Section I

Borrowing constraints

Optimal consumption under certainty

- The standard optimal intertemporal consumption/saving problem:

$$\max_{\{c_s, a_{s+1}\}_{s=t}^{\infty}} U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

$$\text{s.t. } a_{t+1} = (1+i)a_t + y_t - c_t$$

$$\lim_{j \to \infty} \frac{a_{j+1}}{(1+i)^j} \geq 0$$

- The Euler equation:

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1+\rho}{1+i}$$

- Being $u'' < 0$ for the strict concavity of the return function:

$$\begin{cases} \Delta c_{t+1} > 0 \text{ if } i > \rho \\ \Delta c_{t+1} < 0 \text{ if } i < \rho \\ \Delta c_{t+1} = 0 \text{ if } i = \rho \end{cases}$$

- If $i = \rho$, then $c_t = c$; hence:

$$\bar{c} = \frac{i}{1+i}\left[(1+i)a_t + \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}}\right]$$

$$= ia_t + \frac{i}{1+i}\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}}$$

• The NPG condition imposes the **natural borrowing constraint**, i.e. $a_t \geq -b$ for all $t \geq 0$, where:

$$b \equiv \inf_t \left[\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t+1}} \right]$$

• Let us now suppose that our individual faces a more stringent and potentially binding borrowing constraint

• More precisely, let us impose that $a_t \geq -\varphi$ for all $t \geq 0$, where $0 \leq \varphi < b$ is exogenously given (we can set, without loss of generality, $\varphi = 0$)

• The first order and slackness conditions for the dynamic optimization problem can be combined to form the following "Euler inequality:"

$$\begin{cases} u'(c_t) > (1+i)\beta u'(c_{t+1}) & \text{if } a_{t+1} = 0 \\ u'(c_t) = (1+i)\beta u'(c_{t+1}) & \text{if } a_{t+1} > 0 \end{cases}$$

• The implications as far as the dynamics of the optimal consumption path are straightforward, and depend on the relationship between the interest rate i and the rate of intertemporal substitution ρ

• The simplest case arises when $\boldsymbol{i > \rho}$: in this case, $\boldsymbol{c_{t+1} > c_t}$ **for all** $\boldsymbol{t \geq 0}$; hence, $\mathbf{lim}_{t\to\infty}\, \boldsymbol{c_t} = \infty$ and consequently, being the exogenous income level bounded by assumption, $\mathbf{lim}_{t\to\infty}\, \boldsymbol{a_t} = \infty$

- Consider now the case in which $\boldsymbol{i = \rho}$:
  - $\boldsymbol{c_{t+1} > c_t}$ and $c_t = y_t$ whenever $a_{t+1} = 0$
  - $\boldsymbol{c_{t+1} = c_t}$ as soon as $a_{t+1} > 0$
- Note that consumption will never decrease over time, since saving is freely permitted and the individual prefers a constant consumption path
- The natural question that follows is whether c_t will converge or not to a finite limit in the long run; it can be shown that:

$$\lim_{t\to\infty} c_t = \sup_t \bar{y}_t$$

where:

$$\bar{y}_t \equiv \frac{i}{1+i} \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}}$$

- The intuition is simple:
  - The borrowing constraint may bind only when the individual tries to transfer purchasing power from the future to the present because permanent income is expected to increase
  - As soon as permanent income is expected to remain constant or decrease over time, the incentive to borrow disappears
- Hence, the increase in consumption has to stop as soon as permanent income reaches its maximum value
- From that date onwards consumption has to remain constant, being the individual finally free to smooth consumption over time

- Finally, consider the case in which $\boldsymbol{i < \rho}$: in this case, $c_t > c_{t+1}$ as long as the individual is not credit-constrained, i.e. as long as $a_{t+1} > 0$
- Being $c(a)$ strictly increasing in a , it has to be that $a_t > a_{t+1}$ as long as $a_{t+1} > 0$: hence, we can expect the individual to reach the borrowing limit in finite time
- Furthermore, we can conclude that once our individual becomes credit-constrained, she remains constrained forever
  - The intuition is straightforward: if $a_t = 0$ and $a_{t+1} > 0$, then $u'[c(0)] < u'[c(a_{t+1})]$. But if $c(a)$ is a strictly increasing in a, then $c(a_{t+1}) > c(0)$; hence $u'[c(0)] > u'[c(a_{t+1})]$: a contradiction!

• Note however that the Euler equation alone is not enough to characterize the dynamics of c_t when the individual is credit-constrained: we need further pieces of information about the dynamics of exogenous income

• A natural step forward is to assume that income is constant at some $y > 0$: in this case, c_t will converge from above to y in finite time, and will remain constant form that date onwards

• Summary:

1. If $i > \rho$, then $\lim_{t\to\infty} c_t = +\infty$ and $\lim_{t\to\infty} a_t = +\infty$.
2. If $i = \rho$, then $\lim_{t\to\infty} c_t = \sup_t \bar{y}_t$, converging from below.
3. If $i < \rho$ and $y_t = \bar{y}$, then $\lim_{t\to\hat{t}<\infty} c_t = \bar{y}$, converging from above.

Optimal consumption under uncertainty

- The standard optimal stochastic consumption/saving problem:

$$\max_{\{c_s, a_{s+1}\}_{s=t}^{\infty}} U_t = E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} u(c_s)\right]$$

$$\text{s.t. } a_{t+1} = (1+i)a_t + y_t - c_t$$

$$\lim_{j\to\infty} E_t\left[\frac{a_{j+1}}{(1+i)^j}\right] \geq 0$$

- The Euler equation:

$$E_t[u'(c_{t+1})] = \frac{1}{(1+i)\beta} u'(c_t)$$

- We can rewrite the Euler equation as:

$$E_t\left[(1+i)^{t+1}\beta^{t+1}u'(c_{t+1})\right] \leq (1+i)^t\beta^t u'(c_t)$$

- Hence:

$$\boxed{E_t(M_{t+1}) \leq M_t}$$

where:

$$M_t \equiv (1+i)^t\beta^t u'(c_t) \geq 0$$

- Hence, M_t follows a **nonnegative supermartingale**

- A well known convergence theorem of Doob for supermartingales states that:

$$P\left(\lim_{t\to\infty} M_t = \bar{M}\right) = 1$$

where M is a random variable such that $E(M) < +\infty$

- If $\boldsymbol{i} > \boldsymbol{\rho}$, then $\lim_{t\to\infty}(1+i)^t\beta^t = \infty$, and therefore M_t can converge to a finite limit only if $\lim_{t\to\infty} u'(c_t) = 0$
- If u is unbounded this evidently implies that $\lim_{t\to\infty} c_t = \infty$ and consequently that $\lim_{t\to\infty} a_t = \infty$

- If $i = \rho$, then $(1+i)\beta = 1$ and $M_t = u'(c_t)$; can we conclude in this case that c_t will converge to a finite limit? In general the answer is **no**
- We can show that, if the utility function is bounded and the exogenous income process is "sufficiently stochastic:"

$$\operatorname{var}_t\left(\sum_{s=t}^{\infty}\frac{y_s}{(1+i)^{s-t}}\right)\geq\varphi$$

then:

$$P\left(\lim_{t\to\infty} c_t = \infty\right) = 1 \qquad P(\lim_{t\to\infty} a_t = \infty) = 1$$

• The same result obtains when the exogenous income process is *iid* and the utility function unbounded, or when the marginal utility of consumption is convex, i.e. when $u'''(\cdot) > 0$

• Finally, if $\boldsymbol{i < \rho}$, then $\lim_{t\to\infty}(1+i)^t\beta^t = 0$, and therefore M will surely converge to zero

• Hence, the result $P(\lim_{t\to\infty}M_t = 0) = 1$ does not necessarily imply that $\lim_{t\to\infty}c_t = \infty$, but leaves open the possibility that both c_t and a_t converge in the long run to stationary, and finite, random variables

Section II

Markov chains

A time-invariant, discrete-state Markov chain is characterized by:

- An *n*-dimensional state space:

$$S = \{s_1, s_2, \ldots, s_n\}$$

- A $n \times n$ non-negative **transition matrix *Π*** (right stochastic matrix), such that:

$$\sum_{j=1}^{n} \Pi_{ij} = 1 \qquad \Pi_{ij} = \text{Prob}(x_{t+1} = s_j | x_t = s_i)$$

- A $n \times 1$ non-negative vector π_0, such that:

$$\sum_{i=1}^{n} \pi_{i0} = 1 \qquad \pi_{i0} = \text{Prob}(x_0 = s_i)$$

- Note that:

$$\begin{aligned}
&\text{Prob}(x_{t+2} = s_j | x_t = s_i) \\
&= \sum_{m=1}^{n} \text{Prob}(x_{t+2} = s_j | x_{t+1} = s_m)\text{Prob}(x_{t+1} = s_m | x_t = s_i) \\
&= \sum_{m=1}^{n} \Pi_{im}\Pi_{mj} = \Pi_{ij}^{(2)}
\end{aligned}$$

- In general:

$$\text{Prob}(x_{t+k} = s_j | x_t = s_i) = \Pi_{ij}^{(k)}$$

- This implies that the unconditional distribution of x_t is given by:

$$\pi_t = (\Pi')^t \pi_0$$

$$\pi_{it} = Prob(x_{it} = s_i)$$

- Note furthermore that:

$$\pi_{t+1} = \Pi' \pi_t$$

- An **unconditional distribution** is called **stationary (ergodic)** if it remains unaltered over time, i.e. if it satisfies:

$$\pi = \Pi' \pi$$

• Note that:

$$(\mathbf{1} - \Pi')\pi = \mathbf{0}$$

• In other words, $\boldsymbol{\pi}$ is just an eigenvector associated with a unit eigenvalue of $\boldsymbol{\Pi'}$, pinned down by the normalization, $\sum_i \pi_i = 1$

• The matrix $\boldsymbol{\Pi}$ is stochastic, i.e. it has only nonnegative elements and its rows sum up to one: hence, $\boldsymbol{\Pi'}$ has at least one (possibly more) unit eigenvalue, and that there is at least one (again, possibly more) eigenvector satisfying $(1-\boldsymbol{\Pi'})\boldsymbol{\pi} = 0$

• If there is one and only stationary distribution $\boldsymbol{\pi}_\infty$ and $\lim_{t\to\infty}\boldsymbol{\pi}_t = \boldsymbol{\pi}_\infty$ for all possible initial distributions, then the Markov chain is **asymptotically stationary with a unique invariant distribution**

• Let $\boldsymbol{\Pi}$ be a right stochastic matrix with all strictly positive elements, i.e. with $\Pi_{ij} > 0$ for all (i,j): the associated Markov chain is asymptotically stationary and has a unique stationary distribution

- There are four ways to calculate the invariant distribution π_∞:
  - Iterate until convergence on:

$$\boldsymbol{\pi}_{k+1} = \boldsymbol{\Pi}' \boldsymbol{\pi}_k$$

  - Calculate the eigenvalues and eigenvectors of Π' and take the normalized eigenvector associated to $\lambda=1$:

$$\boldsymbol{\pi}_\infty = \frac{v_1}{\sum_{i=1}^{n} v_{1i}}$$

- Define:

$$\hat{\mathbf{A}} \equiv (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}' \qquad \underset{(n+1)\times n}{\mathbf{A}} \equiv \begin{bmatrix} \mathbf{I}_n - \mathbf{\Pi}' \\ \mathbf{1}_n' \end{bmatrix}$$

It turns out that π_∞ is equal to the n+1 column of $\boldsymbol{A}$.

- Note that:

$$\mathbf{1}_n = \pi - \mathbf{\Pi}'\pi + \mathbf{1}_{n\times n}\pi = (\mathbf{I}_n - \mathbf{\Pi}' + \mathbf{1}_{n\times n})\pi$$

Hence:

$$\pi_\infty = (\mathbf{I}_n - \mathbf{\Pi}' + \mathbf{1}_{n\times n})^{-1}\mathbf{1}_n$$

Section III

Bewely models: The basic framework

- **Bewley models** are characterized by a continuum of ex-ante identical and ex-post heterogeneous households that trade a set of non-state-contingent securities
- In the basic framework, there is **no aggregate uncertainty**
- In the steady state, aggregate variables are constant over time, as in a deterministic representative agent economy
- Uncertainty, however, plays an essential role at the individual level: idiosyncratic shocks introduce an incentive for **self insurance**
- The availability of a non-contingent asset allows households to buffer consumption against adverse shocks
- The inability of fully insure against bad shocks will generate **precautionary savings**

• At the individual level, the employment status evolves according to a discrete Markov chain characterized by $\boldsymbol{S}=\{s_1,s_2,\ldots,s_m\}$ and $\boldsymbol{\Pi}$

• Hours are fixed and normalized to unity: hence, labor income amounts to $w_t s_t$, where w_t is the wage rate

• Households are allowed to invest in a single asset, and a_t denotes individual asset holdings at the **beginning** of period t

• Let us **discretize the state space**, and constrain asset holdings on finite-dimensional grid (a more general approach will follow):

$$\mathcal{A} = \{-\phi < a_1 < a_2 < \ldots < a_n\}$$

• The parameter $\varphi>0$ represents a **borrowing constraint**

• Without aggregate dynamics, factor prices remain constant, i.e. $w_t = w$ and $r_t = r$ for all $t \geq 0$

• Hence, given the aggregate factor prices $\{w,r\}$ and the initial conditions $\{a_0,s_0\}$, households solve the following problem:

$$\max_{\{a_{t+1}\}_{t=0}^{\infty}} U = E_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

$$\text{s.t.} \quad a_{t+1} = (1+r)a_t + ws_t - c_t$$

$$a_{t+1} \in \mathcal{A}$$

• Here $0<\beta<1$ represents the intertemporal discount factor, $u(\cdot)$ is C^2, strictly increasing and strictly concave, and such that $\lim_{c\to 0} du(c)/dc = +\infty$; we also impose that **$\beta(1+r)<1$**

• The Bellman equation for the previous recursive problem becomes:

$$v(\mathrm{a}_i, \mathrm{s}_j) = \max_{a' \in \mathcal{A}} \left\{ u[(1+r)\mathrm{a}_i + w\mathrm{s}_j - a'] + \beta \sum_{z=1}^{m} \Pi_{jz} v(a', \mathrm{s}_z) \right\}.$$

• A solution to this problem can be represented as a **value function** $v(a,s)$ and the associated **policy function** $a'=g(a,s)$

• Being the objective function concave and the constraint set convex, the policy function is a deterministic single-value function of the current state vector; we can define an indicator function such that:

$$\mathcal{I}(a_i, a_h, s_j) = \begin{cases} 1 & \text{if } g(\mathrm{a}_h, \mathrm{s}_j) = \mathrm{a}_i \\ 0 & \text{if } g(\mathrm{a}_h, \mathrm{s}_j) \neq \mathrm{a}_i \end{cases}.$$

- To identify the unique solution we can iterate until convergence on the following recursive scheme, given an initial guess for v_0:

$$v_{k+1}(\mathrm{a}_i, \mathrm{s}_j) = \max_{a' \in \mathcal{A}} \left\{ u[(1+r)\mathrm{a}_i + w\mathrm{s}_j - a'] + \beta \sum_{z=1}^{m} \Pi_{jz} v_k(a', \mathrm{s}_z) \right\}.$$

- Define $n \times 1$ vectors $\boldsymbol{v_j}$ and $n \times n$ matrices $\boldsymbol{R_j}$, with $j=1,2,\dots,m$:

$$\mathbf{v}_j(i) = v(\mathrm{a}_i, \mathrm{s}_j),$$
$$\mathbf{R}_j(i,h) = u[(1+r)\mathrm{a}_i + w\mathrm{s}_j - \mathrm{a}_h],$$

- Furthermore, define:

$$\underset{(mn)\times 1}{\mathbf{v}} \equiv \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}, \quad \underset{(mn)\times n}{\mathbf{R}} \equiv \begin{bmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_m \end{bmatrix}.$$

• The recursive scheme can be represented in matrix notation as:

$$\mathbf{v}_{k+1} = \max[\mathbf{R} + \beta(\mathbf{\Pi} \otimes \mathbf{1}_n)\mathbf{v}_k'].$$

• The policy function, and the corresponding indicator function, may be represented by a set of $n \times n$ matrices $\boldsymbol{G_j}$:

$$G_j(i,h) = \begin{cases} 1 & \text{if } g(\mathrm{a}_i, \mathrm{s}_j) = \mathrm{a}_h \\ 0 & \text{if } g(\mathrm{a}_i, \mathrm{s}_j) \neq \mathrm{a}_h \end{cases}.$$

- Denote as λ_t the **unconditional distribution** of $\{a_t, s_t\}$:

$$\lambda_t(\mathrm{a}_i, \mathrm{s}_j) = \mathrm{Prob}(a_t = \mathrm{a}_i, s_t = \mathrm{s}_j)$$

- The exogenous Markov chain and the optimal policy function induce a law of motion for the distribution λ_t:

$$\underbrace{\mathrm{Prob}(a_{t+1} = \mathrm{a}_i, s_{t+1} = \mathrm{s}_j)}_{\text{Unconditional } t+1}$$

$$= \sum_{h=1}^{n} \sum_{z=1}^{2} \underbrace{\mathrm{Prob}(a_{t+1} = \mathrm{a}_i | a_t = \mathrm{a}_h, s_t = \mathrm{s}_z)}_{\text{Policy function}} \times$$

$$\underbrace{\mathrm{Prob}(s_{t+1} = \mathrm{s}_j | s_t = \mathrm{s}_z)}_{\text{Transition probability}} \times$$

$$\underbrace{\mathrm{Prob}(a_t = \mathrm{a}_h, s_t = \mathrm{s}_z)}_{\text{Unconditional } t}$$

- Note that:

$$\text{Prob}(a_{t+1} = \mathrm{a}_i | a_t = \mathrm{a}_h, s_t = \mathrm{s}_z) = \begin{cases} 1 & \text{if } g(\mathrm{a}_h, \mathrm{s}_z) = \mathrm{a}_i \\ 0 & \text{if } g(\mathrm{a}_h, \mathrm{s}_z) \neq \mathrm{a}_i \end{cases}$$

$$\text{Prob}(s_{t+1} = \mathrm{s}_j | s_t = \mathrm{s}_z) = \Pi_{zj}$$

- Hence:

$$\lambda_{t+1}(\mathrm{a}_i, \mathrm{s}_j) = \sum_{z=1}^{m} \sum_{h=1}^{n} G_z(h,i) \Pi_{zj} \lambda_t(\mathrm{a}_h, \mathrm{s}_z) = \sum_{z=1}^{m} \sum_{\{a : \mathrm{a}_i = g(a, \mathrm{s}_z)\}} \Pi_{zj} \lambda_t(a, \mathrm{s}_z).$$

- A **stationary distribution** is a time-invariant distribution $\boldsymbol{\lambda}$ such that:

$$\lambda(\mathrm{a}_i, \mathrm{s}_j) = \sum_{z=1}^{m} \sum_{h=1}^{n} G_z(h,i) \Pi_{zj} \lambda(\mathrm{a}_h, \mathrm{s}_z),$$

- The previous relationship can be written in matrix notation as:

$$\boxed{\mathrm{vec}(\boldsymbol{\lambda}) = \mathbf{Q}' \mathrm{vec}(\boldsymbol{\lambda}),}$$

$$\mathbf{Q} \equiv \begin{bmatrix} \Pi_{11}\mathbf{G}_1 & \Pi_{21}\mathbf{G}_2 & \cdots & \Pi_{m1}\mathbf{G}_m \\ \Pi_{12}\mathbf{G}_1 & \Pi_{22}\mathbf{G}_2 & \cdots & \Pi_{m2}\mathbf{G}_m \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{1m}\mathbf{G}_1 & \Pi_{2m}\mathbf{G}_2 & \cdots & \Pi_{mm}\mathbf{G}_m \end{bmatrix} = (\boldsymbol{\Pi} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_m \end{bmatrix}.$$

- **vec(λ)** can be interpreted as the *ergodic distribution* of a Markov chain characterized by the transition matrix $\boldsymbol{Q}$, and constructed combining the dynamics of both the exogenous and endogenous states

- For the *Law of Large Numbers,* the stationary distribution $\boldsymbol{\lambda}$ will reproduce, in the limit, the fraction of time that individual households spend in each state

- From the aggregate point of view, $\boldsymbol{\lambda}$ reproduces the fraction of the population in state $\{a_i,s_j\}$ along the stationary equilibrium

    - In other words, $\boldsymbol{\lambda}$ can be interpreted as the **steady-state distribution of financial wealth**