PROOF OF LEMMA 2. - Since $p > 2$, there exists $\epsilon < p-2$. If $a = b^{1+\epsilon}$ (1) becomes:

$$\Phi(a, b) = b^{2+\epsilon} \cdot A + b^{p+\epsilon} \cdot B + (b^{2+2\epsilon} + b^p) \cdot C .$$

Since $A > 0$, and $2+\epsilon < \min(p+\epsilon, 2+2\epsilon, p)$,

$$\Phi(a, b) \longrightarrow 0^+$$

$\Phi(a, b)$ is therefore definitively positive when $b \longrightarrow 0$. ■

Let's return to the proof of theorem.

Let $n > m$.

Almost every $x \in [-1/2, 1/2]^N$ is a Lebesgue point of $K_n$ and $K_m$. Moreover

$$\int K_n(x)\, dx = \int K_m(x)\, dx .$$

Therefore there exists $x_0 \in [-1/2, 1/2]^N$ such that

$$K_m(-x_0) > K_n(-x_0)$$

and such that $x_0$ and $-x_0$ are Lebesgue points for both $K_n$ and $K_m$.

Moreover we can assume that the origin is a Lebesgue point for $K_n$ and $K_m$, since all the matter is translation invariant.

Let $a < 1$ and $b \in \boldsymbol{R}$ be s.t. (1) is positive when

$$A = K_m(-x_0) - K_n(-x_0)\,,\ B = K_m(x_0) - K_n(x_0)\,,\ C = K_m(0) - K_n(0)\,.$$

Let $\eta \in \boldsymbol{R}^+$, $\eta < (a \| x_0 \| / 3N)^N$ and $\delta = a\eta$.

Let $B(0, \delta)$ and $B(x_0, \eta)$ be the balls with centers 0 and $x_0$ and measures $\delta$ and $\eta$, respectively.

Let $f \in L^p(\boldsymbol{T}^N)$ be the function:

$$f(x) = \chi_{B(0,\delta)} + b\chi_{B(x_0,\eta)} \tag{2}$$

where $\chi_E$ is the characteristic function of the set $E$.

We prove that, for $\eta$ small enough,

$$D_\eta = \| f - K_n * f \|_p^p - \| f - K_m * f \|_p^p > 0\,.$$

We notice that

$$\begin{aligned} D_\eta \geqslant & \int_{B(0,\delta)} \{ | f(x) - K_n * f(x) |^p - | f(x) - K_m * f(x) |^p \} \, dx + \\ & + \int_{B(x_0,\eta)} \{ | f(x) - K_n * f(x) |^p - | f(x) - K_m * f(x) |^p \} \, dx + \\ & - \left| \int_E \{ | f(x) - K_n * f(x) |^p - | f(x) - K_m * f(x) |^p \} \, dx \right| \end{aligned}$$

where $E$ is the set:

$$E = \boldsymbol{T}^N \setminus (B(0, \delta) \cup B(x_0, \eta))\,.$$

Moreover if $x$ and $x-x_0$ are Lebesgue points for $K_n$ and $K_m$, we obtain, $\forall\, i = m, n$:

$$(K_i * f)(x) = \int_{B(0,\delta)} K_i(x-t)\,dt + b\int_{B(x_0,\eta)} K_i(x-t)\,dt =$$

$$= \delta K_i(x) + b\eta K_i(x-x_0) + \phi_{i,\delta}(x) + \phi_{i,\eta}(x)$$

where

$$\phi_{i,\delta}(x)/\eta = 1/\eta \cdot \int_{B(0,\delta)} [K_i(x-t) - K_i(x)]\,dt \longrightarrow 0, \qquad \eta \longrightarrow 0$$

$$\phi_{i,\eta}(x)/\eta = 1/\eta \cdot \int_{B(x_0,\eta)} (K_i(x-t) - K_i(x-x_0))\,dt =$$

$$= 1/\eta \cdot \int_{B(0,\eta)} [K_i(x-x_0-s) - K_i(x-x_0)]\,ds \longrightarrow 0 \qquad \text{when } \eta \longrightarrow 0\,.$$

a) We notice that

$$\left|\int_E \{|f(x) - K_n * f(x)|^p - |f(x) - K_m * f(x)|^p\}\,dx\right| \leqslant$$

$$\leqslant \left|\int_E \{|K_n * f(x)|^p - |K_m * f(x)|^p\}\,dx\right| \leqslant \tag{3}$$

$$\leqslant (\|K_n\|_p^p + \|K_m\|_p^p) \cdot \|f\|_1^p \leqslant O(\eta^p)\,.$$

b) Let us consider

$$\int_{B(0,\delta)} \{|f(x) - K_n * f(x)|^p - |f(x) - K_m * f(x)|^p\}\,dx$$

on the Lebesgue set of $K_n$ and $K_m$. We notice that

$$\int_{B(0,\delta)} |f - K_i * f|^p\,dx =$$

$$= \int_{B(0,\delta)} \{1 - p\delta K_i(x) - pb\eta K_i(x-x_0) - p\phi_{i,\delta}(x) - p\phi_{i,\eta}(x) + \gamma(x)\}\,dx$$

where

$$\gamma(x) = \sum_{n=2}^{+\infty} (-1)^{n} \binom{p}{n} \left[(K_i * f)(x)\right]^{n} .$$

We prove that:

$$\int_{B(0,\delta)} \phi_{i,\delta}(x)\, dx = o(\eta^2)$$

$$\int_{B(0,\delta)} \phi_{i,\eta}(x)\, dx = o(\eta^2)$$

$$\int_{B(0,\gamma)} \gamma(x)\, dx = o(\eta^2) .$$

In fact

$$\left| (1/\eta^2) \int_{B(0,\delta)} \phi_{i,\delta}(x)\, dx \right| \leqslant a^2/\delta^2 \int_{x\in B(0,\delta)} \int_{t\in B(0,\delta)} \left| K_i(x-t) - K_i(0) \right| dt\, dx +$$

$$+ a^2/\delta^2 \int_{x\in B(0,\delta)} \int_{t\in B(0,\delta)} \left| K_i(x) - K_i(0) \right| dt\, dx \leqslant$$

$$\leqslant 2/2\delta \int_{s\in B(0,2\delta)} \left| K_i(s) - K_i(0) \right| ds + 1/\delta \int_{x\in B(0,\delta)} \left| K_i(x) - K_i(0) \right| dx \longrightarrow 0$$

when $\eta \longrightarrow 0$ .

Analogously

$$1/\eta^2 \int_{B(0,\delta)} \phi_{i,\eta}(x)\, dx \longrightarrow 0 \qquad \text{when } \eta \longrightarrow 0$$

and

$$\int_{B(0,\delta)} \gamma(x)\, dx \leqslant$$

$$\leqslant \int_{x\in B(0,\delta)} \sum_{n=2}^{+\infty} \binom{p}{n} \left[ \int_{t\in B(0,\delta)} \left| K_i(x-t) \right| dt + \int_{t\in B(x_0,\eta)} \left| K_i(x-t) \right| dt \right]^{n} dx \leqslant$$

$$\leqslant \delta^2 \sum_{n=2}^{+\infty} \delta^{n-1} \binom{p}{n} \Big[1/\delta \int_{s\in B(0,2\delta)} |K_i(s) - K_i(0)|\, ds +$$

$$+ 1/\delta \int_{s\in B(x_0,2\eta)} |K_i(s) - K_i(0)|\, ds + 2\,|K_i(0)| + 2/a\,|K_i(x_0)|\Big]^n = o(\eta^2).$$

Therefore, since 0 and $x_0$ are Lebesgue points of $K_i$,

$$\int_{B(0,\delta)} |f - K_i * f|^p\, dx =$$

$$(4) \qquad = \delta - p\delta \int_{B(0,\delta)} K_i(x)\, dx - pb\eta \int_{B(0,\delta)} K_i(x - x_0)\, dx + o(\eta^2) =$$

$$= a\eta - pa^2\eta^2 K_i(0) - pab\eta^2 K_i(-x_0) + o(\eta^2).$$

By the same procedure as in b):

$$(5) \qquad \int_{B(x_0,\eta)} |(f - K_i * f)|^p\, dx = b^p\eta - pab^{p-1}\eta^2 K_i(x_0) - p\eta^2 b^p K_i(0) + o(\eta^2).$$

Therefore, by (3), (4) and (5)

$$D_\eta \geqslant p\eta^2 \{ab\,(K_m(-x_0) - K_n(-x_0)) + ab^{p-1}(K_m(x_0) - k_n(x_0)) +$$

$$+ (a^2 + b^p)(K_m(0) - K_n(0))\} - O(\eta^p) \geqslant p\eta^2\, \Phi(a, b) - O(\eta)^p$$

where $\Phi(a, b)$ is the (1), when

$$A = K_m(-x_0) - K_n(-x_0), \quad B = K_m(x_0) - K_n(x_0), \quad C = K_m(0) - K_n(0).$$

Therefore

$$\| f - K_n * f \|_p^p - \| f - K_m * f \|_p^p \longrightarrow 0^+ \qquad \text{when } \eta \longrightarrow 0$$

since $K_m(-x_0) - K_n(-x_0) > 0$. ■

PROOF OF THEOREM 1. - The case $1 < p < 2$.

LEMMA 3. - *Let* $p > 1$ *and let* $\{K_n\}$ *be an approximate unity s.t.* $K_n \in L^p(T^N)$, $\forall n$. *Then* $\forall m \,\exists\, n > m$ *such that*

$$\| K_n \|_p > \| K_m \|_p . \tag{6}$$

PROOF OF LEMMA 3. - We have only to prove $\| K_n \|_p \longrightarrow \infty$ when $n \longrightarrow \infty$. As it's well known

$$\| K_n \|_p > \| \hat{K}_n \|_q .$$

Moreover, since $\forall j \, | \hat{K}_n(j) | \longrightarrow 1$ $(n \longrightarrow \infty)$, $\| \hat{K}_n \|_q \longrightarrow \infty$. ■

Let's return to the proof of the theorem.

Let $n$ be such that $\| K_n \|_p > \| K_m \|_p$.

As in the previous proof, we can suppose the origin is a Lebesgue point for $K_n$ and $K_m$.

Then, $\forall \eta$, $0 < \eta < mB(0, \pi)$, we consider the ball $B(0, \eta)$ with center 0 and measure $\eta$. Let $f$ be the function:

$$f = \chi_{B(0,\eta)} .$$

We prove that

$$D_\eta = \| f - K_n * f \|_p^p - \| f - K_m * f \|_p^p > 0$$

for $\eta$ small enough.

We notice that, if $x$ is a Lebesgue point for $K_n$ and for $K_m$, $\forall i = m, n$

$$K_i * f(x) = \int_{B(0,\eta)} K_i(x-t)\, dt = \eta K_i(x) + \phi_{i,\eta}(x)$$

where $\phi_{i,\eta}(x) / \eta \longrightarrow 0$ when $\eta \longrightarrow 0$.

Moreover as we noticed above

$$\int_{B(0,\eta)} \phi_{i,\eta}(x) = o(\eta^2).$$

$$\int_{B(0,\delta)} \gamma(x)\, dx = \int_{B(0,\delta)} \sum_{n=2}^{+\infty} (-1)^n \binom{p}{n} [(K_i * f)(x)]^n\, dx = o(\eta^2) \text{ when } \eta \longrightarrow 0.$$

Then we obtain, evaluating $\int_{B(0,\eta)} | f-K_i * f |^p\, dx$ on the Lebesgue set of $K_n$ e $K_m$

$$\int_{B(0,\eta)} | f-K_i * f |^p\, dx =$$

$$(7) \qquad = \int_{B(0,\eta)} (1 - p\eta K_i(x) + p\phi_{i,\eta}(x) + \gamma(x))\, dx =$$

$$= \eta - p\eta^2 K_i(0) + o(\eta^2) \qquad \text{when } \eta \longrightarrow 0$$

since the origin is a Lebesgue point for $K_n$ and for $K_m$.

If we name $N_\eta$ the summability kernel $1/\eta\, \chi_{B(0,\eta)}$

$$\| \phi_{i,\eta}/\eta \|_p = \| K_i - N_\eta * K_i \|_p \longrightarrow 0 \qquad \text{when } \eta \longrightarrow 0.$$

It follows $\forall\, i = n, m$

$$\int_E | K_i * f |^p\, dx = \int_E | \eta K_i(x) + \phi_{i,\eta}(x) |^p\, dx = \eta^p \int_E | K_i(x) |^p\, dx + o(\eta^p).$$

In fact, if

$$E_i = \{x \in E : K_i(x) = 0\},$$

$$E_{i,\eta} = \{x \in E : K_i(x) \neq 0,\ |K_i(x)| < 2\, | \phi_{i,\eta}(x) / \eta |\},$$

we obtain, since $m(E_{i,\eta}) \longrightarrow 0$ when $\eta \longrightarrow 0$,

(8)

$$\eta^p \int_E |K_i(x) + \phi_{i,\eta}(x)/\eta|^p \, dx =$$

$$= \eta^p \int_{E_i} |\phi_{i,\eta}(x)/\eta|^p \, dx + \eta^p \int_{E_{i,\eta}} |K_i(x) + \phi_{i,\eta}(x)/\eta|^p \, dx +$$

$$+ \eta^p \int_{E \setminus E_i \setminus E_{i,\eta}} |K_i(x) + \phi_{i,\eta}(x)/\eta|^p \, dx =$$

$$= \eta^p \int_{E_i} |\phi_{i,\eta}(x)/\eta|^p \, dx + \eta^p \int_{E \setminus E_i \setminus E_{i,\eta}} |K_i(x) + \phi_{i,\eta}(x)/\eta|^p \, dx + o(\eta^p) =$$

$$= \eta^p \|K_i\|_p^p + R_\eta \eta^p$$

where

$$R_\eta \leqslant \int_{E \setminus E_i \setminus E_{i,\eta}} \sum_{n=2}^{+\infty} \binom{p}{n} |\phi_{i,\eta}(x)/\eta|^n \, |K_i(x)|^{p-n} \, dx +$$

$$+ \int_{E \setminus E_i} p \, |\phi_{i,\eta}(x)/\eta| \, |K_i(x)|^{p-1} \, dx \leqslant$$

$$\leqslant \int_{E \setminus E_i \setminus E_{i,\eta}} \sum_{n=2}^{+\infty} \binom{p}{n} 2^{p-n} |\phi_{i,\eta}(x)/\eta|^p \, dx +$$

$$+ \|\phi_{i,\eta}(x)/\eta\|_{2-p} \, \| |K_i|^{p-1} \|_{1/p-1} \longrightarrow 0 \qquad \text{when } \eta \longrightarrow 0 .$$

Then, by (6), (7) and (8)

$$D_\eta = \eta^2 (K_m(0) - K_n(0)) + O(\eta^2) + \eta^p (\|K_n\|_p^p - \|K_m\|_p^p) + o(\eta^p) \longrightarrow 0^+$$

when $\eta \longrightarrow 0 \ \forall \, 1 < p < 2$.

$D_\eta$ is therefore definitively positive when $\eta \longrightarrow 0$. ■

PROOF OF THEOREM 1: The case $p = 1$.

Let $p = 1$, $K_n$ and $K_m$ positive.

We can suppose 0 is a Lebesgue point for $K_n$ and $K_m$ s.t.

$$K_m(0) > K_n(0).$$

Let $f$ be the function:

$$f(x) = \chi_{B(0,\eta)} \qquad \eta \in \boldsymbol{R}, \qquad |\eta| < 1/4.$$

We notice $0 < f * K_i(x) < 1$, $\forall x \in T^N$, $\forall i = m, n$. Therefore

$$\int_{T^N} \{|f(x) - K_n * f(x)| - |f(x) - K_m * f(x)|\}\, dx =$$

$$+ \int_E \{K_n * f(x) - K_m * f(x)\}\, dx + \int_{B(0,\delta)} \{-K_n * f(x) + K_m * f(x)\}\, dx =$$

$$= 2\int_{B(0,\eta)} (K_m - K_n) * f(x)\, dx = 2\eta^2 [K_m(0) - K_n(0)] + o(\eta^2) \longrightarrow 0^+$$

when $\eta \longrightarrow 0$. ■

3. - REMARKS. We already have noticed theorem 1 holds for classical kernels: Féjer, De La Vallée-Poussin, Poisson.

We conclude observing that in our construction only measure theoretical properties of the Lebesgue measure are used; therefore an analogue of theorem 1 holds for general compact groups.

## REFERENCES

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