FUNCTIONAL EQUATIONS

- Let $X \equiv [a,b]$ be a closed subset of R, and let $f \in C[X]$.
- We described several ways to approximate f over X: interpolation, least square regression, and so on.
- Often f itself is unknown, but implicitly defined by a functional equation $g[f(x)] = 0$, where $g{:}R \rightarrow R$ is in $C[X]$.
- We will now discuss methods to solve such functional equations using a **polynomial approximation** of f.
- The various methods we will present in the following can be broadly classified as *projection methods*.

- We start by approximating f with a polynomial in the elements of $\Phi \in C[X]$, where the set Φ is a **basis** for $C[X]$.
- Hence, any $f \in C[X]$ can be written (for some vector $\boldsymbol{c}$) as an infinite sum of its elements:

$$f(x) = \sum_{j=0}^{\infty} c_j \phi_j(x)$$

- The actual approximation will linearly combine only a limited subset of Φ:

$$f(x) \approx h_d(x, \mathbf{c}) \equiv \sum_{j=0}^{d} c_j \phi_j(x) = \mathbf{c}' \boldsymbol{\phi}(x)$$

- The choice of Φ and d is the first critical step: the *Weierstrass theorem* shows that any element of $C[X]$ is approximable by algebraic polynomials.

- Algebraic polynomials can be expressed as sums of Chebyshev polynomials:

$$f(x) = \sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} \alpha_j T_j(x)$$

- If $f \in C^k[X]$ and $k \geq 2$, then:

$$|\alpha_j| \leq \frac{k \| f^{(k)} \|}{j^k}, \quad j = 1, 2, \ldots$$

- Hence, low-degree polynomials should work fine if the function to approximate is smooth enough.

- Polynomials in the T_j are algebraic polynomials themselves: hence, why not using directly the monomials $\{x^j\}$ as our basis functions?
- Because the $\{x^j\}$ are not mutually orthogonal: heuristically, the "information set" carried by x^j overlaps partially with the "information set" carried by x^{j-z} or x^{j+z} for $z \neq j$.
- Each Chebyshev polynomial, instead, is orthogonal to any other member of its family.
- From a purely numerical point of view, the coefficients in sums of Chebyshev polynomials are "better identified."

Chebyshev and Fourier Spectral Methods

Second Edition

John P. Boyd

MORAL PRINCIPLE 1:

(i) When in doubt, use Chebyshev polynomials unless the solution is spatially periodic, in which case an ordinary Fourier series is better.

(ii) Unless you're sure another set of basis functions is better, use Chebyshev polynomials.

(iii) Unless you're really, really sure that another set of basis functions is better, use Chebyshev polynomials.

Orthogonal collocation

- The simplest method to fix the coefficients ***c*** is **collocation**.
- Interpolation requires the interpolating polynomial to cross the approximated function at a given set of points.
- Collocation requires the approximating polynomial to exactly solve the functional equation at some $n \equiv d+1$ distinct points in X, known as **collocation nodes**.
- If we define the **residual function** as $R_d(x,c) \equiv g[h_d(x,c)]$, collocation requires that:

$$R_d(x_j, \mathbf{c}) = 0, \quad j = 1, 2, \ldots, n$$

- Note that we transformed a functional equation into a system of *n* nonlinear equations in *n* unknowns.
- The solution, denoted $h_d(x,c)$, will almost exactly represent f at the given nodes x_j, but will only approximate f over X.
- Hence, a way to evaluate the "goodness of fit" is to compute the residuals $R_d(x_j,c)$ at a certain number of equally spaced points in X (not the collocation nodes, of course).
- This metric is a very indirect one, since it measures only the size of the residuals, not the size of the true approximation error $||f-h_d||$.

- Chebyshev's Theorem shows that interpolating at the zeros of Chebyshev polynomials minimizes the approximation error in the *sup norm*.
- This implies that, if $h_d(x,c)$ is an algebraic polynomial of degree d, then choosing as collocation nodes the n zeros of T_n will minimize the approximation error in the sup norm.
- Collocation performed at the zeros of T_n is called **orthogonal collocation**.

Projection methods

- Let Ψ be the family of all piecewise continuous functions $\phi: X \in R \rightarrow R$ such that $\phi(x) \neq 0$ for some $x \in X$.
- **Projections methods** are based on the following idea: if f solves $g[f(x)] = 0$ for all $x \in X$, then it solves the problem $g[f(x)]\phi(x) = 0$ for all $\phi \in \Psi$ and all $x \in X$ too.
- More generally, a solution to $g[f(x)]=0$ solves the following problem (and vice-versa):

$$\langle g[f(x)], \varphi \rangle = 0, \quad \forall \varphi \in \Psi$$

- This general condition can be approximated by restricting it to a finite-dimensional subset of Ψ.
- In other words, the previous condition can be approximated by a set of $n \equiv d+1$ **orthogonal projections**:

$$\langle g[f(x)], \varphi_j \rangle = 0, \quad j = 0, 1, \ldots, d$$

where $\{\phi_j\} \in \Psi$; the elements in Ψ are called **directions**.

- If f is approximated by a polynomial $h_d(x,c)$ in the elements of a basis Φ for $C[X]$, the orthogonal projections can be rewritten as:

$$\langle R_d(x, \mathbf{c}), \varphi_j \rangle = 0, \quad j = 0, 1, \ldots, d$$

- Note that collocation can be classified as a projection method too.
  - The n conditions can be written as:

$$\langle R_d(x,c), \delta(x - x_j) \rangle = 0$$

for $j=0,1,2,\ldots,d$, where δ is the **Dirac delta function**: $\delta(x- x_j)=1$ for $x=x_j$ and $\delta(x-x_j)=0$ otherwise.

Galerkin's method

- Assume that the basis Φ is a family of **mutually orthogonal** polynomials with respect to the weighting function w.
- Galerkin's method uses the first n elements of Φ as projection directions:

$$\langle R_d(x, \mathbf{c}), \phi_j \rangle = 0, \quad j = 0, 1, \ldots, n$$

- The residuals are projected along **mutually orthogonal** directions, and therefore each condition is made as different as possible from the others.

- Note that $R_d(x,\mathbf{c})$ is a continuous function on X, i.e. member of $C[X]$; hence:

$$R_d(x,\mathbf{c}) = \sum_{j=0}^{\infty} a_j \phi_j(x)$$

where

$$a_j = \frac{\langle R_d, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} = \frac{\langle f, \phi_j \rangle}{\alpha_j}$$

- Being $\alpha_j > 0$, Galerkin's conditions imply that:

$$a_j = 0, \ \ j = 0, 1, 2, \ldots, d$$

- This implies that, if the Galerkin method is used:

$$R_d(x,\mathbf{c}) = \sum_{j=n+1}^{\infty} a_j \phi_j(x)$$

- If Chebyshev polynomials are used as basis functions, and f is sufficiently smooth ($k \geq 2$), then for all $x \in X$:

$$|R_d(x,\mathbf{c})| \leq \sum_{j=n+1}^{\infty} \frac{k \| f^{(k)} \|}{j^k}$$

- The integral can be approximated with Gaussian quadrature:

$$\int_a^b R_d(x,\mathbf{c})\phi_j(x)w(x)dx \approx \sum_{k=1}^{m} \hat{\omega}_k R_d(\hat{x}_k,\mathbf{c})\phi_j(\hat{x}_k)$$

where the *m* nodes $\{x_j\}$ are the zeros of φ_m, with $m>n$.

- If Chebyshev polynomials are used as basis functions, the conditions simplify to:

$$\sum_{k=1}^{m} R_d(\hat{x}_k,\mathbf{c})T_j(\hat{x}_k) = 0, \quad j = 0,1,\ldots,d$$

where the $m>n$ nodes $\{x_j\}$ are the zeros of T_m.

Note that:

$$\begin{bmatrix} \sum_{k=1}^{m} R_d(\hat{x}_k, \mathbf{c}) T_0(\hat{x}_k) \\ \sum_{k=1}^{m} R_d(\hat{x}_k, \mathbf{c}) T_1(\hat{x}_k) \\ \vdots \\ \sum_{k=1}^{m} R_d(\hat{x}_k, \mathbf{c}) T_d(\hat{x}_k) \end{bmatrix} = T^T \begin{bmatrix} R_d(\hat{x}_1, \mathbf{c}) \\ R_d(\hat{x}_2, \mathbf{c}) \\ \vdots \\ R_d(\hat{x}_m, \mathbf{c}) \end{bmatrix}$$

where:

$$T = \begin{bmatrix} T_0(\hat{x}_1) & T_1(\hat{x}_1) & \cdots & T_d(\hat{x}_1) \\ T_0(\hat{x}_2) & T_1(\hat{x}_2) & \cdots & T_d(\hat{x}_2) \\ \vdots & \vdots & \ddots & \vdots \cdots \\ T_0(\hat{x}_m) & T_1(\hat{x}_m) & \cdots & T_d(\hat{x}_m) \end{bmatrix}$$