A Short Course in Dynamic Macroeconomics

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We start our exposition of dynamic macroeconomics by studying the *intertemporal consumption/saving problem* of a single infinitely living, price-taking individual. In other words, we analyze the optimal consumption problem of an individual who lives for an infinite number of periods, being small enough with respect to the economy to consider all market prices as given, and receives each period a strictly positive and exogenous income. We assume that a competitive market for a single homogenous consumption good opens each period: being a price-taker, the individual can purchase any desired quantity at the given market price, normalized to unity for the sake of notational simplicity. Without further assumptions, the individual would entirely allocate her income to consumption, since our intertemporal problem collapses to a sequence of simple static utility maximization problems.

To choose the optimal sequence of consumption levels, *i.e.* the optimal consumption path, becomes a more complex *dynamic* problem if an intertemporal link between consumption possibilities in different periods exist. We assume that a durable good, generally called *asset*, allows the individual to store value and transfer it across periods, *i.e.* to reallocate the purchasing power of her income flow over time. Under these assumptions, the intertemporal utility maximization problem becomes a fully-fledged dynamic optimization problem, since the individual has to optimally choose between consumption and saving, *i.e.* between current and future consumption.

1.2.1 Preferences

The individual's preferences on consumption streams can be summarized by the following intertemporal utility function[1]:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \tag{1.1}$$

where $c_s \in R_+$ is the consumption level at date s, $\beta \in (0,1)$ the intertemporal subjective discount factor, and $u\colon R_+ \to R$ the instantaneous utility function. We define also the intertemporal discount rate as $\rho \equiv (1-\beta)/\beta$. In the optimal control jargon, the intertemporal utility function is known as the objective function, and the instantaneous utility function as the return function. The latter has to satisfy some regularity conditions. Assume[2] that u is C^1, strictly increasing, and strictly concave; furthermore, that $\lim_{c\to 0} u'(c) = +\infty$. Note that the last assumption has the straightforward implication that it will never be optimal to set $c_t = 0$.

1.2.2 Budget constraints

Our individuals may save and accumulate assets through the following technology:

$$a_{t+1} = a_t + s_t \tag{1.2}$$

where $a_t \in R$ is the assets stock at the beginning of date t, measured in units of consumption good, and $s_t \in R$ are savings at date t (note that savings can be negative, *i.e.* the asset stock can be freely disposed of).

For the sake of simplicity, we assume that assets may be held only as consumption loans (debts) contracted with other individuals. By contracting a loan (debt), our individual exchanges a non-negative (non-positive) share of her current income for a possibly higher (lower) one in the next period. In each period, loans (debts) can be traded on a competitive financial market, and pay (cost) a non-negative *interest rate*, which can be interpreted as the market price of current purchasing power. We assume, for the sake of simplicity again, that the interest rate is constant over time and strictly positive, and denote it as $i > 0$.

The individual receives each period an exogenous income flow $y_t \in (0, y_{\max}]$, where $y_{\max} < +\infty$, and faces the following intratemporal budget constraint[3]:

$$c_t + s_t = y_t + i a_t \tag{1.3}$$

[1] Three essential assumptions are hidden in (1.1): *(i)* stationarity, since the return function does not depend on t; *(ii)* additive separability, since the return function evaluated at different dates enters additively the objective function; *(iii)* time impatience, since the discount factor is less than unity.

[2] The instantaneous utility function may be unbounded, *i.e.* we do *not* assume that $|u(c)| < \infty \; \forall c \in R_+$. Without this assumption, the objective function may be unbounded too.

[3] More precisely, the budget constraint is $c_t + s_t \leq y_t + i a_t$. However, as long as the marginal utility of consumption is positive, it would never be optimal to waste resources, and the budget constraint holds with equality.

The natural borrowing limit

It would be unfeasible for any individual to finance her current indebtedness by continuously increasing it; in the long-run, such an outcome would not be sustainable in a competitive market populated by rational agents. To avoid this possibility, we impose the so-called No-Ponzi-Games (NPG) condition:

$$\lim_{j\to\infty} \frac{a_{j+1}}{(1+i)^j} \geq 0 \tag{1.4}$$

for all feasible sequences $\{a_s\}_{s=t}^{\infty}$. The NPG condition states that the present market value of the asset stock cannot be strictly negative in the long-run. In other words, it rules out free lunches.

Substitute (1.3) into (1.2):

$$a_{t+1} = (1+i)\, a_t + y_t - c_t \tag{1.5}$$

and evaluate the result at date $t+1$:

$$a_{t+2} = (1+i)\, a_{t+1} + y_{t+1} - c_{t+1} \tag{1.6}$$

Solve (1.5) for a_t, and substitute (1.6) in the result:

$$a_t = \frac{c_t - y_t}{1+i} + \frac{1}{1+i}\left(\frac{c_{t+1} - y_{t+1}}{1+i} + \frac{a_{t+2}}{1+i}\right) \tag{1.7}$$

Iterate the same procedure to obtain:

$$(1+i)\, a_t = \sum_{s=t}^{\infty} \frac{c_s - y_s}{(1+i)^{s-t}} + \lim_{j\to\infty} \frac{a_{j+1}}{(1+i)^j} \tag{1.8}$$

Imposing the NPG condition and solving for the discounted consumption stream takes us to:

$$\sum_{s=t}^{\infty} \frac{c_s}{(1+i)^{s-t}} \leq (1+i)\, a_t + \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \tag{1.9}$$

Remark 1 *Equation (1.9) states that the present market value of the consumption stream cannot be strictly greater than the present market value of lifetime resources, that is the current asset income plus the present value of the exogenous income stream. Note that, by imposing the NPG, the intratemporal budget constraint becomes an intertemporal one.*

The intertemporal budget constraint (1.9) reveals another clarifying implication of the NPG condition. The inequality (1.9) can be rewritten as:

$$a_t \geq \sum_{s=t}^{\infty} \frac{c_s}{(1+i)^{s-t+1}} - \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t+1}} \tag{1.10}$$

Being consumption positive by assumption, i.e. $c_s \geq 0$ for $s \geq t$, (1.10) implies that:

$$a_t \geq -\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t+1}} \tag{1.11}$$

The inequality (1.11) summarizes the exogenous borrowing constraint implied by the NPG condition, known as the *natural borrowing limit.* This borrowing constraint implicitly defines the *maximum* level of debt that can be repaid from date t onwards setting consumption to zero, i.e. $\sum_{s=t}^{\infty} y_s (1+i)^{-s+t-1}$. In general, the natural borrowing constraint takes the form $a_t \geq -b$ for all $t \geq 0$, where:

$$b \equiv \inf_t \left[\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t+1}} \right] \tag{1.12}$$

Remark 2 *Since $c_t = 0$ for some $t \geq 0$ will never be optimal in equilibrium, the borrowing constraint (1.11) will never actually bind.*

1.2.3 The optimization problem

The individual maximizes (1.1) subject to (1.4) and (1.5), taking a_t, i, and $\{y_s\}_{s=t}^{\infty}$ as given. Formally, she solves a deterministic optimal control problem[4] of the form:

$$\begin{aligned} \max_{\{c_s, a_{s+1}\}_{s=t}^{\infty}} \quad & U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \qquad (1.13) \\ \text{s.t.} \quad & a_{t+1} = (1+i) a_t + y_t - c_t \\ & \lim_{j \to \infty} \frac{a_{j+1}}{(1+i)^j} \geq 0 \end{aligned}$$

We build a present-value Lagrangian:

$$L_t = \sum_{s=t}^{\infty} \beta^{s-t} \left\{ u(c_s) + \lambda_s \left[(1+i) a_s + y_s - c_s - a_{s+1} \right] \right\} \tag{1.14}$$

and partially derive it with respect to c_s, a_{s+1}, and λ_s, taking into account that:

$$\begin{aligned} L_t = \; & ... + \beta^s u(\mathbf{c_s}) + \beta^s \left[\lambda_s (1+i) \mathbf{a_s} + y_s - c_s - a_{s+1} \right] + \qquad (1.15) \\ & + \beta^{s+1} u(c_{s+1}) + \beta^{s+1} \left[\lambda_{s+1} (1+i) \mathbf{a_{s+1}} + y_{s+1} - c_{s+1} - a_{s+2} \right] + ... \end{aligned}$$

The first order condition are the following (a hat identifies the optimal path):

$$u'(\hat{c}_t) = \hat{\lambda}_t \tag{1.16}$$

$$\beta \hat{\lambda}_{t+1} (1+i) = \hat{\lambda}_t \tag{1.17}$$

$$\hat{a}_{t+1} = (1+i) \hat{a}_t + y_t - \hat{c}_t \tag{1.18}$$

We know that these conditions are simply necessary but not sufficient for problem (1.13).

Under our strict concavity assumption, a well known result (see Maffezzoli, 2001, par. 6.4.2) states that, if:

$$\lim_{j \to \infty} \beta^j \hat{\lambda}_j a_{j+1} \geq 0 \tag{1.19}$$

[4] The consumption level is the only control variable, and the asset stock the only state variable. The control and state variables take values in convex subsets of R; the return function and the equation of motion are C^1 and strictly concave. The problem is fully recursive; even if the planning horizon starts in date 0, the individual is allowed to optimize again each period, solving the same problem with a different initial condition.

for all feasible sequences $\{a_t\}_{t=0}^{\infty}$, then the first order conditions together with the so called transversality condition[5] (TVC):

$$\lim_{j\to\infty} \beta^j \hat{\lambda}_j \hat{a}_{j+1} = 0 \tag{1.20}$$

are jointly necessary *and* sufficient. The TVC has a clear economic interpretation:

Remark 3 *The Envelope Theorem shows that the costate variable at date t represents the current value of the state variable denominated in time t utils (units in which utility is measured). The TVC implies that the present value in utils of the asset stock cannot be strictly positive in the limit. Heuristically, since the asset stock may be freely consumed, it is not optimal to keep a positively valued asset stock when the economy reaches its "end", i.e. when time goes to infinite.*

To summarize, the sequence (if any exists) $\left\{\hat{c}_s \in R_+, \hat{a}_{s+1} \in R, \hat{\lambda}_s \in R\right\}_{s=t}^{\infty}$ that satisfies the non-linear system of first order conditions, together with the initial condition a_t and the boundary condition (1.20), is the unique solution[6] to problem (1.13).

Exercise 4 *Show that (1.4), (1.16), and (1.17) jointly imply (1.19).*

The previous Exercise shows that the NPG is actually equivalent to condition (1.19).

1.2.4 The Euler equation

Consider (1.16) at dates t and $t+1$. Substitute the result into (1.17) to get (hats are omitted for notational convenience):

$$u'(c_t) = (1+i)\,\beta u'(c_{t+1}) \tag{1.21}$$

Equation 1.21 is known as the Euler equation. The Euler equation formalizes the so-called Keynes-Ramsey rule:

Remark 5 *If we decrease consumption at date t by dc_t, the utility loss is equal to $u'(c_t)\,dc_t$. Next period, we can consume more, since the decrease in consumption at date t implies an increase in savings. In particular, in date $t+1$ consumption increases by $(1+i)\,dc_t$. If the plan is optimal, then there is no advantage in reallocating consumption, and (1.21) holds.*

We may rewrite the Euler equation as:

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1+i} \tag{1.22}$$

[5] As stated in Obstfeld and Rogoff (1996, p. 65, note 4), only the NPG condition is a true constraint on the individual. It is certainly possible not to consume all lifetime resources if one wishes; the TVC implies only that doing so would not be optimal, as long as the marginal utility of consumption is positive.

[6] Since the return function may be unbounded, we should impose the assumption that U_t is bounded for all feasible consumption plans. This assumption may be slightly weakened, since what we need is simply that the superior of U_t on R_+^{∞} is bounded (equivalently, that is bounded in the sup norm).

Equation (1.22) states that the marginal rate of substitution between c_t and c_{t+1} along an optimal plan is equal to the ratio between $(1+i)^{-1}$, the relative price of consumption[7] at date $t+1$, and 1, the relative price of consumption at date t.

Rewrite (1.22) as:

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1+\rho}{1+i} \tag{1.23}$$

Being $u'' < 0$ for the strict concavity of the return function, equation (1.23) implies that:

$$\begin{cases} \Delta c_{t+1} > 0 & \text{if } i > \rho \\ \Delta c_{t+1} < 0 & \text{if } i < \rho \\ \Delta c_{t+1} = 0 & \text{if } i = \rho \end{cases} \tag{1.24}$$

Remark 6 *If the interest rate is greater than the intertemporal discount rate then the optimal consumption path is increasing over time. If, instead, the interest rate is lower, the optimal consumption path is decreasing. Finally, if they are the same, the optimal consumption path is constant.*

1.2.5 The optimal consumption path

Focus on the last case[8], $i = \rho$. By imposing the TVC, we make the intertemporal budget constraint hold with equality:

$$\sum_{s=t}^{\infty} \frac{c_s}{(1+i)^{s-t}} = (1+i)\,a_t + \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \tag{1.25}$$

Being $c_t = \bar{c}\ \forall t$, we can solve (1.25) for $\bar{c}$, the optimal constant level of consumption:

$$\begin{aligned} \bar{c} &= \frac{i}{1+i}\left[(1+i)\,a_t + \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}}\right] \\ &= ia_t + \frac{i}{1+i}\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \end{aligned} \tag{1.26}$$

We can rewrite (1.26) as:

$$\bar{c} = ia_t + \bar{y}_t \tag{1.27}$$

where ia_t is the *current asset income* and:

$$\bar{y}_t \equiv \frac{i}{1+i}\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \tag{1.28}$$

is the *current permanent income*[9]. The current transitory income is defined as $\tilde{y}_t \equiv y_t - \bar{y}_t$.

[7] To obtain one unit of consumption good next period, the individual has to save $(1+i)^{-1}$ units of consumption good. This is the price of future consumption in terms of present consumption.

[8] For the moment, we are still in a partial equilibrium framework, since the interest rate is constant and exogenous. If the interest rate were really different from the intertemporal discount rate, our individual would either starve in the long-run, or consume the worldwide resources. These outcomes are quite unlikely.

[9] We adapt the terminology introduced by Friedman (1957).

Note that, by evaluating (1.26) at date t and $t+1$, we obtain:

$$a_{t+1} + \sum_{s=t+1}^{\infty} (1+i)^{t-s} y_s = a_t + \sum_{s=t}^{\infty} (1+i)^{t-s-1} y_s \tag{1.29}$$

Reorganizing (1.29) leads to:

$$s_t = -\sum_{s=t}^{\infty} \frac{\Delta y_{s+1}}{(1+i)^{s-t+1}} \tag{1.30}$$

Equations (1.27) and (1.30) tell us many interesting things:

Remark 7 ***Consumption depends only on asset income and permanent income.*** *The individual saves to smooth consumption over time; in other words, the individual accumulates assets in order to transfer purchasing power from high income periods to low income periods. She prefers to consume a constant amount of goods across time, instead of large amounts in "good" periods and low ones in "bad" periods. This is basically what is known as the life-cycle/permanent-income hypothesis, introduced by Modigliani, Brumberg, and Friedman.*

Remark 8 ***Saving depends only on transitory income****, since $s_t = \tilde{y}_t - ia_t$. Saving compensates the present value of future changes in exogenous income: in other words, it anticipates future declines in disposable income. Assume that $y_t = y \; \forall t$. Equations (1.27) and (1.30) tell us that, in this case, $\bar{c} = ia_t + y$ and $s_t = 0$. In words, when the exogenous income level is constant, the individual does not save. Assume now that exogenous income decreases by one unit in period $t+1$, and then returns to its previous, constant, value. Savings increase in period t by $(1+i)^{-1}$, the exact amount needed to increase future asset income by one unit, and compensate the decrease in exogenous income: the individual saves for the "rainy days."*

Remark 9 ***The propensity to consume out of a transitory shock to income is less than one.*** *The effect on consumption of a windfall gain in period t, i.e. a marginal and unexpected increase in y_t, equals $\partial \bar{c} / \partial y_t = i/(1+i) < 1$.[10] The reason is clear: a marginal increase in y_t, leaving the future income stream unaffected, has only a very limited effect on permanent income, and so a very limited effect on consumption. This result has a somewhat Keynesian flavor, even if the theoretical underpinnings are radically different.*

Exercise 10 *Which of the many assumptions imposed on the utility function is directly responsible for the consumption smoothing behavior?*

Isoelastic utility

Let us be more specific, and assume that the instantaneous utility function is isoelastic:

$$u(c_t) = \frac{c_t^{1-\frac{1}{\mu}} - 1}{1 - \frac{1}{\mu}} \tag{1.31}$$

[10] Note that $i/(1+i) < i$ as long as $i > 0$.

The isoelastic utility function is characterized by a constant elasticity of intertemporal substitution, defined as $-u'(c) / [cu''(c)]$ and equal in this case to μ.[11] Substitute $u'(c) = c^{-1/\mu}$ in (1.21) and solve interactively for c_s as a function of c_t, where $s \geq t$:

$$c_s = [\beta^\mu (1+i)^\mu]^{s-t} c_t \tag{1.32}$$

Substitute[12] (1.32) into (1.25):

$$c_t = \frac{(1+i) a_t + \sum_{s=t}^{\infty} (1+i)^{t-s} y_s}{\sum_{s=t}^{\infty} \left[\beta^\mu (1+i)^{\mu-1}\right]^{s-t}} \tag{1.33}$$

and simplify to obtain:

$$c_t = (i+\varphi) a_t + \frac{i+\varphi}{1+i} \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \tag{1.34}$$

where $\varphi \equiv 1 - \beta^\mu (1+i)^\mu$. By defining the current permanent income as:

$$\bar{y}_t = \frac{i+\varphi}{1+i} \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \tag{1.35}$$

we can generalize the statement made in the previous Section: current consumption depends only on the current asset income and the current permanent income.

Substitution, income, and wealth effects

A change in the constant interest rate, being a change in the relative price of consumption in all future dates[13], has a substitution effect, an income effect, and a wealth effect on the overall consumption path. These effects may or may not go in the same direction. In other words, an increase (decrease) in the interest rate causes certainly an increase (decrease) in the slope of the consumption path, as implied by (1.23), but its effect on the current and future consumption levels is a priori indeterminate.

To develop a clearer intuition, we focus on a simple, two period intertemporal consumption problem, depicted in Figure 1.1. The starting optimal allocation is point a, located at the right of the endowment point. The slope of the intertemporal budget constraint is $-(1+i)$. An increase in the interest rate (a decrease in the relative price of c_2) directly increases the budget constraint's negative slope; of course, the new budget constraint intersects the old one at the given endowment point e. The substitution effect, induced by the decrease in the relative price of c_2, leads our individual to consume more c_2 and less c_1, moving the allocation from a to b. The decrease in the relative price of future consumption implies an increase in the purchasing power of any given level of lifetime resources, generating an income effect that moves the allocation from b to c. In

[11] The isoelastic utility function converges to a logarithmic utility function when $\mu \to 1$. Note that the -1 in (1.31) is essential for this result, being otherwise completely redundant.

[12] Substituting (1.31) and (1.32) into (1.1) gets $U_t = \iota \sum_{s=t}^{\infty} \left[\beta^\mu (1+i)^{\mu-1}\right]^{s-t}$, where ι is a constant depending on c_0. The objective function converges to a real number for any initial choice of c_0, *i.e.* it is bounded, if and only if $\beta^\mu (1+i)^{\mu-1} < 1$. We impose this assumption.

[13] To consume one unit in period $t+j$, where $j \geq 1$, the individual has to save $(1+i)^{-j}$ units in period t. So, $(1+i)^{-j}$ is the price of consumption at date $t+j$ in terms of consumption at date t.

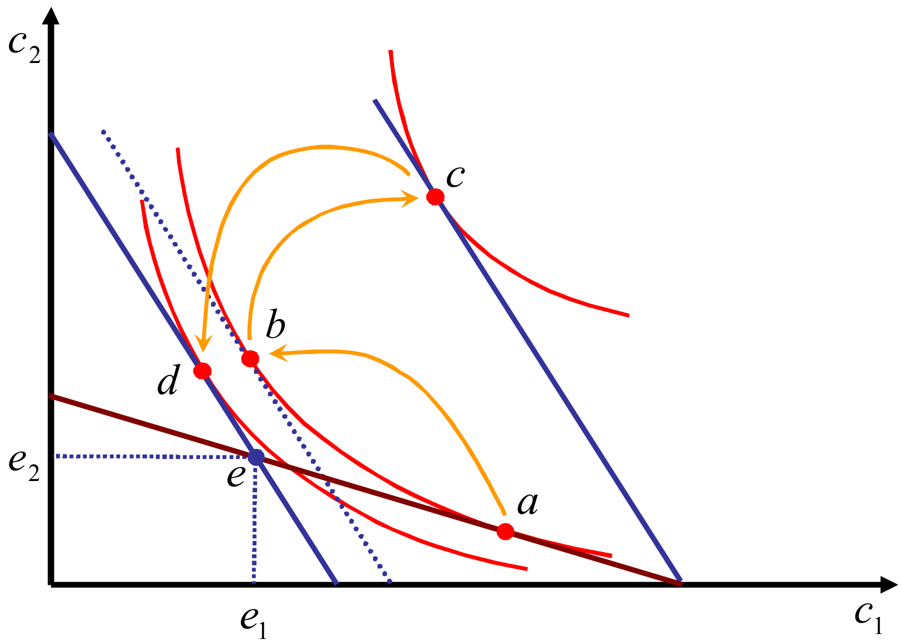

Figure 1.1: Intertemporal substitution, income and wealth effects

our example, both c_2 and c_1 do increase; in general, however, we cannot determine a priori the sign of the income effect. Finally, the decrease in the relative price of future consumption decreases the value of the individual's lifetime resources, generating a wealth effect. Again, in our example, the wealth effect moves the allocation from c to d, the new optimal allocation, decreasing again both c_2 and c_1. In general, however, also the sign of the wealth effect is a priori undetermined, even if it is directly linked to the sign of the income effect.

Summary 11 *To summarize, an increase (decrease) in the interest rate implies:*

1. *A negative (positive) substitution effect that increases the slope of the optimal consumption path, induced by the decrease in the relative price of future consumption;*

2. *An income effect, induced by the increase in the purchasing power of a given discounted income stream;*

3. *A wealth effect, induced by the decrease in the present value of the future income stream.*

4. *The overall effect on current and future consumption is a priori undetermined.*

Assume however that the instantaneous utility function is isoelastic, as in the previous Section, with $\mu = 1$ (logarithmic utility). Equation (1.34) becomes:

$$c_t = (1-\beta)\left[(1+i)\,a_t + \sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}}\right] \tag{1.36}$$

Remark 12 *In this very particular case, the substitution and income effects cancel themselves out. Consumption is simply equal to a constant share, depending on the intertemporal discount factor, of the present market value of lifetime resources.*

Changes in the interest rate affect consumption only via their effect on the current asset income and the prices of future consumption, i.e. the prices at which future endowments can be sold on the market for consumption loans. The wealth effect is mixed: there is a positive component, given by the increase in current asset income, and a negative effect, given by the decrease in the present value of the future income stream. The negative component is likely to dominate.

Exercise 13 *The government introduces a proportional tax on asset income, paying the revenues back trough a lump-sum transfer. How does this policy affect consumption? Discuss separately the substitution and income effects.*

Potentially binding borrowing constraints

Let us now suppose that our individual faces a more stringent and potentially binding borrowing constraint. More precisely, let us impose that $a_t \geq -\phi$ for all $t \geq 0$, where $0 \leq \phi < b$ is exogenously given. We can set, without loss of generality, $\phi = 0$.[14]

The first order and slackness conditions for the dynamic optimization problem can be combined to form the following "Euler inequality":

$$\begin{cases} u'(c_t) > (1+i)\,\beta u'(c_{t+1}) & \text{if } a_{t+1} = 0 \\ u'(c_t) = (1+i)\,\beta u'(c_{t+1}) & \text{if } a_{t+1} > 0 \end{cases} \tag{1.37}$$

The implications of (1.37) as far as the dynamics of the optimal consumption path are relatively straightforward, and evidently depend on the relationship between the interest rate i and the rate of intertemporal substitution ρ. There are three possible cases: let us analyze them in turn:

1. The simplest case arises when $i > \rho$. In that case, (1.37) implies that $c_{t+1} > c_t$ for all $t \geq 0$; hence, $\lim_{t\to\infty} c_t = \infty$ and consequently, being the exogenous income level bounded by assumption, $\lim_{t\to\infty} a_t = \infty$.

2. Consider now the case in which $i = \rho$. Evidently, (1.37) implies that $c_{t+1} > c_t$ and $c_t = y_t$ whenever $a_{t+1} = 0$, and $c_{t+1} = c_t$ as soon as $a_{t+1} > 0$.[15] The natural question that follows is whether c_t will converge or not to a finite limit in the long run. The answer is reassuring: it can be shown[16] that, in this case, $\lim_{t\to\infty} c_t = \sup_t \bar{y}_t$, where $\bar{y}_t$, the current permanent income level, has been defined in (1.28). The intuition behind this result is simple: the borrowing constraint may bind only when our individual tries to transfer purchasing power from the future to the present because income - and consequently the permanent income level - is expected to increase; as soon as permanent income is expected to remain constant or decrease over time, the incentive to borrow disappears. Hence, the increase in consumption has to stop as soon as $\bar{y}_t$ reaches its maximum value, and from that date onwards consumption has to remain constant at $\sup_t \bar{y}_t$, being the individual finally free to smooth consumption over time.

[14] Chamberlain and Wilson (2000, p. 370) show that any problem characterized by an arbitrary borrowing limit can be mapped into an equivalent problem in which income is nonnegative and borrowing is not permitted.

[15] Note that consumption will never *decrease* over time, since saving is freely permitted and the individual prefers a constant consumption path.

[16] See Chamberlain and Wilson (2000, Th. 3, p. 375).

3. Finally, consider the case in which $i < \rho$. In this case, (1.37) clearly implies that $c_t > c_{t+1}$ as long as the individual is not credit-constrained, i.e. as long as $a_{t+1} > 0$. Since consumption is a strictly increasing function of the current asset stock - more precisely, the policy function for consumption, $c(a)$, is strictly increasing in a,[17] i.e. $c'(a) > 0$ - the straightforward consequence is that $a_t > a_{t+1}$ as long as $a_{t+1} > 0$: hence, we can expect the individual to reach the borrowing limit in finite time. Furthermore, we can conclude that once our individual becomes credit-constrained, she remains constrained forever. The intuition is straightforward: if $a_t = 0$ and $a_{t+1} > 0$, then (1.37) implies that $u'[c(0)] < u'[c(a')]$. But if $c(a)$ is a strictly increasing in a, then $c(a') > c(0)$; hence $u'[c(0)] > u'[c(a')]$: a contradiction! Note however that (1.37) alone is not enough to characterize the dynamics of c_t when the individual is credit-constrained, i.e. when $a_{t+1} = 0$: we need further pieces of information about the dynamics of exogenous income. A natural step forward is to assume that income is constant over time at some strictly positive level $\bar{y}$. Under this assumption, the previous discussion suggests that c_t will converge from above to $\bar{y}$ in finite time, and will remain constant form that date onwards.

Summary 14 *Three possible cases:*

1. *If $i > \rho$, then $\lim_{t\to\infty} c_t = +\infty$ and $\lim_{t\to\infty} a_t = +\infty$.*
2. *If $i = \rho$, then $\lim_{t\to\infty} c_t = \sup_t \bar{y}_t$, converging from below.*
3. *If $i < \rho$ and $y_t = \bar{y}$, then $\lim_{t\to\hat{t}<\infty} c_t = \bar{y}$, converging from above.*

1.3 Stochastic setting

Assume now that the sequence $\{y_s\}_{s=t}^{\infty}$ is an exogenous stochastic process[18]. The future income stream becomes uncertain. To deal with uncertainty, the framework has to be slightly extended. In particular, we have to modify our assumptions on the individual's preferences, since the future consumption stream, depending on the income stream, becomes a stochastic sequence too.

1.3.1 Preferences

We assume that preferences of our individual may be represented by:

$$U_t = E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} u(c_s)\right] \tag{1.38}$$

where E_t is the mathematical expectation operator conditional on information available at date t. The information set contains the present and past values of all variables in the model. The return function is C^3, strictly increasing, and strictly concave, again with $\lim_{c\to 0} u'(c) = +\infty$.

[17] More precisely, the policy function for consumption, $c(a)$, is a strictly increasing in a, i.e. $c'(a) > 0$. For a formal proof, see Krueger (2002, Ch. 10, Proposition 102, p. 259).

[18] We follow Sargent (1987a, p. 364) and assume that $\{y_s\}_{s=t}^{\infty}$ is a stochastic process of mean exponential order less than β^{-1}. This implies that $E_t(\beta^j y_{t+j}) \to 0$ as $j \to \infty$. Intuitively, we are assuming that expected future incomes are not "too big".

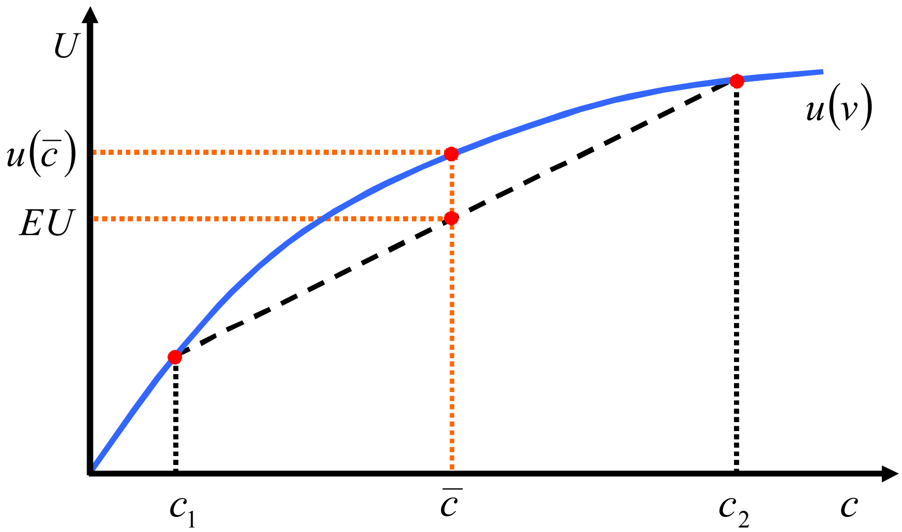

Figure 1.2: Risk aversion

The objective function can be interpreted as a *von Neumann-Morgenstern* utility function, and the return function as a *Bernoulli* utility function reflecting the individual's attitude toward risk. The strict concavity of the instantaneous utility function implies *risk aversion* (see Mas-Colell *et al.*, 1995, ch. 6). Such a Bernoulli utility function is depicted in Figure 1.2.

Under risk aversion, the utility generated by a certain level of consumption, say $\bar{c}$, is always higher than the expected utility generated by a lottery over two (or more) possible consumption levels, say c_1 and c_2, giving the same expected outcome. We define the *Arrow-Pratt coefficient of relative risk aversion* as:

$$\sigma \equiv -\frac{cu''(c)}{u'(c)} \tag{1.39}$$

The coefficient of relative risk aversion corresponds clearly to the reciprocal of the elasticity of intertemporal substitution, and is therefore related to the curvature of the instantaneous utility function.[19]

1.3.2 The optimization problem

Being now c_t and y_t random variables, the NPG condition and the intertemporal budget constraint have to hold in expectations. The individual maximizes (1.38) subject to (1.4) and (1.5), taking again a_t, i, and the stochastic process governing y_t as given. Having no control over the actual sequences, she maximizes over the contingency plans for c_t and

[19] The strict link between elasticity of intertemporal substitution and relative risk aversion is a strong property of time-separable intertemproal utility functions.

a_{t+1}. Formally, she solves a stochastic optimal control problem of the form:

$$\max_{\{c_s,a_{s+1}\}_{s=t}^{\infty}} \quad U_t = E_t\left[\sum_{s=t}^{\infty}\beta^{s-t}u(c_s)\right] \tag{1.40}$$
$$\text{s.t.} \quad a_{t+1} = (1+i)\,a_t + y_t - c_t$$
$$\lim_{j\to\infty} E_t\left[\frac{a_{j+1}}{(1+i)^j}\right] \geq 0$$

We can build a present-value Lagrangian in expectations. Under some regularity conditions[20], we are allowed to partially derive it with respect to c_s, a_{s+1}, and λ_s, obtaining the following first order conditions:

$$u'(\hat{c}_t) = \hat{\lambda}_t \tag{1.41}$$
$$\beta(1+i)\,E_t\left(\hat{\lambda}_{t+1}\right) = \hat{\lambda}_t \tag{1.42}$$
$$\hat{a}_{t+1} = (1+i)\,\hat{a}_t + y_t - \hat{c}_t \tag{1.43}$$

We impose also a stochastic TVC:[21]

$$\lim_{j\to\infty}\beta^j E_t\left(\hat{\lambda}_j\hat{a}_{j+1}\right) = 0 \tag{1.44}$$

1.3.3 The stochastic Euler equation

Substituting (1.41) into (1.42) we obtain a stochastic version of the Euler equation discussed in the previous Section:

$$(1+i)\,\beta E_t\left[u'(c_{t+1})\right] = u'(c_t) \tag{1.45}$$

Equation (1.45) is our key result. By rewriting it as:

$$E_t\left[u'(c_{t+1})\right] = \frac{1}{(1+i)\,\beta}u'(c_t) \tag{1.46}$$

we reproduce the seminal result in Hall (1978). He concluded that, since the marginal utility of consumption follows a univariate first order Markov process, no other variables should Granger-cause it: quite a remarkable implication![22]

Remark 15 *If $i=\rho$, equation (1.46) collapses to:*

$$E_t[u'(c_{t+1})] = u'(c_t) \tag{1.47}$$

In this special case, the marginal utility of consumption follows a ***martingale***[23]*.*

[20] We have to justify the interchange of limits and integration. In our case, these conditions are generally satisfied.

[21] Note that the NPG holds in expectations too; this implies that the first order conditions and the TVC are jointly necessary and sufficient.

[22] A stochastic process y_t is said not to Granger-cause a process x_t if $E(x_{t+1} \mid x_t, x_{t-1}, ..., y_t, y_{t-1}, ...) = E(x_{t+1} \mid x_t, x_{t-1}, ...,)$.

[23] A martingale is a stochastic process such that $E_t(x_{t+1}) = x_t$. See Shiryæv (1989, ch. VII, Def. 1, p. 474).

Exercise 16 *Assume that income is completely diversifiable, i.e. assume that the individual can buy property claims to other individual's incomes, and sell property claims to her own one. Furthermore, assume that the Bernoulli utility function is isoelastic. Obtain a closed form solution for* c_t *and interpret the result.*

Potentially binding borrowing constraints

More generally, under a potentially binding borrowing constraint, the first order and slackness conditions can be combined to obtain the following inequality:

$$E_t\left[u'\left(c_{t+1}\right)\right] \leq \frac{1}{(1+i)\beta}u'\left(c_t\right) \tag{1.48}$$

Note that, being $(1+i)^t\beta^t \geq 0$ for all $t \geq 0$, we can rewrite (1.48) as:

$$E_t\left[(1+i)^{t+1}\beta^{t+1}u'\left(c_{t+1}\right)\right] \leq (1+i)^t\beta^t u'\left(c_t\right) \tag{1.49}$$

Defining $M_t \equiv (1+i)^t\beta^t u'\left(c_t\right) \geq 0$, the previous expression becomes the following:

$$E_t\left(M_{t+1}\right) \leq M_t \tag{1.50}$$

Hence, the random variable M_t behaves as a *nonnegative supermartingale*.[24]

This simple fact has some strong implications for the dynamics of the optimal consumption level. A well known convergence theorem of Doob for supermartingales states that:[25]

$$P\left(\lim_{t\to\infty} M_t = \bar{M}\right) = 1 \tag{1.51}$$

where $\bar{M}$ is a *random variable* such that $E\left(\bar{M}\right) < +\infty$.

1. If $i > \rho$, then $\lim_{t\to\infty}(1+i)^t\beta^t = \infty$, and therefore M_t can converge to a finite limit only if $\lim_{t\to\infty} u'\left(c_t\right) = 0$. This evidently implies that $\lim_{t\to\infty} c_t = \infty$ and consequently that $\lim_{t\to\infty} a_t = \infty$.

2. If $i = \rho$, then $(1+i)\beta = 1$ and $M_t = u'\left(c_t\right)$. Can we conclude in this case that c_t will converge to a finite limit, i.e. to a bounded random variable? Quite surprisingly, in general we cannot. It can be shown[26] that, if the exogenous income process is "sufficiently stochastic," i.e. if there is a $\varphi > 0$ such that:

$$\operatorname{var}_t\left(\sum_{s=t}^{\infty}\frac{y_s}{(1+i)^{s-t}}\right) \geq \varphi \tag{1.52}$$

for all $t \geq 0$, and the Bernoulli function $u(\cdot)$ is bounded, then:

$$P\left(\lim_{t\to\infty} c_t = \infty\right) = 1 \tag{1.53}$$

and consequently $P\left(\lim_{t\to\infty} a_t = \infty\right) = 1$ too. The same result obtains when the

[24] A nonnegative supermartingale is a stochastic process such that: (i) $x_t \geq 0$; (ii) $E\left(x_t\right) < +\infty$; (iii) $E_t(x_{t+1}) \leq x_t$. See again Shiryæv (1989, ch. VII, Def. 1, p. 475).

[25] See Shiryæv (1989, ch. VII, Th. 1, p. 508).

[26] For further details and formal proofs, see Chamberlain and Wilson (2000, Corollary 2, p. 381).

exogenous income process is *iid* and the Bernoully function unbounded, or when the marginal utility of consumption is convex, i.e. when $u'''(\cdot) > 0$.

Remark 17 *This result contrasts sharply with its counterpart under certainty: when income is non-stochastic, consumption tends to a finite limit; as soon as income becomes sufficiently stochastic, consumption has to diverge to infinite.*

3. Finally, if $i < \rho$, then $\lim_{t\to\infty}(1+i)^t \beta^t = 0$, and therefore $\bar{M}$ will surely converge to zero. Hence, the result $P(\lim_{t\to\infty} M_t = 0) = 1$ does not necessarily imply that $\lim_{t\to\infty} c_t = \infty$, but leaves open the possibility that both c_t and a_t converge in the long run to stationary, and finite, random variables.

Summary 18 *To summarize:*

1. *If $i > \rho$, then $P(\lim_{t\to\infty} c_t = \infty) = 1$ and $P(\lim_{t\to\infty} a_t = \infty) = 1$.*
2. *If $i = \rho$, then $P(\lim_{t\to\infty} c_t = \infty) = 1$ and $P(\lim_{t\to\infty} a_t = \infty) = 1$ if (i) income is "sufficiently stochastic" and $u(\cdot)$ bounded, or (ii) income is iid and $u(\cdot)$ unbounded, or (iii) $u'''(\cdot) > 0$.*
3. *If $i < \rho$, then $P(\lim_{t\to\infty} c_t = c) = 1$ and $P(\lim_{t\to\infty} a_t = a) = 1$, where c and a are two random variables such that $E(c) < +\infty$ and $E(a) < +\infty$.*

1.3.4 Certainty equivalence

To easily obtain a close form solution, assume that the instantaneous utility function is linear-quadratic in consumption[27]:

$$u(c_t) = c_t - \frac{\alpha}{2}c_t^2 \tag{1.54}$$

where $\alpha > 0$, so that $u'(c_t) = 1 - \alpha c_t$. Under this assumption, (1.46) becomes:

$$E_t(c_{t+1}) = \psi_0 + \psi_1 c_t \tag{1.55}$$

where $\psi_0 \equiv [(1+i)\beta - 1]/[(1+i)\beta\alpha]$ and $\psi_1 \equiv [(1+i)\beta]^{-1}$. If we furthermore assume that $i = \rho$, we conclude that the consumption level itself follows a martingale:

$$E_t(c_{t+1}) = c_t \tag{1.56}$$

Consider now the intertemporal budget constraint (1.25), in expectations:

$$\sum_{s=t}^{\infty} \frac{E_t(c_s)}{(1+i)^{s-t}} = (1+i)a_t + \sum_{s=t}^{\infty} \frac{E_t(y_s)}{(1+i)^{s-t}} \tag{1.57}$$

[27] The linear-quadratic model is a useful example, but has some unpleasant properties (see Sargent, 1987a, p. 366, n. 5). For instance, note that there is a satiation level of consumption, $c = 1/\alpha$. To bypass this problem, we assume that the satiation level is large compared to average income. More specifically, we assume that $E_t\left[\sum_{j=0}^{\infty}(1+i)^{-j}y_{t+j}\right] \leq 1/\alpha$ holds with probability one. Furthermore, note that $u' \to 1$ as $c \to 0$: hence, consumption can be zero, and even negative, in equilibrium.

Substitute (1.56) into (1.57); consider that, for the Law of Iterated Expectations[28], $E_t(c_s) = c_t \; \forall s \geq t$, and solve the result for c_t:

$$c_t = ia_t + \frac{i}{1+i}\sum_{s=t}^{\infty}\frac{E_t(y_s)}{(1+i)^{s-t}} \tag{1.58}$$

Define the *current expected permanent income* as:

$$\overline{y}_t \equiv \frac{i}{1+i}\sum_{s=t}^{\infty}\frac{E_t(y_s)}{(1+i)^{s-t}} \tag{1.59}$$

Equation (1.58) shows that, under our simplifying assumptions, current consumption depends only on the current asset income and the current expected permanent income.

Remark 19 *Compare (1.26) and (1.59); the similarity is striking. We are contemplating what is known as the* ***certainty equivalence principle****. With linear-quadratic utility, the individual acts under uncertainty as if future random variables will turn out equal to their conditional mean.*

Evaluating (1.58) at date $t+1$ and taking the first difference:

$$\begin{aligned} \Delta c_{t+1} &= i\Delta a_{t+1} + \sum_{s=t+1}^{\infty} i(1+i)^{t-s} E_{t+1}(y_s) + \\ &\quad -\sum_{s=t}^{\infty} i(1+i)^{t-s-1} E_t(y_s) \end{aligned} \tag{1.60}$$

Substitute (1.5) into (1.60):

$$\begin{aligned} \Delta c_{t+1} &= i(ia_t + y_t - c_t) + \\ &\quad + \sum_{s=t+1}^{\infty} i(1+i)^{t-s} E_{t+1}(y_s) - \sum_{s=t}^{\infty} i(1+i)^{t-s-1} E_t(y_s) \end{aligned} \tag{1.61}$$

Substitute now (1.58) into (1.61) and simplify:

$$\Delta c_{t+1} = iy_t + \sum_{s=t+1}^{\infty} i(1+i)^{t-s}E_{t+1}(y_s) - \sum_{s=t}^{\infty} i(1+i)^{t-s}E_t(y_s) \tag{1.62}$$

A further reorganization of (1.62) takes us to:

$$\Delta c_{t+1} = \frac{i}{1+i}\sum_{s=t+1}^{\infty}\frac{E_{t+1}(y_s) - E_t(y_s)}{(1+i)^{s-t-1}} \tag{1.63}$$

Remark 20 *Equation (1.63) shows that the change in consumption between dates t and $t+1$ depends on the difference between the present value of the expected future income stream, conditional on information available at date $t+1$, and the present value of the same expected income stream, conditional on information available at date t.*

[28] The Law of Iterated Expectations states that $E_t[E_{t+j}(x_{t+j})] = E_t(x_{t+j})$.

In other words, consumption changes from dates t to $t+1$ only if further pieces of information on the future income stream become available at date $t+1$.

To further extend our analysis, we need to be more specific as far as the stochastic process governing exogenous income is concerned. For the sake of simplicity[18], assume that the income level follows a simple white noise process of the form $y_t = \bar{y} + \varepsilon_t$, where ε_t is a zero-mean *iid* innovation. Clearly $E_i(y_s) = \bar{y}$ for all $s \geq i+1$. Substituting the last result into (1.58) gets:

$$c_t = ia_t + \bar{y} + \frac{i}{1+i}\varepsilon_t \tag{1.64}$$

Here we see the certainty equivalence principle at work: the individual behaves as if all future incomes will turn out equal to their conditional mean.

We easily show that:

$$\Delta c_{t+1} = \frac{i}{1+i}\varepsilon_{t+1} \tag{1.65}$$

The change in consumption is a simple linear function of the unpredictable changes in income. The propensity to consume out of the innovation is positive but less that one (recall point 9, p. 10).

Exercise 21 *Show that* $c_t = \overline{y} + \frac{i}{1+i}\sum_{j=0}^{\infty}\varepsilon_{t-j}$.

Exercise 22 *Define total income as* $Y_t = ia_t + y_t$. *Show that* $c_t = \frac{i}{1+i-L}Y_t$.

We may slightly generalize these results by assuming that the income level follows a stationary AR(1) process of the form $y_{t+1} = \phi y_t + \varepsilon_{t+1}$, where ε_t is again a zero-mean *iid* innovation and $\phi \in (0,1]$. This implies that $E_j(y_s) = \phi^{s-j}y_j$ for all $s \geq j$ and $E_{j+1}(y_s) - E_j(y_s) = \phi^{s-j-1}\varepsilon_{j+1}$ for all $s \geq j+1$. Substituting the last result into (1.63) gets:

$$\Delta c_{t+1} = \frac{i}{1+i-\phi}\varepsilon_{t+1} \tag{1.66}$$

Compare equations (1.65) and (1.66). The propensity to consume out of a unexpected shock implied by (1.66) is higher than the same propensity implied by (1.65). The reason is clear.

Remark 23 *If income follows an autoregressive process, each innovation has a transitory but long-lasting effect that dies out only in the limit, since* $y_t = \sum_{j=-\infty}^{t}\phi^{t-j}\varepsilon_j$. *The degree of income's persistence depends positively on* ϕ. *The more persistent income is, the more reactive the expected permanent income is to unexpected shocks, and the higher the propensity to consume.*

Exercise 24 *Assume that the income follows a random walk of the form* $y_{t+1} = y_t + \varepsilon_{t+1}$, *where* ε_t *is a zero-mean iid innovation. Obtain an expression for* Δc_{t+1} *(easy!) and interpret the result.*

Exercise 25 *Assume that the first difference of income follows a stationary AR process of the form* $\Delta y_{t+1} = \phi\Delta y_t + \varepsilon_{t+1}$, *where* ε_t *is a zero-mean iid innovation and* $\phi \in (0,1)$ *Obtain an expression for* Δc_{t+1}. *Discuss the relative volatility of consumption vs. income and interpret your results.*

[18] Sargent (1987a, pp. 366-368) uses the general Wold representation.

1.4 Precautionary savings

The previous Section was completely devoted to the so-called linear-quadratic case. We showed that linear-quadratic utility implies certainty equivalence, *i.e.* that the individual acts as if all future incomes will turn equal to their expected values. This kind of behavior is a crude (and somewhat misleading) approximation of the observed consumption behavior. Available empirical evidence suggests that consumption reacts also to changes in the variability of future income streams. We extend our analysis in this direction, providing a intuitive characterization of what is known as *precautionary saving*.

We already stressed that the strict concavity of the return function implies a positive degree of risk aversion. A risk averter always prefers a certain outcome to a lottery that yields the same expected outcome. In other words, under risk aversion, a higher degree of uncertainty about future consumption decreases expected utility. However, equation (1.46) shows that the individual's consumption choices depend exclusively on the marginal utility of consumption. An increase in uncertainty will affect the optimal consumption path only through its effect on marginal utility.

In the literature on choice under uncertainty, it is customary to assume a non-increasing level of relative risk aversion (see Mas-Colell *et al.*, 1995, p. 193). Under this assumption, the individual becomes less risk averse regard lotteries that are proportional to her wealth as her wealth increases.[29] A necessary (but not sufficient) condition for the relative risk aversion to be non-increasing is a strictly positive third derivative of the Bernoulli utility function, *i.e.* $u''' > 0$. In other words, we assume that the marginal utility of consumption is a strictly convex function.

Under this assumption, the expected marginal utility of consumption depends positively on the degree of uncertainty. Since the first derivative of marginal utility is negative, the Euler equation requires the optimal consumption path to become steeper if uncertainty about the future income stream increases. In other words, under non-increasing relative risk aversion, an increase in uncertainty leads to an increase in precautionary saving.

1.4.1 A simple example

To obtain a clearer intuition, assume that there are only two periods, $t \in \{1, 2\}$, and only two possible states of the world, $s \in \{1, 2\}$, where the first state is the "bad" one, and the second is the "good" one. The future income in the "bad" state is lower than in the "good" one, *i.e.* $y_2^1 < y_2^2$, where y_2^s is future income in state s. The probability of the "bad" state happening is equal to 1/2.

The expected future income, $y_2^e = \frac{1}{2}\left(y_2^1 + y_2^2\right)$, is equal to the current income level, y_1. The intertemporal budget constraint holds with probability one:

$$c_1 + \frac{c_2^s}{1+i} = y_1 + \frac{y_2^s}{1+i}, \ \forall s \tag{1.67}$$

The Euler equation becomes (assuming $i = \rho$):

$$\frac{1}{2}\left[u'\left(c_2^1\right) + u'\left(c_2^2\right)\right] = u'\left(c_1\right) \tag{1.68}$$

Under certainty equivalence, *i.e.* when the marginal utility is linear in consumption,

[29] We know from p. 15 that the degree of relative risk aversion is equal to the inverse of the elasticity of intertemporal substitution.

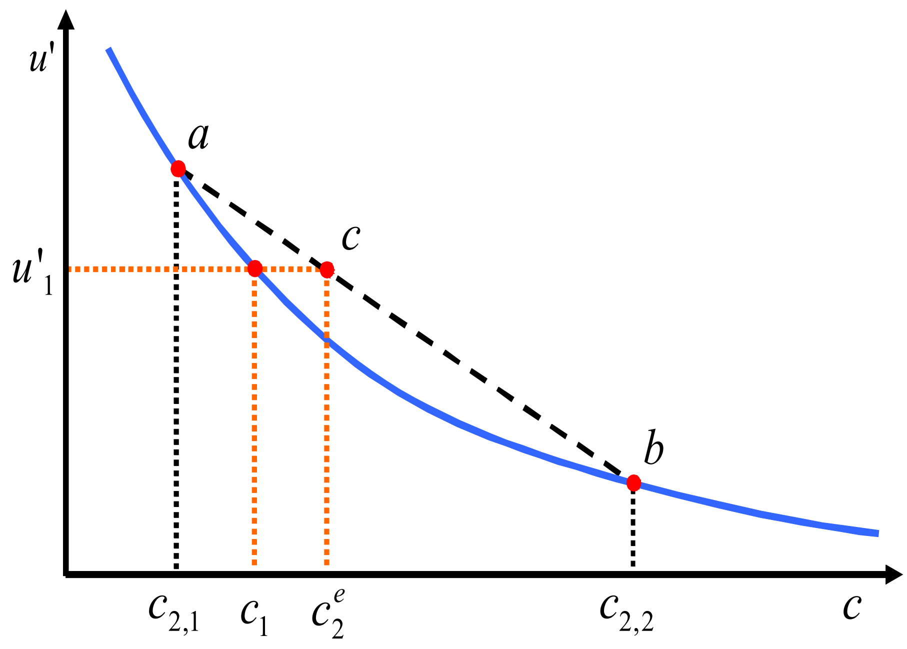

Figure 1.3: Precautionary savings

equation (1.68) implies that $c_2^e = c_1$, where $c_2^e \equiv \frac{1}{2}(c_2^1 + c_2^2)$ is the future expected consumption level. Since the intertemporal budget constraint holds with probability one, it holds in expectations too. Substituting $c_2^e = c_1$ into the intertemporal budget constraint in expectations, we show that, under certainty equivalence, $c_1 = y_1$ and $c_2^s = y_2^s \;\; \forall s$. In other words, given our assumptions, the individual would simply consume her exogenous income without saving anything.

Assume now that the third derivative of the instantaneous utility function is strictly positive, *i.e.* that marginal utility is a convex function of consumption. Figure 1.3 represents a possible allocation of present and future contingent consumption levels that satisfies (1.68).

For the strict convexity of marginal utility, the optimal level of c_1 has to be lower than c_2^e, and this can be achieved only by saving a positive amount in period 1. In other words, the individual saves a positive share of her income in the first period just because she dislikes the variability of income in the second period, *i.e.* only for precautionary motives.

In this simplified framework, we can easily increase the variability of future income without affecting its expected value; we just decrease y_2^1 and increase y_2^2 by the same proportion. The intertemporal budget constraint (1.67) implies that, for any level of current consumption c_1, the future consumption levels contingent on state 1 and 2 have respectively to decrease and increase by the same proportion. The previous allocation does not satisfy the Euler equation anymore, as shown in Figure 1.4. Since the probability does not change, the point c moves straightly upwards; the marginal utility of current consumption is now lower than the expected marginal utility of future consumption.

The new optimal allocation of present and future contingent consumption, represented in Figure 1.5, is reached by decreasing the current consumption level and thereby increasing by the same proportion both future contingent consumption levels. In other words, an increase in uncertainty that leaves the expected value of future income unaffected,

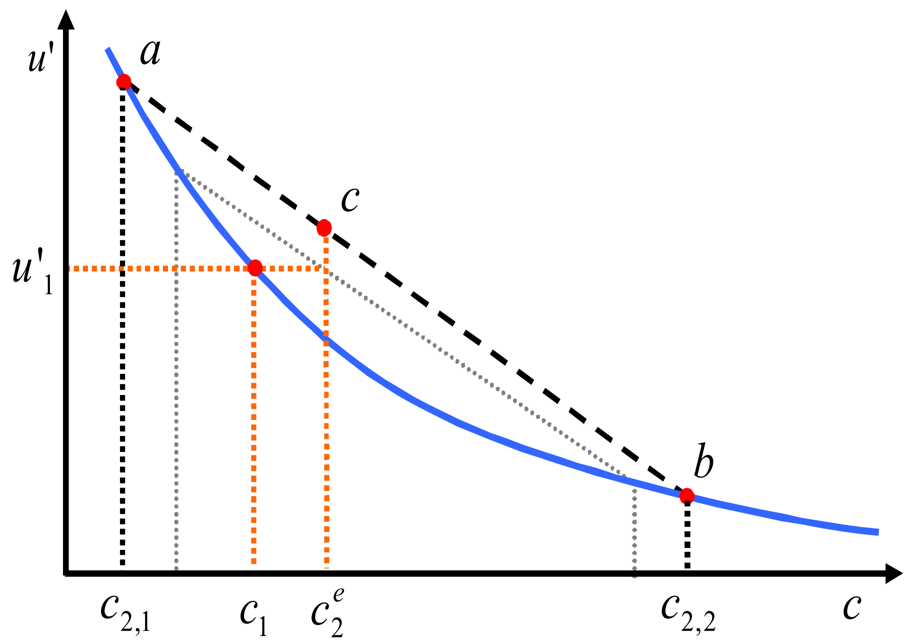

Figure 1.4: Increase in variability

decreases current consumption and increases expected future consumption. Again, the corresponding increase in current savings is only due to a precautionary motive.

Unfortunately, we can hardly push further our qualitative analysis of the precautionary saving behavior, since it's generally impossible to obtain closed form solutions for optimal consumption plans under risk aversion. The following exercise asks the reader to solve indeed one of the few cases in which such a solution exists.

Exercise 26 *Assume that* $u = (-1/\alpha)\exp(-\alpha c_t)$ *and that income follows a random walk,* $y_{t+1} = y_t + \varepsilon_t$*, where* $\varepsilon_t \sim N(0, \sigma^2)$ *is a iid innovation. Obtain a closed form solution for* c_t*. Show that consumption depends negatively on an increase in uncertainty. Verify that your solution satisfies the intertemporal budget constraint for any realization of income.*

1.5 A numerical exercise

We will now numerically solve and simulate a very simple stochastic dynamic *partial equilibrium* consumption model. The scope of this exercise is to quantitatively evaluate the main theoretical predictions developed in the previous Sections.

We assume that the economy is populated by a *continuum* of identical and infinitely living households of measure zero that can be aggregated into a single *representative household*. These households can freely trade consumption loans and debts at the given exogenous interest rate on a competitive asset market, and receive each period a stochastic flow of exogenous income. To make the problem computationally manageable and empirically sensible, we assume that:

- the households' instantaneous utility function is isoelastic;

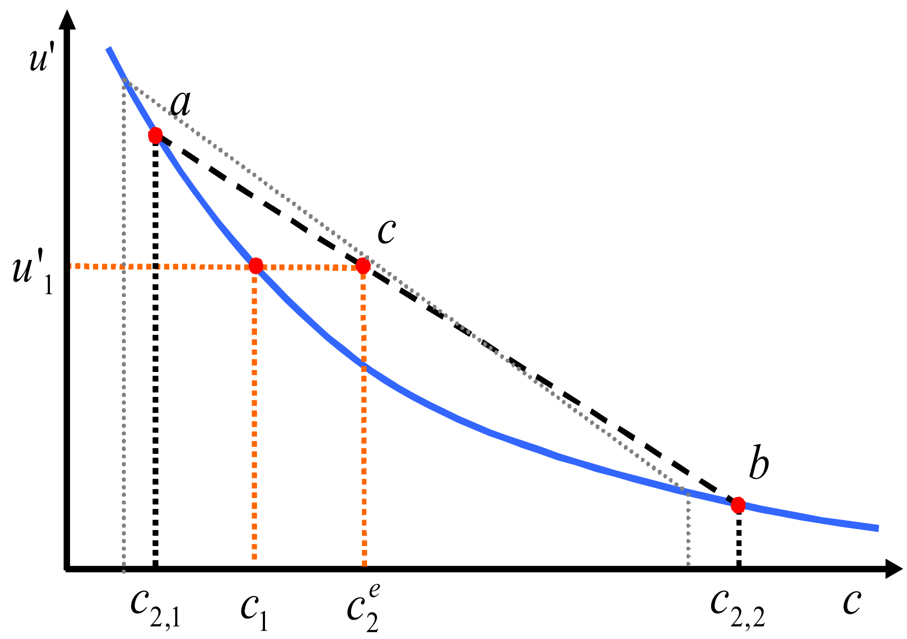

Figure 1.5: New optimal allocation

- the stochastic process driving exogenous income follows a trend-stationary AR(1) process;
- there is an extremely small quadratic cost of holding assets that pins down the steady state.

More formally, we will study the optimal contingent consumption plan for a representative household that solves the following stochastic dynamic optimization problem:

$$\max_{\{C_s, A_{s+1}|Y_t\}_{s=t}^{\infty}} \quad U_t = E_t\left(\sum_{s=t}^{\infty}\beta^{s-t}\frac{C_s^{1-\zeta}}{1-\zeta}\right) \tag{1.69}$$

$$s.t. \quad A_{s+1} = \left(1+i-\frac{\psi}{2}A_s\right)A_s + Y_s - C_s$$

$$\lim_{j\to\infty} E_t\left[\frac{A_{j+1}}{(1+i)^j}\right] \geq 0$$

where C_s is per-capita consumption at date t, A_s the per-capita stock of assets held by the representative household, and Y_s the exogenous income; $\zeta > 0$ is the reciprocal of the elasticity of intertemporal substitution, and measures the degree of relative risk aversion; β the intertemporal discount factor; $i > 0$ is the exogenous interest rate; and $\psi > 0$ a parameter governing the quadratic cost of holding assets.

We assume that the exogenous income can be decomposed into a deterministic component that grows at a constant rate $\gamma - 1 > 0$ and a stationary stochastic process. More precisely, we assume that the logarithm of income can be decomposed into a linear time trend and a stationary AR(1) component:

$$\ln Y_{t+1} = c + (t+1)\ln\gamma + \phi \ln Y_t + \varepsilon_t \tag{1.70}$$

where $\varepsilon_t \sim N(0, \sigma^2)$ is an *iid* innovation.

Intuitively, if income grows exogenously at a positive and constant rate, all other aggregate variables will grow at the same rate in the long-run. Hence, the original model is *non-stationary*, *i.e.* it will converge to a balanced-growth path, not to a steady state.[30] However, our numerical procedures require the model to be stationary: hence, we need to normalize it with respect to the growing component. In other words, we need to divide everything by the deterministic component of Y_t; note that:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \frac{Z_s^{1-\zeta}}{1-\zeta} \left(\frac{C_s}{Z_s}\right)^{1-\zeta} = Z_t^{1-\zeta} \sum_{s=t}^{\infty} \tilde{\beta}^{s-t} \frac{c_s^{1-\zeta}}{1-\zeta} \tag{1.71}$$

where $Z_s \equiv c\gamma^s$, $c_s \equiv C_s/Z_s$, and $\tilde{\beta} \equiv \beta\gamma^{1-\zeta}$; furthermore, note that $A_{t+1}/Z_t = (A_{t+1}/Z_{t+1})(Z_{t+1}/Z_t) = \gamma a_{t+1}$. Hence, the normalized, and stationary, model can be rewritten as:

$$\begin{aligned} \max_{\{c_s, a_{s+1} | y_t\}_{s=t}^{\infty}} \quad & U_t = E_t \left(\sum_{s=t}^{\infty} \tilde{\beta}^{s-t} \frac{c_s^{1-\zeta}}{1-\zeta} \right) \\ s.t. \quad & \gamma a_{s+1} = \left(1 + i - \frac{\psi}{2} a_s\right) a_s + y_s - c_s \\ & \ln y_{s+1} = \phi \ln y_s + \varepsilon_s \end{aligned} \tag{1.72}$$

We can easily obtain the following Euler equation:

$$E_t \left[\tilde{\beta} c_{t+1}^{-\zeta} (1 + i - \psi a_t) \right] = \gamma c_t^{-\zeta} \tag{1.73}$$

In a deterministic setting, the previous equation evaluated at the steady-state would imply that:

$$a = \frac{\gamma - \tilde{\beta}(1+i)}{\psi \tilde{\beta}} \tag{1.74}$$

The only steady-state allocation that is logically consistent with our assumptions is $a = 0$. This statement can be motivated in two related ways: (i) an allocation such that $a \neq 0$ would imply an ever increasing or decreasing asset stock A_t, and this clearly contrasts with our assumption of a constant interest rate (in other words, such a possibility would be just an artifact of our partial equilibrium approach); (ii) in a deterministic steady state, income and consumption remain both constant over time, and therefore the incentive to keep a positive asset stock for precautionary reasons disappears, while the cost of holding it does not.

Hence, to guarantee that asset holdings in the deterministic steady state were zero, we impose that $\tilde{\beta} = \gamma/(1+i)$. Under this assumption, the first order conditions for the

[30] For more technical details, see Sec. 3.1.2, p. 43.

model can be rewritten as:

$$E_t\left[c_{t+1}^{-\zeta}\left(\gamma-\tilde{\beta}\psi a_t\right)\right]=\gamma c_t^{-\zeta} \tag{1.75}$$

$$\gamma a_{t+1}=a_t\left(\frac{\gamma}{\tilde{\beta}}-\frac{\psi}{2}a_t\right)+y_t-c_t \tag{1.76}$$

$$\lim_{t\to\infty}\tilde{\beta}^t E_t\left(c_t^{-\zeta}a_{t+1}\right)=0 \tag{1.77}$$

Given the recursive and concave structure of our problem, the unique solution can be characterized by a time-invariant policy function $c(a,y)$. Since no closed form solution is generally available, we may exploit this property and use a proper numerical method to approximate the policy function over a chosen interval. Hence, note that, under our assumptions, the policy function has to satisfy the following functional equation:

$$E\left[c(a',y')^{-\zeta}\left(\gamma-\tilde{\beta}\psi a\right)\mid a,y\right]=\gamma c(a,y)^{-\zeta} \tag{1.78}$$

where:

$$a'=\frac{a}{\tilde{\beta}}+\frac{y-c(a,y)-\frac{\psi}{2}a^2}{\gamma}$$
$$\ln y'=\phi\ln y+\varepsilon$$
$$\varepsilon\sim N(0,\sigma^2)$$

To obtain an approximated solution to (1.78), we will apply the simplest of the *projection methods* advocate by Judd (1992): *orthogonal collocation* (see the Appendix for details). Before doing that, however, we need to parameterize the model, i.e. find empirically sensible values for the parameters.

Following well established standards in the literature, we set $\zeta=2$ and $\beta=0.9875$. The parameter ψ, which is directly related to the cost of holding assets, is arbitrarily set to 0.0001. We are left with the parameters governing the stochastic properties of the exogenous income component. We estimate them using U.S. quarterly data from 1947:I to 2002:III for the Real Gross Domestic Product (GDP) and the Real Personal Consumption Expenditure (PCE), both expressed in chained 1996 prices.

The theoretical structure of our model implies that income and consumption share the same determinist trend, at least in the long-run. To extract this common trend, we can estimate using the Seemingly Unrelated Regression (SUR) technique the following system:

$$\begin{bmatrix} GDP_t \\ PCE_t \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}+\ln\gamma\begin{bmatrix} t \\ t \end{bmatrix}+\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} \tag{1.79}$$

where the z_{jt}'s are the two cyclical components. The estimated common long-run quarterly growth rate is $\gamma=1.0086$. The cyclical components obtained from this multivariate detrending procedure are plotted in Figure 1.6: note that they represent by construction percentage deviations from the determinist trend, and that the latter can be interpreted as the long-run balanced growth path.

We can now estimate the stochastic properties of y_t by fitting an AR(1) process on the cyclical component of real GDP:

$$z_{1t+1}=\phi z_{1t}+\varepsilon_t \tag{1.80}$$

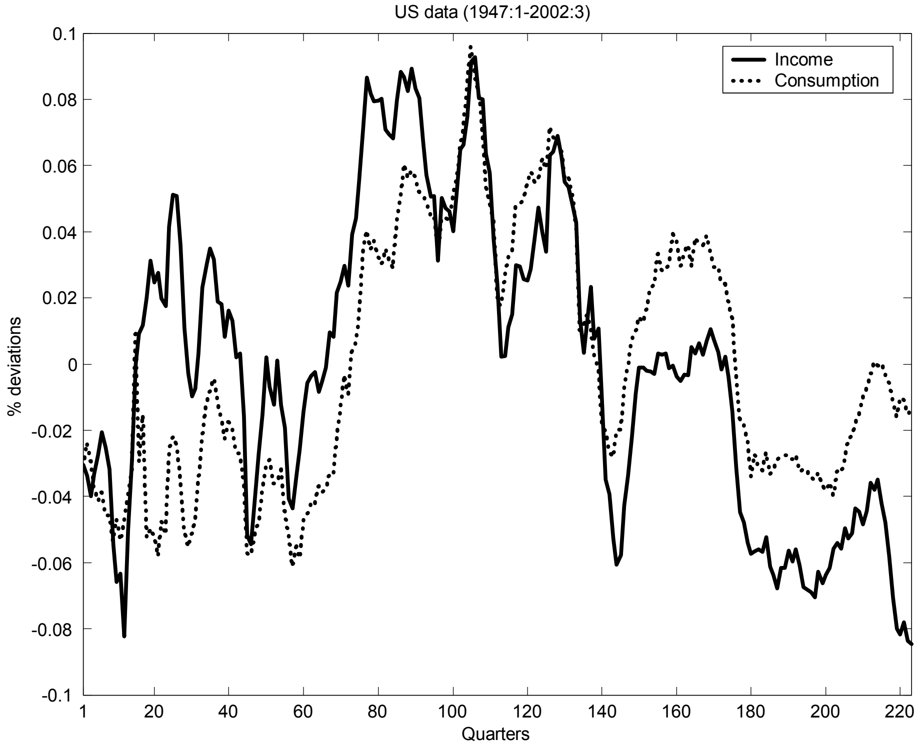

Figure 1.6: Cyclical components of income and consumption in the US.

The estimated parameters are $\phi = 0.982$ and $\sigma = 0.01$. At the same time, this procedure allows us to estimate the sequence of shocks ε_t that actually generated the observed cyclical component. Our benchmark parameterization is therefore the following:

$$\zeta = 2, \quad \beta = 0.9875, \quad \psi = 0.0001$$
$$\gamma = 1.0086, \quad \phi = 0.982, \quad \sigma = 0.01$$

We can finally solve the model numerically and obtain an approximated policy function $\hat{c}(a, y)$. Once the policy function is available, we can iteratively simulate all aggregate time series for a given initial level of asset holdings and a given sequence of shocks.

We already noticed that the steady state of a deterministic version of the model will be characterized by zero asset holdings. Our representative household is clearly risk averse: hence, we may expect a positive level of precautionary savings. In other words, the long-run asset holdings in the stochastic model, *i.e.* the unconditional mean of a_t, will be positive on average. To quantitatively asses the importance of precautionary savings, we will now simulate our model for a very long time horizon, say 3000 quarters, starting from the deterministic steady state. The simulated series, in levels, are plotted in Figure 1.7.

The long-run assets stock held by the representative household is clearly positive, and actually equal to 54% of total long-run income, defined as $q_t \equiv y_t + ia_t$. Precautionary savings amount to 0.47% of total income. The unconditional means of asset holdings, total income, and consumption are respectively equal to $a = 0.553$, $q = 1.017$, and $c = 1.012$. Under our benchmark parameterization, the steady-state cost of holding assets is just a tiny share of total income, equal to 0.0015%.

To estimate the stochastic properties of exogenous income, we fitted an AR(1) model to the cyclical component of GDP: as a by-product, we obtained the sequence of shocks

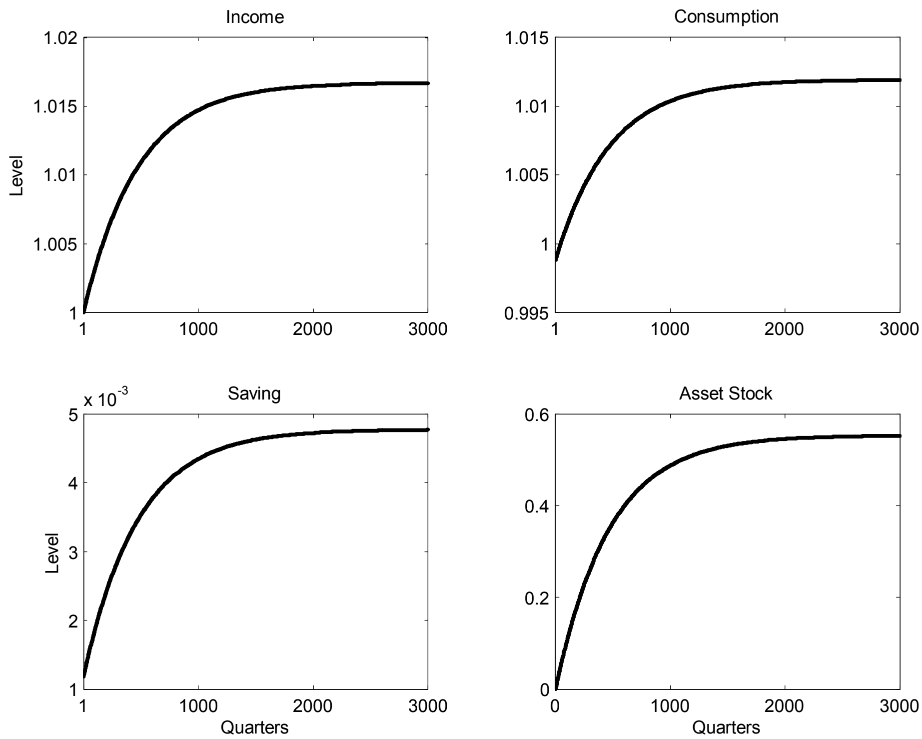

Figure 1.7: Precautionary savings.

that generated that particular time series. We can now feed our model with this observed innovations to exogenous income, and evaluate how well it can fit the time series for the cyclical component of consumption. The simulated time series, expressed in deviations from the steady state, are plotted in Figure 1.8.

By carefully examining the figures, we reach the following conclusions:

1. All time series are highly correlated; to be more precise, consumption, saving, and asset holdings are positively correlated with income. In other words, the "shape" of the simulated paths is extremely similar. This is not surprising, since innovations to income are the only driving force in this simplified model.

2. The consumption level is less volatile than income, while the opposite is true for saving. We are admiring consumption smoothing at work: the percentage deviation of consumption from its mean is relatively smooth over time, when compared to the percentage deviation of income. The reason, again, is clear: innovations to income are highly persistent ($\phi = 0.98$) but still transitory, and this limits to a certain extent the reaction of consumption.

In Figure 1.9 we jointly plot the observed and simulated series for the cyclical component of aggregate consumption. The two series are remarkably similar, but the overlap is clearly not perfect: the simulated series seems to be less volatile that the observed one. Table 1.1 summarizes some stochastic properties of the simulated and observed series: as we can see, the simulated series for consumption is highly correlated with the observed one, but the correlation is less than unit. Note furthermore that the simulated series for consumption and total income are much more correlated than the corresponding observed

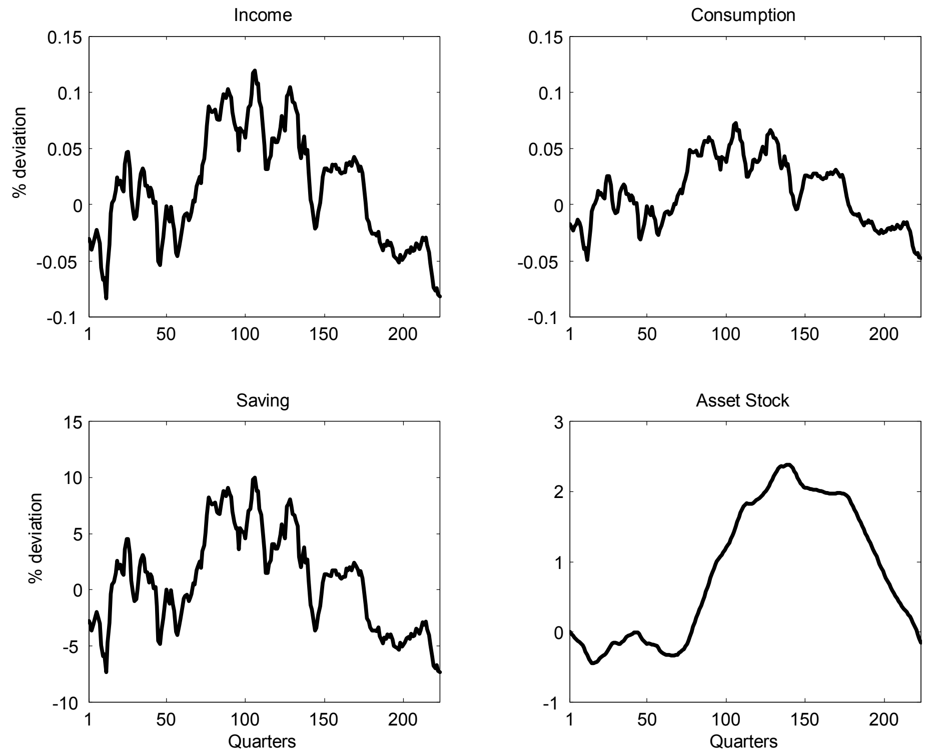

Figure 1.8: Simulated income, consumption, saving, and asset stock.

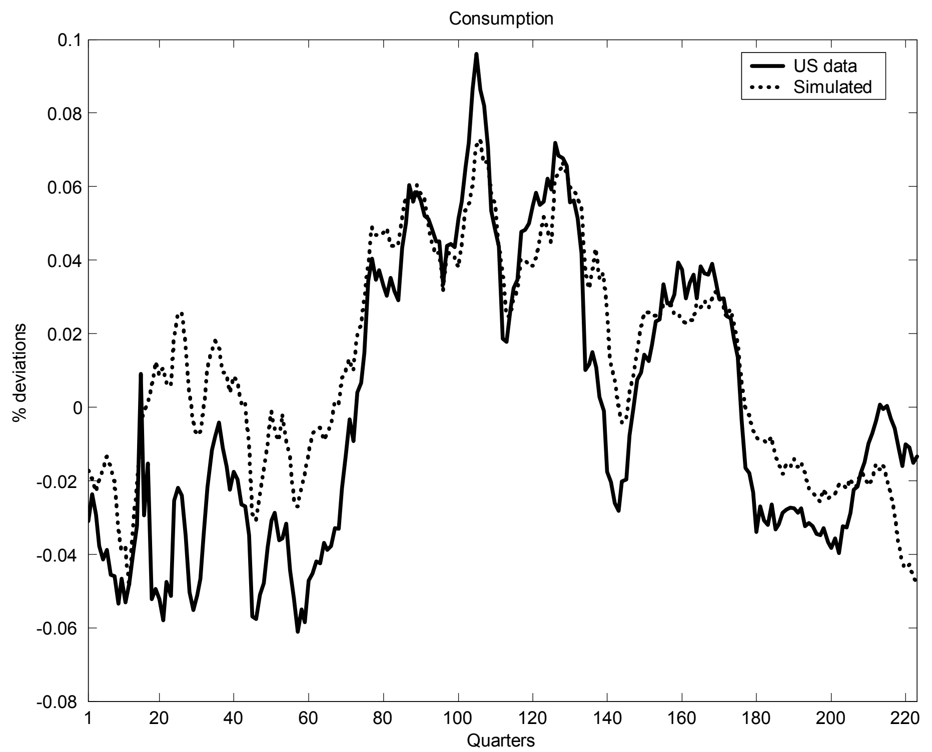

Figure 1.9: Simulated vs. observed consumption.

	$\hat{c}, c$	$\hat{q}, \hat{c}$	q, c
Cor.	**0.87**	0.99	0.71
Rel. Vol.	-	0.60	0.84

Table 1.1: Simulated vs. observed consumption

one, while simulated consumption is less volatile with respect to total income than its observed counterpart. In summary, our simple model fits the data surprisingly well, but simulated consumption is too highly correlated with income and not volatile enough.

The reasons for these inconsistencies can be many: first of all, we are considering aggregate data, not individual level ones, and heterogeneity among individuals may enhance variability of aggregate consumption; then, our model clearly misses many important aspects of reality, as the variability of real interest rates, the labor/leisure choice, the presence of habit persistence in consumption, and so on. All these extensions may introduce mechanisms that amplify the reaction of the marginal utility of consumption to exogenous shocks.

By changing the value of some parameters, we can perform what is known as *sensitivity analysis*. Table 1.2 shows what happens when the value of some of the parameters are slightly changed.

	Std.	$\phi = 0.95$	$\sigma = 0.011$	$\zeta = 1$	$\beta = 0.98$
a/q	54%	11%	65%	6.7%	67%
i/q	0.47%	0.10%	0.56%	0.06%	0.60%

Table 1.2: Sensitivity analysis

As we can see, a drop in persistence generates a sharp decrease in the long-run asset stock and investment share: the degree of overall uncertainty associated with a given standard deviation of the shocks is positively related to their persistence, and this amplifies the incentive to save for a precautionary motive. Symmetrically, a rise in the shocks' volatility clearly stimulates an increase in precautionary savings, just because the environment becomes more uncertain. A drop in ζ, the reciprocal of the elasticity of intertemporal substitution, i.e. the degree of risk aversion, has evidently to translate in a sharp decrease in precautionary savings: the effect however is surprisingly strong. Finally, a slight drop in β, the intertemporal discount factor, makes the representative household more impatient: the current consumption level becomes more important, and its variability more undesirable; hence, precautionary savings increase.

1.6 Appendix: the collocation method

Following Judd (1992), we approximate the policy function over a rectangle $D \equiv [\underline{a}, \overline{a}] \times [\underline{y}, \overline{y}] \in R^2$ with a linear combination of multidimensional basis functions taken from a *2*-fold tensor product of *Chebyshev polynomials*. In other words, we approximate $c(a, y)$ with:

$$\widehat{c}(a, y; \boldsymbol{\theta}) = \sum_{i=0}^{d} \sum_{j=0}^{d} \theta_{ij} \psi_{ij}(a, y) \tag{1.81}$$

where:

$$\psi_{ij}(a,y) \equiv T_i\left(2\frac{a-\underline{a}}{\overline{a}-\underline{a}}-1\right)T_j\left(2\frac{y-\underline{y}}{\overline{y}-\underline{y}}-1\right) \tag{1.82}$$

Each T_n represents a n-order Chebyshev polynomial, defined over [-1,1] as $T_n(x) = \cos(n \arccos x)$, while the parameter d denotes the higher polynomial order used in our approximation.

The functional equation (1.78) becomes:

$$E\left[c(a',y';\boldsymbol{\theta})^{-\zeta}\left(\gamma-\tilde{\beta}\psi a\right) \mid a,y\right] = \gamma c(a,y;\boldsymbol{\theta})^{-\zeta} \tag{1.83}$$

where:

$$a' = \frac{a}{\tilde{\beta}} + \frac{y - c(a,y;\boldsymbol{\theta}) - \frac{\psi}{2}a^2}{\gamma} \tag{1.84}$$

Taking into account that $\ln y'$ is distributed as $\phi \ln y + \sigma z$, where $z \sim N(0,1)$, we can rewrite (1.83) as:

$$\int_{-\infty}^{\infty} c\left[a', \exp(\phi \ln y + \sigma z); \boldsymbol{\theta}\right]^{-\zeta}\left(\gamma-\tilde{\beta}\psi a\right)\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz = c(a,y;\boldsymbol{\theta})^{-\zeta} \tag{1.85}$$

We are left now with a critical problem: how to select the appropriate value for the vector of coefficients $\boldsymbol{\theta}$. The collocation method, extremely simple in principle, takes n^2 points in D (as many points as coefficients in $\boldsymbol{\theta}$) and requires equation (1.85) to exactly hold at this points. In other words, it transforms the functional equation (1.85) into a system of n^2 non-linear equations in n^2 unknowns. The *Chebyshev Approximation Theorem* shows that these points in D can be optimally chosen among the zeros of Chebyshev polynomials; *i.e.* we can find n zeros of Chebyshev polynomials in $[-1,1]$, reverse the normalization and transform them into the corresponding values respectively in $[\underline{a},\overline{a}]$ and $[\underline{y},\overline{y}]$, and finally obtain the vector of n^2 points in D as their Cartesian product.

The non-linear system of equations can be numerically solve for $\hat{\boldsymbol{\theta}}$, using any standard algorithm, like the Newton one; as an intermediate step, the integral in (1.85) can be numerically approximated using Gauss-Hermite quadrature. All these algorithms are usually part of the standard toolbox of any computer program specialized in numerical analysis, like MATLAB or GAUSS.

Of course, the policy function $\hat{c}\left(a,y;\hat{\boldsymbol{\theta}}\right)$ will exactly satisfy the Euler equation only on an extremely limited subset of D; the hope is that it approximates the actual policy function on the reaming part of D with a satisfying degree of precision. Some comparison exercises show that collocation is a surprisingly efficient solution method, at least when the number of state variables is low.

Chapter 2

Investment

The subject of our analysis will be a single, profit-maximizing competitive firm producing a homogeneous consumption good using labor and capital. We assume that, while the labor services are purchased from the households, the capital stock is directly owned by the firm itself. Being physical capital a durable good, the firm faces a *dynamic* profit maximization problem; its current investment decisions will influence its future productive capacity.

More formally, the firm's production technology can be summarized by a constant-returns-to-scale *production function* of the form $y_t = F(n_t, k_t)$, where y_t is output at date t, $k_t \in R_+$ the current physical capital stock, and $n_t \in R_+$ the labor input, hired on a competitive market at a given wage rate w_t.

We assume that:

- $F : R_+^2 \to R_+$ is C^2, with $\nabla F > 0$;
- F is homogenous of degree one (constant-returns-to-scale), *i.e.* $ty = F(tn, tk)$;
- F is strictly quasi-concave;[1]
- $F(n, 0) = 0 \; \forall n \in R_+$;
- $F(0, k) = 0 \; \forall k \in R_+$;
- $\lim_{k \to 0} \frac{\partial F(n,k)}{\partial k} = +\infty \; \forall n \in R_{++}$;
- $\lim_{k \to \infty} \frac{\partial F(n,k)}{\partial k} = 0 \; \forall n \in R_{++}$.

The last two assumptions are known as *Inada conditions.*

For the sake of simplicity, assume furthermore that $n_t = 1 \; \forall t$. We define the intensive production function as $y_t = f(k_t)$; note that $f(0) = 0$, $f' > 0$, $f'' < 0$, $\lim_{k \to 0} f(k) = \infty$, and $\lim_{k \to \infty} f(k) = 0$.

The firm sells its output and purchases the investment good on competitive markets, at the given prices p_t and ν_t. Physical capital accumulates over time according to $\Delta k_{t+1} = i_t$, where i_t represents net investment (depreciation is ruled out for the sake of notational simplicity).

[1] Recall that the strict quasi-concavity of the production function implies the strict convexity of the isoquants.

2.1 The user cost model

The firm maximizes the market present value of its stream of future profits. Formally, the firm solves the following optimal control problem[2], taking $\{p_s, v_s, w_s\}_{s=t}^{\infty}$, R, and the initial condition k_t as given:

$$\max_{\{k_{s+1}\}_{s=t}^{\infty}} \Pi_t = \sum_{s=t}^{\infty} R^{s-t} \left[p_s f(k_s) - w_s - \nu_s \Delta k_{s+1}\right] \tag{2.1}$$

where R is a market discount factor, constant for simplicity. The first order condition is the following:

$$R\left[p_{t+1} f'\left(\hat{k}_{t+1}\right) + \nu_{t+1}\right] = \nu_t \tag{2.2}$$

Remark 27 *We can interpret (2.2) as a no-arbitrage condition; along the optimal path, the given market interest rate $r \equiv (1-R)/R$ has to equal the rate of return on a marginal unit of capital, given by the corresponding "dividend" (the value of its marginal product) plus the future price of investment goods over their current price:*

$$1 + r = \frac{p_{t+1} f'\left(\hat{k}_{t+1}\right) + \nu_{t+1}}{\nu_t} \tag{2.3}$$

We define the *user cost of capital* as:

$$c_t \equiv (1+r)\,\nu_t - \nu_{t+1} \tag{2.4}$$

where v_t corresponds to the direct cost of purchasing and additional unit of capital, rv_t measures the forgone asset income, while v_{t+1} represents the future revenue from reselling the unit of capital.

Solving (2.2) for $\hat{k}_{t+1}$, we conclude that:

$$\hat{k}_{t+1} = (f')^{-1}\left(\frac{c_t}{p_{t+1}}\right) \tag{2.5}$$

Being $f'' < 0$, the future optimal capital stock depends negatively on the user cost of capital and positively on the future price of output, as expected.

Note that (2.5) pins down the optimal *level* of future capital only; investment adjust immediately to variations in c_t. The available empirical evidence, however, suggests that investment adjust slowly to changes in the user cost of capital. Appealing to a "time to build" argument, the traditional literature relaying on the user cost model assumes that investment is proportional to changes in the optimal capital stock:

$$i_t = \sum_{s=0}^{\infty} \alpha_s \Delta \hat{k}_{t-s} \tag{2.6}$$

The previous formulation, however, has the strong implication that the actual capital stock tends to the optimal level only in the limit.

[2] The Inada conditions guarantee *(i)* a bounded objective function; *(ii)* an interior solution, *i.e.* $\left\{\hat{y}_s, \hat{k}_{s+1}\right\}_{s=t}^{\infty} \in R^2_{++}$.

2.2 Adjustment costs and Tobin's q

In the 60's, James Tobin introduced an alternative theory of investment. He argued that, if the market value of a firm exceeds the cost of capital, then the firm may increase its value by investing more. In other words, investment should be positive as long as the firm's market value exceeds the replacement cost of capital. The ratio of a firm's value to the replacement cost of capital is known as *Tobin's average* q. From another point of view, firms should invest as long as the value of an extra marginal unit of capital exceeds its cost. The ratio between the value of a marginal unit of installed capital and its cost is known as *Tobin's marginal* q. Hayashi (1982) shows how Tobin's results can be incorporated in the neoclassical framework by assuming the existence of some kind of adjustment costs.

2.2.1 The Hayashi model

The key feature characterizing the Hayashi model is a cost of adjusting capital, that can be interpreted as a deadweight installation cost. To be more precise, we assume that the function $\psi\left(i_t/k_t\right)k_t$ summarizes the adjustment cost, in terms of additional units of the investment good, corresponding to a given investment rate, with $\psi \geq 0$, $\psi(0) = 0$, $\psi' > 0$, and $\psi'' > 0$. In other words, the process of installing i_t units of new capital simply dissipates $\psi\left(i_t/k_t\right)k_t$ more units. The last two assumptions imply that adjustment costs are strictly convex; the more rapidly the firm adjust its capital stock, the more costly the investment process is. Note furthermore that the function ψ is homogeneous of degree zero in its arguments. Finally, we assume for the sake of notational simplicity that $p_t = 1\ \forall t$ and $\nu_t = \nu\ \forall t$.

Under these assumptions, profits at date t are given by:

$$\pi_t = f\left(k_t\right) - w_t - \nu\left[i_t + \psi\left(\frac{i_t}{k_t}\right)k_t\right] \tag{2.7}$$

Again, the firm maximizes the market present value of its stream of future profits, taking $\{w_s\}_{s=t}^{\infty}$, ν, R , and the initial condition $k_t > 0$ as given:

$$\max_{\{i_s,k_{s+1}\}_{s=t}^{\infty}} \quad \Pi_t = \sum_{s=t}^{\infty} R^{s-t}\left\{f\left(n_s,k_s\right) - w_t - \nu\left[i_s + \psi\left(\frac{i_s}{k_s}\right)k_s\right]\right\} \tag{2.8}$$
$$\text{s.t.} \quad \Delta k_{s+1} = i_s$$

As usual, to solve the problem we build a Lagrangian and partially derive it with regard to i_s and k_{s+1}. The first order conditions are the following:

$$\lambda_t = \nu\left[1 + \psi'\left(\frac{i_t}{k_t}\right)\right] \tag{2.9}$$

$$\lambda_t = R\left[f'\left(k_{t+1}\right) + \nu\tilde{\psi}_{t+1} + \lambda_{t+1}\right] \tag{2.10}$$

where:

$$\tilde{\psi}_t \equiv \psi'\left(\frac{i_t}{k_t}\right)\frac{i_t}{k_t} - \psi\left(\frac{i_t}{k_t}\right) \tag{2.11}$$

The trasversality condition for this problem is:

$$\lim_{j\to\infty} R^j \lambda_j k_{j+1} = 0 \tag{2.12}$$

The Lagrange multiplier λ_t has a precise economic interpretation.

Remark 28 *Thanks to the Envelope theorem, we know that λ_t can be interpreted as the shadow value of capital at the end of date t, i.e. the shadow price of installed capital. From this point of view, the TVC simply states that it cannot be optimal for the firm to "end" its life with a positively valued capital stock.*

Equation (2.9) can be written as:

$$i_t = (\psi')^{-1} (q_t - 1) k_t \tag{2.13}$$

where $q_t \equiv \lambda_t / v$ is *Tobin's marginal q, i.e.* the ratio between the value to the firm of increasing capital by one unit and the cost of purchasing the latter. Equation (2.13) is a microfounded version of the investment equation proposed by Tobin.[3]

The firm's Euler equation (2.10) states that the current shadow price of capital is equal to the discounted future marginal product of capital, plus the future marginal contribution of capital to lower installation costs, plus the future shadow price of capital. Rewriting it as:

$$1 + r = \frac{f'(k_{t+1}) + \nu\tilde{\psi}_{t+1} + \lambda_{t+1}}{\lambda_t} \tag{2.14}$$

shows that, along an optimal path, the rate of return on a marginal unit of installed capital, defined as the corresponding "dividend" (marginal product plus decrease in adjustment costs) plus the future price over the current price, has to equal the given interest rate $r \equiv (1 - R)/R$.

2.2.2 Dynamics

Consider now equations (2.9) and (2.10), and rewrite them as:

$$q_t = 1 + \psi'\left(\frac{i_t}{k_t}\right) \tag{2.15}$$

$$q_t = R\left[\frac{f'(k_{t+1})}{v} + \tilde{\psi}_{t+1} + q_{t+1}\right] \tag{2.16}$$

In steady-state, $k_t = \bar{k}$ and $i_t = 0$ for all t. Equation (2.15) implies that $\bar{q} = 1$; this confirms Tobin's intuition: investment is positive only as long as $q_t > 1$. Equation (2.16), instead, implies that $\bar{k} = (f')^{-1}(\nu r)$; the steady-state capital stock depends negatively on the price of investment goods and on the exogenous interest rate.

Substitute now (2.13) into the equation of motion to obtain:

$$\Delta k_{t+1} = (\psi')^{-1} (q_t - 1) k_t \tag{2.17}$$

[3] Note that the marginal q is a sufficient statistic for the investment rate.

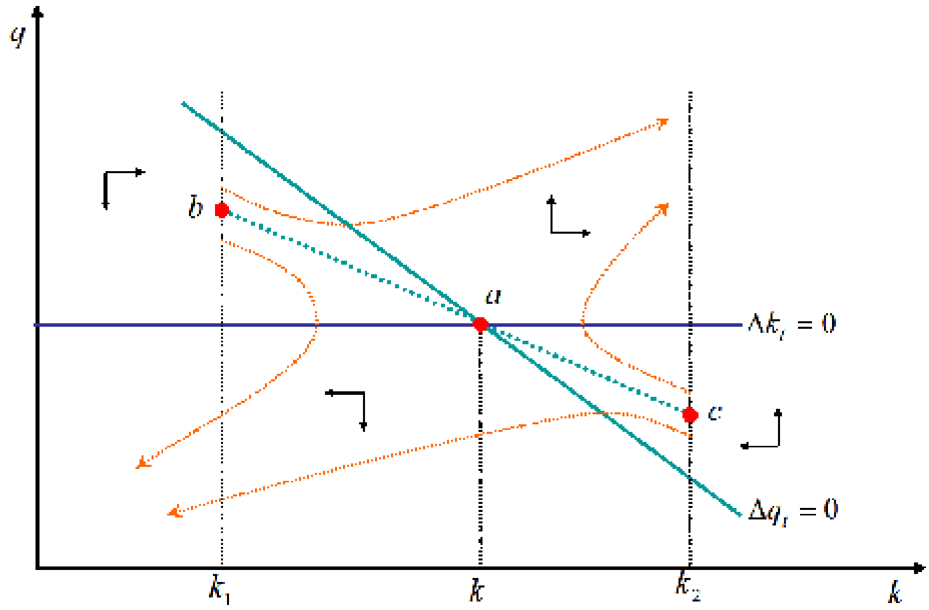

Figure 2.1: Dynamics of the q-model

Furthermore, rewrite (2.16) as:

$$\Delta q_{t+1} = r q_t - \frac{1}{v} f' \left[(\psi')^{-1} (q_t - 1) k_t + k_t \right] - \tilde{\psi}_{t+1} \tag{2.18}$$

where:[4]

$$\tilde{\psi}_{t+1} = \psi' \left[(\psi')^{-1} (q_{t+1} - 1) \right] (\psi')^{-1} (q_{t+1} - 1) - \psi \left[(\psi')^{-1} (q_{t+1} - 1) \right] \tag{2.19}$$

We can linearize the system of difference equations in q_t and k_t given by (2.17) and (2.18) around the steady-state (quite a boring task), and show that the system is saddle-path stable, *i.e.* that, for each initial condition $k_t > 0$, there is a unique path converging to the steady-state. The phase plan for the linearized system is depicted in Figure 2.1.

By assuming $\Delta k_{t+1} = 0$ and $\Delta q_{t+1} = 0$ in equations (2.17) and (2.18), we obtain the functions describing the curves drawn in our figure. Point a corresponds to the steady-state. If the initial capital stock is lower than $\bar{k}$, and equal for instance to k_1, the system jumps immediately to point b; Tobin's q is greater than one, investment becomes positive, and the capital stock increase until the steady-state is reached.

If the initial level of q_t is lower than the one implied by b, the system will converge to the origin at an increasing speed; the physical capital stock (and/or the marginal q) becomes negative in a finite time, violating a feasibility constraint. This can not be an optimal path. If, instead, the initial value of q_t is higher, the system follows a path characterized by ever increasing values of q_t and k_t; this violates the TVC, and the corresponding path can not be optimal. To summarize, any diverging path can be ruled out, for violating the TVC and/or the feasibility constraints. Similar arguments apply to paths starting with a initial capital stock greater that $\bar{k}$, as for instance k_2.

[4]Simply solve (2.13) for i_t/k_t and substitute the result into (2.11).

2.2.3 Marginal and average q

Solving equation (2.10) forward, we obtain:

$$\lambda_t = \sum_{s=t}^{\infty} R^{s-t+1} \left[f'(k_{s+1}) + \nu \tilde{\psi}_{s+1} \right] + \lim_{j \to \infty} R^j \lambda_{t+j} \tag{2.20}$$

In the long-run, the capital stock k_t converges to a strictly positive constant. The TVC then implies that $R^j \lambda_{t+j} \to 0$ as $j \to \infty$, ruling out bubbles in the shadow price of installed capital.

Equation (2.20), then, simplifies to:

$$\lambda_t = \sum_{s=t}^{\infty} R^{s-t+1} \left[f'(k_{s+1}) + \nu \tilde{\psi}_{s+1} \right] \tag{2.21}$$

Remark 29 *The current shadow price of installed capital is equal to the discounted stream of future marginal products of capital plus the marginal contributions to the reduction in adjustment costs.*

Now, multiply both sides of (2.10) by k_{t+1}, and take into account the linear homogeneity of the production function[5]:

$$\lambda_t k_{t+1} = R \left[f(k_{t+1}) - w_{t+1} + \nu \tilde{\psi}_{t+1} k_{t+1} + \lambda_{t+1} k_{t+1} \right] \tag{2.22}$$

Substitute the definition of $\tilde{\psi}_{t+1}$ together with $\Delta k_{t+2} = i_{t+1}$ into (2.22):

$$\begin{aligned} \lambda_t k_{t+1} = R \Bigg[& f(k_{t+1}) - w_{t+1} - \nu \psi \left(\frac{i_{t+1}}{k_{t+1}} \right) k_{t+1} + \\ & + \nu \psi' \left(\frac{i_{t+1}}{k_{t+1}} \right) i_{t+1} + \lambda_{t+1} k_{t+2} - \lambda_{t+1} i_{t+1} \Bigg] \end{aligned} \tag{2.23}$$

Evaluating (2.9) at date $t+1$, and multiplying by i_{t+1}, we obtain:

$$\lambda_{t+1} i_{t+1} = \nu \left[1 + \psi' \left(\frac{i_{t+1}}{k_{t+1}} \right) \right] i_{t+1} \tag{2.24}$$

Substituting this result into (2.23) and simplifying gets:

$$\lambda_t k_{t+1} = R \left[f(k_{t+1}) - w_{t+1} - \nu i_{t+1} - \nu \psi \left(\frac{i_{t+1}}{k_{t+1}} \right) k_{t+1} + \lambda_{t+1} k_{t+2} \right] \tag{2.25}$$

By iterating on (2.25) and imposing the TVC, we obtain:

$$\lambda_t k_{t+1} = \sum_{s=t}^{\infty} R^{s-t+1} \left[f(k_{s+1}) - w_{s+1} - \nu i_{s+1} - \nu \psi \left(\frac{i_{s+1}}{k_{s+1}} \right) k_{s+1} \right] \tag{2.26}$$

Remark 30 *In other words, the shadow value of installed capital equals the current value of the firm, i.e. the discounted stream of future profits,* Π_{t+1}.

[5] The Euler theorem states that $f(k_t) = f'(k_t) k_t + w_t$.

Dividing both sides of (2.26) by νk_{t+1} we obtain another important result.

$$\frac{\lambda_t}{v} = \frac{\Pi_{t+1}}{v k_{t+1}} \tag{2.27}$$

Remark 31 *Tobin's marginal q is equal to Tobin's average q, i.e. the ratio between the firm's market value and the replacement cost of its installed capital. Hayashi (1982) shows that this result holds as long as:*

1. *the production function presents constant returns to scale;*
2. *perfect competition holds on product and factor markets;*
3. *the adjustment cost function is homogeneous of degree zero;*
4. *the are no bubbles in the valuation of firms.*

Chapter 3

The Ramsey-Cass-Koopmans model

In previous Chapters we developed the dynamic theory of consumption and investment separately. Furthermore, we presented a fully-fledged dynamic stochastic general equilibrium (DSGE) model, the Lucas' "tree model". The latter, however, was still based, for the sake of simplicity, on a simple endowment economy. We will now merge these analytical tools into a complete dynamic macroeconomic model, the well-known neoclassical optimal growth model, or Ramsey-Cass-Koopmans (RCK) model, initially proposed by Frank Ramsey in 1928, and then fully developed by David Cass and Tjalling Koopmans in 1965. The RCK model and its stochastic extensions can be considered the workhorses of modern macroeconomics and growth theory.

Figure 3.1 briefly summarizes the model's role in the history of economic thought. Cass (1965) and Koopmans (1965) independently developed the neoclassical optimal growth model by combining Ramsey's optimal intertemporal consumption/saving model with the neoclassical growth model introduced by Solow (1956) and characterized by an exogenous constant saving rate. Brock (1974) showed that the Pareto-optimal solution, obtained under the benevolent planner assumption, could be easily decentralized in a competitive setting; Brock and Mirman (1972), in turn, developed a stochastic version of the deterministic RCK model. Generalizations of the Brock-Mirman model have been used by Kydland and Prescott (1982) and Long and Plosser (1984) to explain the stochastic properties of US macroeconomic aggregates; their work lies at the heart of the well-known *Real Business Cycle* literature. Finally, Romer (1986), Lucas (1988), and Rebelo (1991) extended the RCK model by introducing, respectively, learning-by-doing and human capital accumulation; their models, being able to counterbalance the decreasing marginal productivity of physical capital and endogenously generate a positive long-run growth rate, were the first contributions to the so-called *Endogenous Growth* literature.

3.1 The basic framework

In its basic version, the RCK model is not decentralized, *i.e.* the existence of competitive markets is not assumed. Instead, an omnipotent and benevolent social planner maximizes the agents' welfare, controlling the individual consumption streams, subject to the aggregate feasibility constraints. The solution to this planning problem characterizes the Pareto-optimal intertemporal allocation of consumption and investment. Under some (well, many) regularity conditions, a Pareto-optimal allocation can be decentralized as a competitive recursive equilibrium; in other words, by studying the properties of a solution to the planner problem, we indirectly study a competitive equilibrium. A more direct link

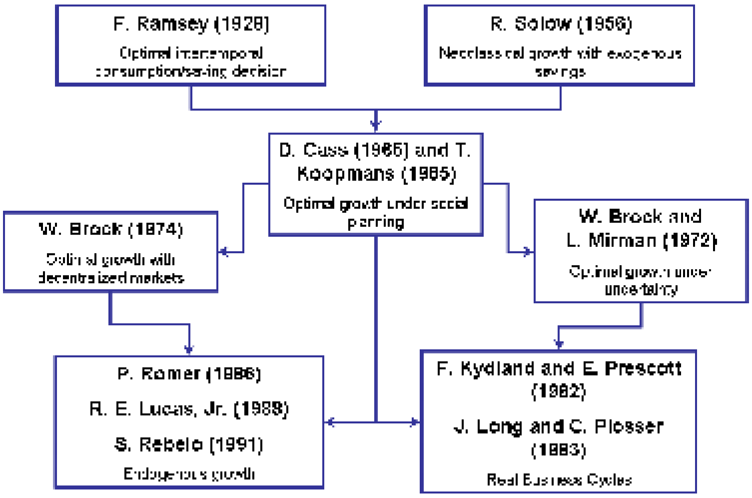

Figure 3.1: A brief history of the RCK model and its extensions.

between the two formulations will be established in the following.

3.1.1 Building blocks

Preferences

The economy is inhabited by a *continuum* of infinitely-living identical agents, each of measure zero. Being all agents identical, they can be aggregated into a single *representative agent*, whose preferences on infinite consumption streams $\{c_t\}_{t=0}^{\infty}$ may be represented by the standard *intertemporal utility function*:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.1}$$

where $c_t \in R_+$ is the per-capita consumption level at date s, $\beta \in (0,1)$ the intertemporal subjective discount factor, and $u\colon R_+ \to R$ the instantaneous utility function. As usual (see Section 1.2.1, p. 5), we assume that $u(\cdot)$ is C^2, strictly increasing, and strictly concave, with $\lim_{c \to 0} u'(c) = +\infty$.

Technology

A single consumption good is produced using a *Constant-Returns-to-Scale* (CRS) technology summarized by the following neoclassical production function:

$$y_t = AF(k_t, n_t L) \tag{3.2}$$

where $y_t \in R_+$ is the per-capita output level at date t, $k_t \in R_+$ the per-capita physical capital stock, $n_t \in [0,1]$ the time share devoted to labor, $A \in R_{++}$ the so called *Total Factor Productivity* (TFP), and $L \in R_{++}$ the fixed time endowment (from now on, for the sake of simplicity, $L = A = 1$). The production function enjoys all the properties listed in Section 2, p. 32.

Note that, since leisure is not valued in the utility function, it would never be optimal to consume it in a positive amount. We can safely anticipate this equilibrium outcome, and assume from the beginning that $n_t = 1 \ \forall t$. This allows us to define the *intensive production function* as $f(k_t) \equiv F(k_t, 1)$; note that $f(0) = 0$, $f' > 0$, $f'' < 0$, $\lim_{k \to 0} f(k) = \infty$, and $\lim_{k \to \infty} f(k) = 0$.

Resource constraints

Physical capital is the only durable good in the economy. An aggregate intratemporal resource constraint, which is also an accumulation equation for physical capital, holds in each period:

$$k_{t+1} = (1-\delta) k_t + y_t - c_t \tag{3.3}$$

where $\delta \in [0, 1]$ is the depreciation rate.[1] We assume the existence of a strictly positive initial capital stock k_0.

The consumption good can be freely transformed into physical capital, and vice-versa, physical capital can be freely transformed into the consumption good (hence, investments may be negative).

The planner's problem

We assume the existence of a *benevolent social planner* who governs the economy by choosing the per-capita consumption plans. Being benevolent, the planner aims to maximize the intertemporal utility function of the representative agent. Formally, she solves the following deterministic optimal control problem:

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad & U = \sum_{t=0}^{\infty} \beta^t u(c_t) && (3.4) \\ \text{s.t.} \quad & k_{t+1} = f(k_t) + (1-\delta) k_t - c_t \\ & k_0 > 0 \text{ given} \end{aligned}$$

The corresponding present-value Lagrangian can be written as:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[f(k_t) + (1-\delta) k_t - c_t - k_{t+1} \right] \right\} \tag{3.5}$$

Optimality conditions

Deriving the Lagrangian with respect to c_t, k_{t+1}, and λ_t, we obtain the (by now) familiar first order conditions (a hat identifies the optimal plan):

$$u'(\hat{c}_t) = \hat{\lambda}_t \tag{3.6}$$

$$\hat{\lambda}_t = \beta \hat{\lambda}_{t+1} \left[f'\left(\hat{k}_{t+1}\right) + 1 - \delta \right] \tag{3.7}$$

$$\hat{k}_{t+1} = f\left(\hat{k}_t\right) + (1-\delta) \hat{k}_t - \hat{c}_t \tag{3.8}$$

[1] At this stage, depreciation is a purely physical, exogenous process.

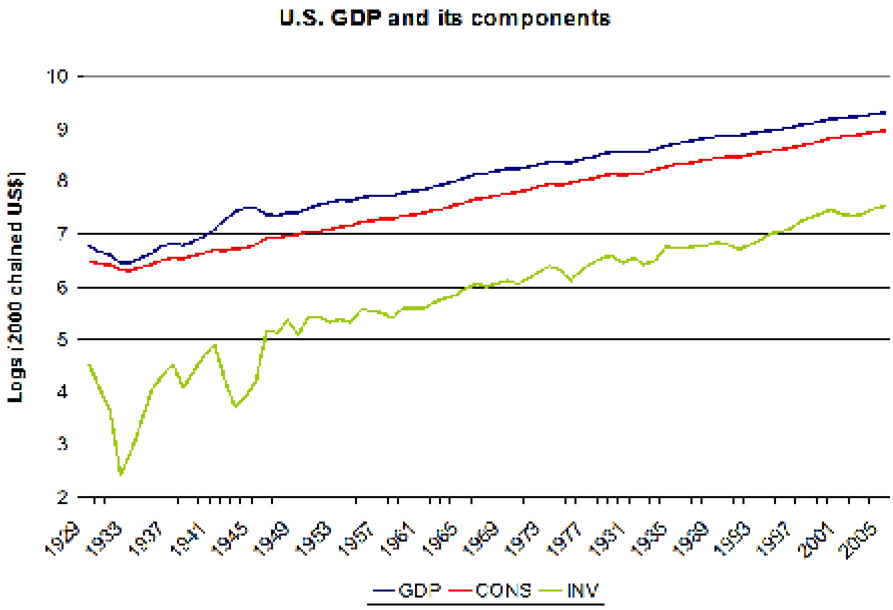

Figure 3.2: Long-run balanced growth in the United States.

These first order conditions, together with the TVC:

$$\lim_{t\to\infty} \beta^t \hat{\lambda}_t \hat{k}_{t+1} = 0 \tag{3.9}$$

are necessary and sufficient for the planner's problem.

Reorganizing the first order conditions, we easily obtain the (by now) well-known Euler equation:

$$\beta u'\left(\hat{c}_{t+1}\right)\left[f'\left(\hat{k}_{t+1}\right)+1-\delta\right] = u'\left(\hat{c}_t\right) \tag{3.10}$$

Remark 32 *The Euler equation has the usual interpretation. If we decrease consumption at date t by dc_t, the welfare loss equals $U'(c_t)dc_t$. At date $t+1$, consumption will increase, since the drop in current consumption implies higher investments and therefore an increase in k_{t+1}. In particular, future consumption will increase by $\left[f'\left(k_{t+1}\right)+1-\delta\right]dc_t$. If the plan is optimal, then there should be no advantage in reallocating consumption, hence (3.10) holds.*

3.1.2 The steady state

A clear and pervading empirical evidence, discussed in Kaldor (1963), suggests that real economies tend to a balanced growth path in the long run. In other words, all main macroeconomic variables, in particular real GDP and its components, seem to growth at the same, constant, and possibly positive growth rate in the long run. Figure 3.2 shows the clear long-run balanced growth experienced by the U.S. over the 1929-2005 period. Any dynamic macroeconomic model, in order to reproduce this basic stylized fact, should be reducible to a stationary dynamic system, *i.e.* to a system converging in the long run to a balanced growth path.

Existence

Physical capital is the only durable good in the basic RCK model. Obviously enough, then, the process of capital accumulation is the only possible source of long-run growth. We will now show that the RCK model follows a balanced growth path whenever the growth rate of capital is constant, and, furthermore, that the only admissible constant growth rate is zero. In other words, we conclude that the basic version of the RCK model is unable to converge to any balanced growth path where all variables grow at a strictly positive rate, being therefore unable to reproduce the Kaldorian stylized facts.

The dynamics of our model is driven by a system of two difference equations: (3.8) and (3.10). Consider the latter, and rewrite it as:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[f'(k_{t+1}) + 1 - \delta \right] \tag{3.11}$$

Define the capital stock growth rate as $\gamma_t^k \equiv (k_{t+1} - k_t)/k_t$, and assume that $\lim_{t\to\infty} \gamma_t^k = \gamma^k > 0$. Under this assumption, $\lim_{t\to\infty} k_t = \infty$, while $\lim_{k\to\infty} f'(k) = 0$ for the Inada conditions. Take the limit of (3.11) as $t \to \infty$:

$$\lim_{t\to\infty} \frac{u'(c_t)}{u'(c_{t+1})} = \beta (1 - \delta) \tag{3.12}$$

Being $u'' < 0$ and $0 < \beta(1-\delta) < 1$, equation (3.12) implies that $\lim_{t\to\infty} \gamma_t^c = \gamma^c < 0$.

A constant, strictly negative long-run growth rate of consumption is incompatible with a constant, strictly positive long-run growth rate of the capital stock. To see why, substitute (3.6) in the TVC to obtain:

$$\lim_{j\to\infty} \beta^j u'(c_j) k_{j+1} = 0 \tag{3.13}$$

If $\lim_{t\to\infty} c_t = 0$, then $\lim_{t\to\infty} u'(c_t) = \infty$; being $\lim_{t\to\infty} k_t = \infty$ by assumption, condition (3.13) cannot be satisfied.

Remark 33 *Assuming a strictly positive long-run growth rate of capital along the optimal path generates a contradiction. There is only one possibility left: γ^k necessarily equals zero along the optimal path.*

Consider (3.8), and rewrite it as:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \tag{3.14}$$

Divide (3.14) by k_t to obtain:

$$\gamma_t^k = \frac{f(k_t)}{k_t} - \delta - \frac{c_t}{k_t} \tag{3.15}$$

If $\lim_{t\to\infty} \gamma_t^k = 0$, equation (3.15) holds only if $\gamma^k = \gamma^y = \gamma^c = 0$.

Remark 34 *If the growth rate of capital is constant and equal to zero, the system follows a balanced growth path characterized by a common zero growth rate, i.e. it reaches a steady state.*[2]

[2] A direct consequence of this result is that the return function for our optimal control problem will

Properties

Evaluate (3.8) and (3.10) at the steady-state:

$$u'(c) = [f'(k) + 1 - \delta]\beta u'(c) \tag{3.16}$$
$$0 = f(k) - \delta k - c \tag{3.17}$$

Simplify (3.16) and (3.17), and solve them for the steady-state allocation:

$$k = (f')^{-1}\left(\frac{1}{\beta} - 1 + \delta\right) \tag{3.18}$$
$$c = f(k) - \delta k \tag{3.19}$$

We note that:

- being $f'' < 0$, the steady-state capital stock depends positively on β and negatively on δ (why?);
- steady-state investment is just large enough to counterbalance depreciation.

Consider now equation (3.19). Does the steady-state capital stock maximize steady-state consumption?. To answer the question, consider the first order condition for the corresponding maximization problem:

$$f'(\bar{k}) = \delta \tag{3.20}$$

Equation (3.20) is known as the *Golden rule*, and $\bar{k}$ is the capital stock that would maximize consumption as expressed in (3.19).

Consider now (3.18) and rewrite it as:

$$f'(k) = \delta + \rho \tag{3.21}$$

where ρ is the intertemporal discount rate. Being $f'' < 0$ and $\rho > 0$, evidently $k < \bar{k}$ and $c < \bar{c}$.

Remark 35 *The planner does not follow the Golden rule, i.e. does not maximize the steady-state consumption level.*

There is a simple intuition behind this results: since the representative agent has a strictly positive intertemporal discount rate (she is impatient) it would be sub-optimal for the planner to decrease current consumption (saving more and increasing the steady-state capital stock) beyond a certain point to get a higher consumption level in steady state.

3.1.3 Dynamics

The set of admissible points (or allocations) $\{c_t, k_t\}$ coincides with R^2_+. Any sequence $\{c_t, k_t\}_{t=0}^{\infty}$ in R^2_+ that satisfies (3.8) and (3.10) for a given initial point $\{c_0, k_0\}$ such that $c_0 \geq 0$ and $k_0 > 0$ is called a *trajectory*. Note that, under our assumptions:

be bounded in the long run; this implies that the objective function will be bounded (along the optimal path) too.

- for each point $\{c_t, k_t\} \in R^2_+$ there is one and only one point $\{c_{t+1}, k_{t+1}\} \in R^2$ satisfying (3.8) and (3.10);
- points in R^2_+ belong to one and only one trajectory;
- any trajectory contains infinitely many other ones, defined as $\{c_j, k_j\}_{j=s}^{\infty} \subset \{c_t, k_t\}_{t=0}^{\infty}$ for $s \geq 1$.

Any trajectory satisfying (3.8), (3.10), and the TVC is defined as optimal, being the unique solution to the planner's problem. This section aims to study the *qualitative* properties of these optimal trajectories from a graphical point of view.

The phase diagram

Consider again equation (3.14):

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \tag{3.22}$$

We note that:

- $\Delta k_{t+1} = 0$ if and only if $c_t = f(k_t) - \delta k_t$;
- $\Delta k_{t+1} < 0$ *iff* $c_t > f(k_t) - \delta k_t$;
- $\Delta k_{t+1} > 0$ *iff* $c_t < f(k_t) - \delta k_t$.

Reconsider also (3.11):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[f'(k_{t+1}) + 1 - \delta \right] \tag{3.23}$$

Equation (3.23) depends on k_{t+1}, which in turn depends on k_t and c_t through (3.22). Substitute (3.22) into (3.23):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left\{ f' \left[\underbrace{k_t + f(k_t) - \delta k_t - c_t}_{k_{t+1}} \right] + 1 - \delta \right\} \tag{3.24}$$

If $\Delta c_{t+1} = 0$, then $u'(c_t) = u'(c_{t+1})$, and the previous equation can be rewritten as:

$$\frac{1}{\beta} - 1 + \delta = f' \left[k_t + f(k_t) - \delta k_t - c_t \right] \tag{3.25}$$

Note that:

$$(f')^{-1} \left(\frac{1}{\beta} - 1 + \delta \right) = k = k_t + f(k_t) - \delta k_t - c_t \tag{3.26}$$

Being $u'' < 0$ by assumption, we conclude that:

- $\Delta c_{t+1} = 0$ if and only if $c_t = f(k_t) - \delta k_t - (k - k_t)$;
- $\Delta c_{t+1} < 0$ *iff* $c_t < f(k_t) - \delta k_t - (k - k_t)$;
- $\Delta c_{t+1} > 0$ *iff* $c_t > f(k_t) - \delta k_t - (k - k_t)$.

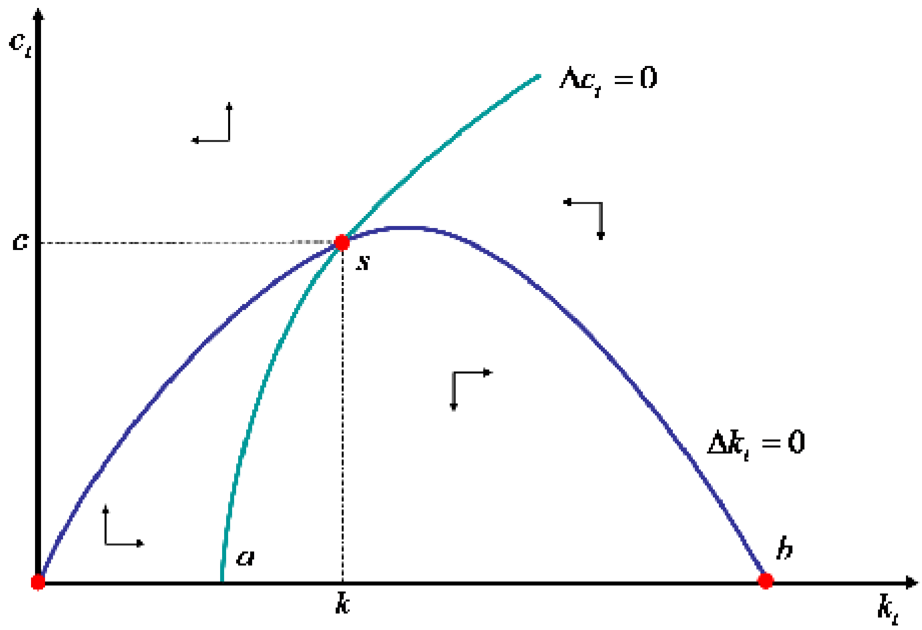

Figure 3.3: Dynamics in the RCK model.

Now, $c_t = f(k_t) - \delta k_t$ defines the *locus* in R_+^2 where $\Delta k_{t+1} = 0$, while $c_t = f(k_t) - \delta k_t - (k - k_t)$ the *locus* where $\Delta c_{t+1} = 0$. If we measure consumption on the vertical axis, we can represent the first equation as a strictly concave (why?) curve starting from the origin and intersecting the horizontal axis again at $b \equiv \{0, \hat{k}\}$, and the second equation as a ever-increasing, strictly concave (again, why?) curve starting from $a \equiv \{0, \tilde{k}\}$, where $\hat{k}$ is defined implicitly by $f(\hat{k}) = \delta \hat{k}$ and $\tilde{k}$ by $f(\tilde{k}) - \delta \tilde{k} = k - \tilde{k}$. The curves intersect at $\{c, k\}$, the steady-state. Both curves, together with the adjustment dynamics implied by the previous results (summarized by a set of arrows), are shown in Figure 3.3.

Stability

Remark 36 *The system is* ***saddle-path stable****, i.e. for each initial value of k_0 there is one and only one trajectory converging to the steady state (point S); all other trajectories diverge.*[3]

There are three obvious possibilities:

- If $k_0 = k$, then $k_t = k \; \forall t$.
- If $k_0 < k$, there will be one and only one trajectory converging "from below" to the steady state.
- Finally, if the system starts with $k_0 > k$, there will be one and only one trajectory converging "from above".

These three trajectories are represented in Figure 3.4.

[3] To prove formally the system's saddle path stability, we could linearize it around the steady-state, show the saddle-path stability of the resulting linear system, and then refer to the Poincaré-Lyapunov-Perron theorem to extend this result to the original non-linear system. This procedure, simple but cumbersome, is omitted here.

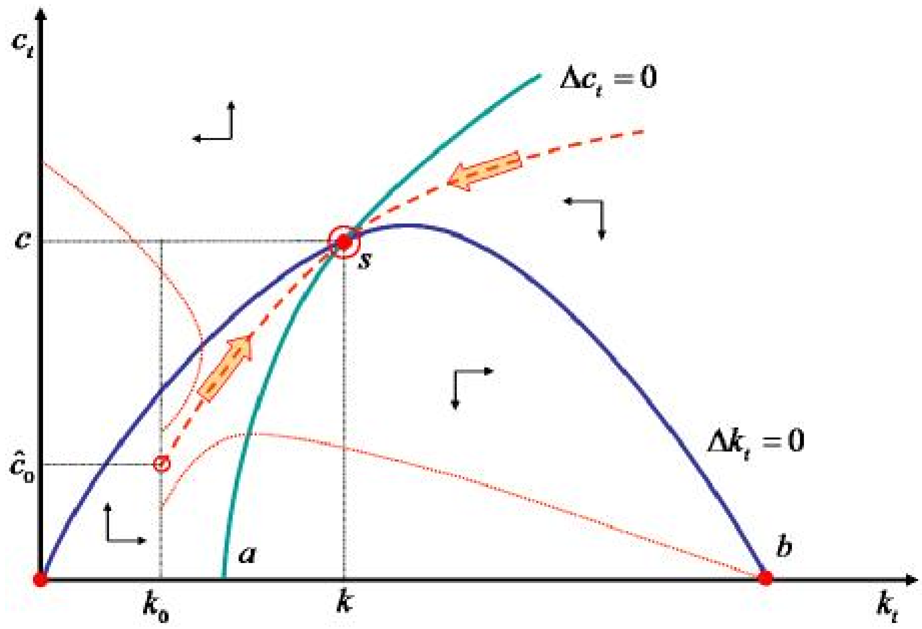

Figure 3.4: Saddle-path convergence to the steady state.

We will now show that these converging trajectories are the optimal ones, since all the other trajectories violate the TVC or the feasibility constraints.

Exercise 37 *Check that the convergent trajectories satisfy the TVC.*

Assume that $k_0 < k$. At date 0, the planner has to chose the optimal initial level of consumption, c_0:

- If she picks a value for c_0 above $\hat{c}_0$, the system jumps on a diverging trajectory, and the physical capital stock would necessarily become negative in a finite time span. Clearly, such a diverging trajectory would violate the feasibility constraints.

- If, instead, she picks a value for c_0 below $\hat{c}_0$, the system jumps on a trajectory that will converge to b in the long run. Along this trajectory, the consumption level tends asymptotically to zero, and this violates the TVC (try to get an intuition!).

Summary 38 *For all $k_0 < k$, the unique optimal plan that solves the planner's problem for the given initial condition belongs to the trajectory converging "from below" to the steady state. It is easy to show that, for all $k_0 > k$, the unique optimal plan belongs to the trajectory that converges to the steady state "from above". If $k_0 = k$, the optimal solution will simply be: don't move!*[4]

[4] There is a further possibility, typical of discrete systems, as noted in Gandolfo (1996). The unique solution to the planner's problem could be a sequence of points belonging alternatively to both convergent trajectories. In other words, the model may oscillate while converging to the steady state without violating the first order conditions and the TVC.

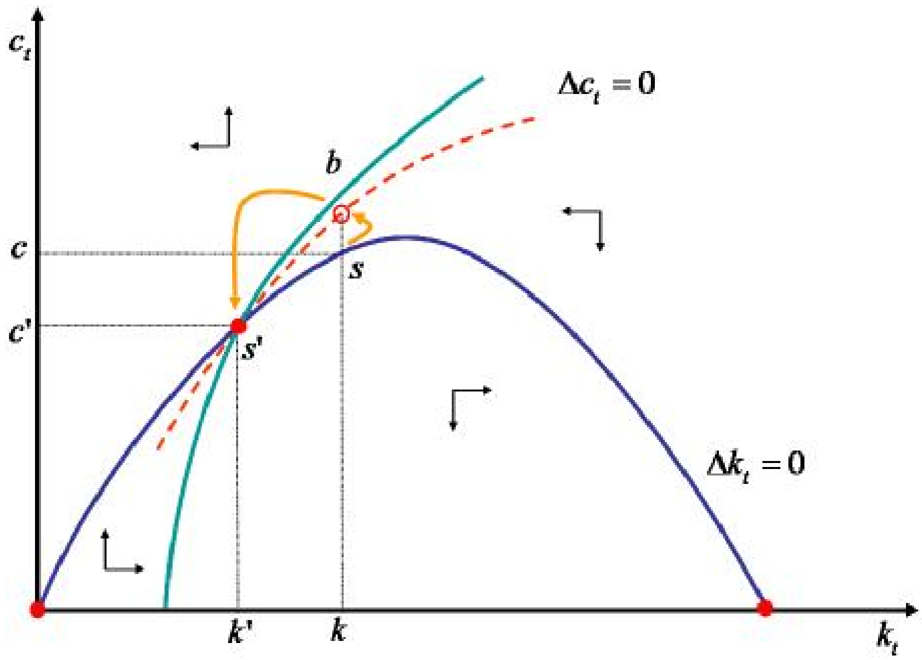

Figure 3.5: Transitional dinamics.

Transitional dynamics

Assume now that the system has already reached the steady-state. Suddenly the representative agent becomes unexpectedly more impatient, *i.e.* β decreases. The steady-state capital stock decreases. How does the system behave during the transition to the new steady-state? The transitional dynamics is sketched in Figure 3.5. The system starts in a; the sudden decrease in β moves the $\Delta c_t = 0$ *locus* on the left. Consumption can freely jump at date t, while capital is fixed. Therefore, the system jumps immediately to b on the new saddle path, and then adjust slowly to the new steady-state c.

3.1.4 The Bellman equation

The planner's problem represented in (3.4) can be easily generalized as:

$$\begin{aligned} \max_{\{c_s,k_{s+1}\}_{s=t}^{\infty}} \quad & U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \\ \text{s.t.} \quad & k_{t+1} = f(k_t) + (1-\delta) k_t - c_t \\ & k_t > 0 \text{ given} \end{aligned} \tag{3.27}$$

Define the *value function* $v_t \colon R_+ \to R$ as the maximal value attainable at date t by the representative agent's intertemporal utility function, along the optimal path, if the current capital stock were $k_t > 0$; in other words:

$$\begin{aligned} v_t(k_t) \quad \equiv \quad & \max_{\{c_s,k_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \\ & \text{s.t. } k_{t+1} = f(k_t) + (1-\delta) k_t - c_t \end{aligned} \tag{3.28}$$

Assume, for the sake of exposition, that $v_{t+1}(\cdot)$ were known. In this case, the previous problem could be restated as:

$$\begin{aligned} v_t(k_t) &\equiv \max_{\{c_t, k_{t+1}\}} u(c_t) + \beta v_{t+1}(k_{t+1}) \\ &\text{s.t. } k_{t+1} = f(k_t) + (1-\delta) k_t - c_t \end{aligned} \tag{3.29}$$

Now, the structure of the RCK model is completely *recursive*: *i.e.*, the optimization problem starting at date 0 for given k_0 is structurally identical to the problem starting at date 1 for given k_1. There is only one thing that matters: the current value of the state variable, k_t. Hence, the value function has to be *time invariant*, i.e. $v_t(\cdot) = v_{t+1}(\cdot)\ \forall t$.

The planner's problem, therefore, can be restated in the following recursive and time-invariant form, known as the *Bellman equation*:

$$\begin{aligned} v(k) &\equiv \max_{c \in R_+} u(c) + \beta v(k') \\ &\text{s.t. } k' = f(k) + (1-\delta) k - c \end{aligned} \tag{3.30}$$

where k' represents the next period capital stock. The value function is strictly related to a time-invariant *policy function* that relates the optimal consumption choice to the current capital stock:

$$\begin{aligned} c(k) &\equiv \arg\max_{c \in R_+} u(c) + \beta v(k') \\ &\text{s.t. } k' = f(k) + (1-\delta) k - c \end{aligned} \tag{3.31}$$

Note that (3.30) and (3.31) are *functional equations*: they *implicitly* define both the value and policy functions; these equations, in general, do not have close form solutions but have to be solved numerically (one well-known exception to this rule is discussed below).

Hence, we concluded that the solution to the planner's problem can be represented as a value function, together with its associated policy function.

3.1.5 A closed form solution

Let us be more specific as far as the involved functional forms are concerned, and assume that $u(c) = \ln c$, $f(k) = k^\alpha$ with $\alpha \in (0,1)$, and $\delta = 1$. This specification is one of the few for which the RCK model admits a closed form solution.

Optimal control

We start by considering the corresponding system of first order conditions:

$$c_{t+1} = \alpha\beta k_{t+1}^{\alpha-1} c_t \tag{3.32}$$

$$k_{t+1} = k_t^\alpha - c_t \tag{3.33}$$

Rewrite (3.32) as:

$$\frac{k_{t+1}}{c_t} = \alpha\beta \frac{k_{t+1}^\alpha}{c_{t+1}} \tag{3.34}$$

Substitute (3.33) into (3.34), and solve the result interactively for k_{t+1}/c_t, obtaining:

$$\frac{k_{t+1}}{c_t} = \sum_{j=1}^{\infty} (\alpha\beta)^j + \lim_{j\to\infty} (\alpha\beta)^j \frac{k_{t+j+1}}{c_{t+j}} \tag{3.35}$$

Now, the TVC implies that:

$$\lim_{j\to\infty} \beta^j \frac{k_{t+j+1}}{c_{t+j}} = 0 \tag{3.36}$$

Imposing the TVC on (3.35) and reorganizing gets:

$$\frac{k_{t+1}}{c_t} = \alpha\beta \sum_{j=0}^{\infty} (\alpha\beta)^j = \frac{\alpha\beta}{1-\alpha\beta} \tag{3.37}$$

since $0 < \alpha\beta < 1$. Substituting (3.37) into (3.33) and simplifying leads us to:

$$c_t = (1-\alpha\beta)\, k_t^{\alpha} \tag{3.38}$$

Equation (3.38) is the optimal policy function we were looking for. Together with (3.34), it completely characterize the optimal plan solving the RCK model under this very particular parameterization.

Dynamic programming

We can also tackle the problem with dynamic programming. In general, the Bellman functional equation associated[5] with problem (3.4) is:

$$v(k) = \max_{k'} \; u\left[f(k) + (1-\delta)k - k'\right] + \beta v(k') \tag{3.39}$$

where k' represents next period's capital stock. Feasibility requires that $\{c, k'\} \in R_+^2$.

Under our parameterization, the Bellman equation specializes to:

$$v(k) = \max_{k'} \; \ln(k^{\alpha} - k') + \beta v(k') \tag{3.40}$$

In steady-state, both the investment-capital ratio, i_t/k_t, and the investment share, i_t/y_t, are constant. This implies that $k = \eta y = \eta k^{\alpha}$, where $\eta = i/y$ since $i = k$. A good guess for the stationary policy function may then be $\hat{k}' = \eta\hat{k}^{\alpha}$.

The Benveniste-Sheinkman rule states that $v'(k) = \alpha/[(1-\eta)k]$. Integrate both sides with respect to k to get:

$$v(k) = \int \frac{\alpha}{(1-\eta)k} dk = a + b\ln k \tag{3.41}$$

where $b \equiv \alpha/(1-\eta)$, while a is a constant to be determined.

[5] In note 2, p. 44, we stressed that the objective function was actually bounded along the optimal path. As we know, this implies that there is a one-to-one relation between optimal control and dynamic programming.

The value function at date 0 can be expressed as:

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t \ln(1-\eta)\,\hat{k}_t^{\alpha} = \sum_{t=0}^{\infty} \beta^t \ln(1-\eta) + \alpha \sum_{t=0}^{\infty} \beta^t \ln \hat{k}_t \tag{3.42}$$

Iterating on our guess and taking logs, we obtain:

$$\ln \hat{k}_t = \frac{1-\alpha^t}{1-\alpha} \ln \eta + \alpha^t \ln k_0 \tag{3.43}$$

Substituting (3.43) into (3.42) we get:

$$\begin{aligned} v(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln(1-\eta) + \alpha \sum_{t=0}^{\infty} \beta^t \left(\frac{1-\alpha^t}{1-\alpha} \ln \eta + \alpha^t \ln k_0 \right) = \\ &= \frac{\ln(1-\eta)}{1-\beta} + \frac{\alpha \ln \eta}{1-\alpha} \sum_{t=0}^{\infty} \beta^t (1-\alpha^t) + \frac{\alpha}{1-\alpha\beta} \ln k_0 \end{aligned} \tag{3.44}$$

Comparing (3.41) and (3.44) we conclude that $b = \alpha/(1-\alpha\beta)$ and, of course, that $\eta = \alpha\beta$.

Finally, since:

$$\sum_{t=0}^{\infty} \beta^t \left(1-\alpha^t\right) = \frac{(1-\alpha)\beta}{(1-\beta)(1-\alpha\beta)} \tag{3.45}$$

we obtain:

$$a = \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta \ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)} \tag{3.46}$$

3.1.6 Decentralization

In the previous Sections we analyzed the benevolent planner's problem, focusing on the Pareto-efficient optimal plan without explicitly characterizing the corresponding market structure and competitive equilibrium. Let us now introduce the simplest decentralization scheme that allows us to easily highlight (once again) the strict relationship between Pareto-efficient allocations and competitive equilibria in the RCK framework.

Assume that the representative household owns all factors of production, i.e. capital and labor. Physical capital is the only asset available in the economy. Hence, the representative household makes consumption/investment decisions under perfect foresight, and rents capital and labor to a representative firm. The latter uses the services of capital and labor to produce the final consumption (investment) good. All markets are competitive, and all trades occur at date 0. For this economy, a price system is a sequence of prices $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ where (*i*) q_t is the date-0 price of one unit of consumption at date t; (*ii*) r_t is the date-0 price of one unit of capital rented at date t; (*iii*) w_t is the date-0 price of one unit of labor rented at date t.

Households

The representative household faces the following intratemporal budget constraint:

$$q_t \left[c_t + k_{t+1} - (1-\delta)\,k_t\right] = r_t k_t + w_t \tag{3.47}$$

Iterating on (3.47), we get the present-value intertemporal budget constraint:

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} \{w_t + [q_t (1-\delta) + r_t - q_{t-1}] k_t\} - \lim_{t\to\infty} q_t k_{t+1} \tag{3.48}$$

where $q_{-1} = 0$. Note that, if $u'(\cdot) > 0$, then it will never be optimal to held a strictly positive capital stock in the limit, because this would decrease the amount of resources available for consumption.

Remark 39 *Optimality requires the following terminal condition:*

$$\lim_{t\to\infty} q_t k_{t+1} = 0 \tag{3.49}$$

This implies that the household's problem can be reformulated as:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad U = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.50}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} q_t [c_t + k_{t+1} - (1-\delta) k_t] = \sum_{t=0}^{\infty} (r_t k_t + w_t)$$

where the sequence of prices $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and the initial condition $k_0 > 0$ are taken as given.

The corresponding present-value Lagrangian can be written as (note the single Lagrange multiplier):

$$L_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left\{ \sum_{t=0}^{\infty} (r_t k_t + w_t) - \sum_{t=0}^{\infty} q_t [c_t + k_{t+1} - (1-\delta) k_t] \right\} \tag{3.51}$$

Deriving the Lagrangian with respect to c_t and k_{t+1} we obtain the (by now) familiar first order conditions (a hat identifies the optimal plan):

$$\beta^t u'(\hat{c}_t) = q_t \hat{\lambda} \tag{3.52}$$

$$r_{t+1} = q_t - q_{t+1}(1-\delta) \tag{3.53}$$

Equation (3.53) above, a no-arbitrage condition, is known as "user cost of capital" formula. The interpretation is straightforward: along an optimal path, the future return on one additional unit of capital installed today has exactly offset its "user cost," i.e. the difference between the price at which the unit has to be purchased today and the price at which what remains after depreciation has been take into account may be sold next period.

Firms

The representative firm maximizes the present value of economic profits, given by:

$$\Pi = \sum_{t=0}^{\infty} [q_t f(k_t) - w_t - r_t k_t] \tag{3.54}$$

The first order conditions for the services of labor and capital, as usual, equate the value of the marginal productivity of each factor to its price:

$$w_t = q_t \left[f\left(\hat{k}_t\right) - f_k\left(\hat{k}_t\right)\hat{k}\right] \quad (3.55)$$
$$r_t = q_t f_k\left(\hat{k}_t\right) \quad (3.56)$$

Note that optimal control problem of the representative firm effectively boils down to a sequence of static profit maximization problems.

Equilibrium

Definition 40 *In the economy outlined in this section, a* ***competitive equilibrium****, for a given* $k_0 > 0$*, is made of an allocation* $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ *and a price system* $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ *such that:*

1. *the allocation* $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ *solves the representative household's problem for the given price system;*

2. *the allocation* $\{c_t, k_t\}_{t=0}^{\infty}$ *maximizes the profits of the representative firm for the given price system;*

3. *demand equals supply in all markets, and in particular in the market for the final consumption good:*
$$\underbrace{c_t + k_{t+1} - (1-\delta)k_t}_{demand} = \underbrace{f(k_t)}_{supply} \quad (3.57)$$

By combining equations (3.49), (3.52), (3.53), and (3.56) we can easily show that the competitive equilibrium is characterized by the following dynamic system:

$$u'(\hat{c}_t) = \beta u'(\hat{c}_{t+1})\left[f_k\left(\hat{k}_{t+1}\right) + 1 - \delta\right] \quad (3.58)$$
$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + f\left(\hat{k}_t\right) - \hat{c}_t \quad (3.59)$$
$$\lim_{t\to\infty} \beta^t u'(\hat{c}_t)\hat{k}_{t+1} = 0 \quad (3.60)$$

Remark 41 *Quite evidently, and not surprisingly, there is a one-to-one relationship between the Pareto-efficient optimal plan identified in the previous Sections and the competitive dynamic equilibrium analyzed here.*

3.2 Exogenous growth

In Section 3.1.2, p. 43, we argued that the basic RCK model is incompatible with long-run growth. But we also know that long-run growth is clearly supported by empirical evidence. How can we "convince" the RCK model to follow a balanced growth path in the long-run?

The simplest way is to assume the existence of an exogenous engine of growth. This mechanism may be naturally identified with technological progress, *i.e.* with a process that improves the productivity of some factors over time. It is well known (we omit the simple but cumbersome proof; see King, Plosser, and Rebelo, 1987, or McCallum, 1996, Appendix B) that only labor-augmenting technological progress is compatible with balanced growth. This means that only production functions of the form:

$$y_t = F(k_t, Z_t n_t) \tag{3.61}$$

where Z_t is a technology index, are compatible with balanced growth. If Z_t grows at a positive exogenous rate $\gamma > 0$, then $Z_t = Z_0 (1+\gamma)^t$.

3.2.1 Constant elasticity of substitution

Another well know result (again, see King, Plosser, and Rebelo, 1987, or McCallum, 1996, Note 11, p. 46) states that only the utility functions characterized by a constant elasticity of marginal utility with respect to consumption, defined as $\xi_{cc} \equiv [u''(c)c]/u'(c)$, are compatible with balanced growth.

The proof is rather simple. Along a balanced growth path, consumption grows at the common constant rate. This implies that, in the long-run, $c_{t+1} = (1+\gamma) c_t$. Consider (3.10):

$$u'(c_t) = [f'(k_{t+1}, Z_{t+1}) + 1 - \delta] \beta u'(c_{t+1}) \tag{3.62}$$

Note that, for consumption to grow at a constant rate, the term $f'(k_{t+1}, Z_{t+1}) + 1 - \delta$ has to be constant in the long-run. We can then rewrite (3.62) as:

$$u'(c_t) = \omega u'[(1+\gamma) c_t] \tag{3.63}$$

Differentiate equation (3.63) with regard to c_t :

$$u''(c_t) = \omega\gamma u''[(1+\gamma) c_t] \tag{3.64}$$

Dividing (3.64) by (3.63), multiplying both sides by c, and simplifying we obtain:

$$\frac{u''(c_t) c_t}{u'(c_t)} = \frac{u''(c_{t+1}) c_{t+1}}{u(c_{t+1})} \tag{3.65}$$

The sole family of utility functions that satisfies this requirement is the so-called isoelastic family, defined as $u(c) = c^{1-1/\mu}/(1-1/\mu)$ for $\mu \neq 1$, and[6] as $u(c) = \log(c)$ for $\mu = 1$.[7] The strict concavity assumption requires $1/\mu > 0$. It is easy to show that $\xi_{cc} = -1/\mu$, and that the intertemporal elasticity of substitution is equal to μ.

3.2.2 Normalization

The introduction of technological progress has a very important consequence: the system will not converge to a steady-state in the long run (by the way, this was exactly our goal!). However, to study its dynamics from a qualitative point of view, we need the phase-plane techniques introduced in the previous Section. These techniques require the model to have a steady-state.

A convenient solution to our problem is the following. We can normalize the model with regard to some variable that grows at the common rate in the long-run. A natural

[6] It is easy to show, applying the l'Hopital rule, that $\lim_{\mu \to 1} \left(c^{1-\mu} - 1\right) / (1-\mu) = \ln(c)$. The constant factor is dropped in our analysis for the sake of national simplicity.

[7] This family of CES utility functions is not invariant to monotone transformations. Hence, we are moving from ordinal to cardinal preferences. This is a commonly overlooked fact.

candidate may be Z_t, the exogenous technical progress, since it can be easily recovered knowing the initial condition Z_0 and the constant grow rate γ. In this case, there would be a one-to-one relationship between the original and the normalized systems (*i.e.* they would be *isomorphic*).

Denoting with a tilde the normalized variables, the intertemporal utility function becomes:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \frac{Z_s^{1-\frac{1}{\mu}}}{1-\frac{1}{\mu}} \left(\frac{c_s}{Z_s} \right)^{1-\frac{1}{\mu}} = Z_t^{1-\frac{1}{\mu}} \sum_{s=t}^{\infty} \tilde{\beta}^{s-t} \frac{\tilde{c}_s^{1-\frac{1}{\mu}}}{1-\frac{1}{\mu}} \tag{3.66}$$

where $\tilde{\beta} \equiv \beta (1+\gamma)^{1-\frac{1}{\mu}}$.

Taking into account that the production function is homogenous of degree one, and that $k_{t+1}/Z_t = (k_{t+1}/Z_{t+1})(Z_{t+1}/Z_t) = (1+\gamma)\,\tilde{k}_{t+1}$, we can write the normalized accumulation equation as:

$$(1+\gamma)\,\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + f\left(\tilde{k}_t\right) - \tilde{c}_t \tag{3.67}$$

The normalized system is qualitatively similar to the basic RCK model (Check!). The normalized system tends a steady-state in the long run.[8] If the normalized system tends to a steady-state, the original model tends to a balanced growth path, where all variables grow at the exogenous growth rate γ.

3.3 Endogenous growth

In the previous Section, we concluded that the mechanism at the heart of basic RCK model, i.e. physical capital accumulation, is essentially incompatible with a positive rate of long-run economic growth without further assumptions. Let us analyze the issue more in detail. Consider the basic RCK framework again, and recall the accumulation equation (3.3), rewriting it as:

$$\gamma_t^k = \frac{f(k_t)}{k_t} - \delta - \frac{c_t}{k_t} \tag{3.68}$$

In the long run:

$$\lim_{t\to\infty} \gamma_t^k = \lim_{t\to\infty} \frac{f(k_t)}{k_t} - \delta - \lim_{t\to\infty} \frac{c_t}{k_t} \tag{3.69}$$

Note that $\lim_{t\to\infty} \gamma_t^k = \gamma^k > 0$ only if:

$$\begin{cases} \lim_{t\to\infty} \frac{c_t}{k_t} = \frac{c}{k} > 0 \\ \lim_{t\to\infty} \frac{f(k_t)}{k_t} = \eta > 0 \\ \eta > \delta + \frac{c}{k} \end{cases} \tag{3.70}$$

Let us examine this three conditions in turn:

1. The first one, $\lim_{t\to\infty} \frac{c_t}{k_t} = \frac{c}{k} > 0$, simply requires that $\lim_{t\to\infty} \gamma_t^c = \gamma^k > 0$, i.e. growth along a *balanced growth path*.

2. The second one, again requires that $\lim_{t\to\infty} \gamma_t^y = \gamma^k > 0$. Furthermore, note that if

[8] Is there anything missing? Well, actually yes: what about the boundedness of the objective function? Since the normalized model reaches a steady-state, we simply need to assume that $\tilde{\beta} < 1$.

$\gamma^k > 0$ then $\lim_{t\to\infty} k_t = \infty$. Hence the condition can be rewritten as:

$$\lim_{t\to\infty} \frac{f(k_t)}{k_t} = \lim_{k\to\infty} \frac{f(k_t)}{k_t} \overset{H}{=} \lim_{k\to\infty} f'(k_t) = \eta > 0 \tag{3.71}$$

This amounts to a blatant violation of the second Inada condition.

3. Finally, the third one, $\lim_{k\to\infty} f'(k_t) = \eta > \delta + c/k$.

Remark 42 *Long-run growth in the basic RCK model is admissible along a balanced growth path only, and requires the existence of a lower bound (left, for the moment, unspecified) for the marginal productivity of physical capital.*

This condition is however only necessary, not sufficient, for long-run growth. Consider the Euler equation (3.11) and assume a isoelastic utility function of the form $u(c) = c^{1-1/\mu}/(1-1/\mu)$:

$$\gamma_t^c = \beta^\mu [f'(k_{t+1}) + 1 - \delta]^\mu - 1 \tag{3.72}$$

In the long run:

$$\gamma^c = \beta^\mu [\eta + 1 - \delta]^\mu - 1 \tag{3.73}$$

Hence, $\gamma^c > 0$ if and only if:

$$\eta > \delta + \rho \tag{3.74}$$

where $\rho = (1-\beta)/\beta$ is the intertemporal discount rate. Note that if $0 < \eta < \delta + \rho$ then the only possible long-run growth rate remains $\gamma^k = 0$.

3.3.1 The AK model

The simplest (simplistic?) way to impose condition (3.74) is to assume that the production function enjoys constant returns to scale in the accumulable factor, i.e. in our case, that it is *linear* in physical capital:

$$f(k_t) = Ak_t \tag{3.75}$$

where $f'(k) = A > \delta + \rho > 0$.

If the utility function is isoelastic, the first order conditions for the planner's problem become the following ones:

$$c_{t+1} = \phi c_t \tag{3.76}$$
$$k_{t+1} = \psi k_t - c_t \tag{3.77}$$

where:

$$\psi \equiv A + 1 - \delta > 1 \tag{3.78}$$
$$\phi \equiv (\beta\psi)^\mu > 1 \tag{3.79}$$

These conditions are necessary and sufficient to identify the optimal path, together with the usual TVC:

$$\lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0 \tag{3.80}$$

Note that the Euler equation does not depend on k_t!

Remark 43 *The AK model has no transitional dynamics: once c_0 has been optimally chosen, the system starts immediately to evolve along a balanced growth path.*

Iterating on the FOCs (and dropping the hats for notational simplicity) we obtain:

$$c_t = \phi^t c_0 \tag{3.81}$$

$$k_{t+1} = \psi^{t+1} k_0 - \sum_{j=0}^{t} \psi^{t-j} c_j \tag{3.82}$$

Note that:

$$\lambda_t = c_t^{-\frac{1}{\mu}} = \left(\phi^t c_0\right)^{-\frac{1}{\mu}} \tag{3.83}$$

Hence, the TVC can be rewritten as:

$$\lim_{t\to\infty} \beta^t \left(\phi^t c_0\right)^{-\frac{1}{\mu}} \left(\psi^{t+1} k_0 - \sum_{j=0}^{t} \psi^{t-j} c_j\right) = 0 \tag{3.84}$$

Note that:

- being $c_0 > 0$ along an optimal path, we can safely divide both sides of (3.84) by c_0;
- $\phi^{\frac{1}{\mu}} = \beta\psi$ by definition.

Hence (3.84) becomes:

$$\lim_{t\to\infty} \beta^t (\beta\psi)^{-t} \left(\psi^{t+1} k_0 - \psi^t \sum_{j=0}^{t} \psi^{-j} \phi^j c_0\right) = 0 \tag{3.85}$$

Simplifying:

$$c_0 \lim_{t\to\infty} \sum_{j=0}^{t} \left(\frac{\phi}{\psi}\right)^j = \psi k_0 \tag{3.86}$$

Being $\phi/\psi < 1$, we know that:

$$\lim_{t\to\infty} \sum_{j=0}^{t} \left(\frac{\phi}{\psi}\right)^j = \frac{\psi}{\psi - \phi} \tag{3.87}$$

This allows us to solve for c_0:

$$c_0 = \zeta k_0 \tag{3.88}$$

where $\zeta = \psi - \phi$.

Let us now check that c_t and k_t actually move together along a balanced growth path. Consider that:

$$\frac{c_t}{k_t} = \frac{\phi^t \zeta k_0}{\psi^t k_0 - \sum_{j=0}^{t-1} \psi^{t-1-j} c_j} = \frac{\phi^t \zeta k_0}{\psi^t k_0 - \sum_{j=0}^{t-1} \psi^{t-1-j} \phi^j \zeta k_0} =$$

$$\frac{\phi^t \zeta}{\psi^t - \psi^{t-1}(\psi - \phi)\sum_{j=0}^{t-1}\left(\frac{\phi}{\psi}\right)^j} = \frac{\phi^t \zeta}{\psi^t \left[1 - \sum_{j=0}^{t-1}\left(\frac{\phi}{\psi}\right)^j + \frac{\phi}{\psi}\sum_{j=0}^{t-1}\left(\frac{\phi}{\psi}\right)^j\right]} =$$

$$\frac{\phi^t \zeta}{\psi^t \left[1 - \sum_{j=0}^{t-1}\left(\frac{\phi}{\psi}\right)^j + \sum_{j=1}^{t}\left(\frac{\phi}{\psi}\right)^j\right]} = \frac{\phi^t \zeta}{\psi^t \left(\frac{\phi}{\psi}\right)^t} = \zeta$$

This obviously implies that $c_t = \zeta k_t \ \forall t$.[9] The two variables share the same growth rate; hence:

$$\gamma = \phi - 1 = [\beta (A + 1 - \delta)]^{\mu} - 1 \tag{3.89}$$

Remark 44 *This long-run, positive growth rate depends exclusively on the deep parameters of the model: growth is endogenous, being a result of the internal workings of the model, i.e. capital accumulation in the presence of non-decreasing returns to capital.*

Note that the investment share in income is given by:

$$s_i = \frac{i}{y} = \frac{y - c}{y} = \frac{A - \zeta}{A} = \frac{\gamma + \delta}{A} \tag{3.90}$$

There is a one-to-one relationship between the investment share and the long-run growth rate. This strong implication has been used by Jones (1995) and others to test the empirical validity of the AK model.

3.4 Numerical experiments

As in Section 1.5, p. 26, we can exploit the model's recursive structure and numerically solve it with respect to the time-invariant policy function for consumption, which depends on the physical capital stock, the only state variable. As far as the actual functional forms is concerned, we assume an isoelastic utility function, $u(c) = c^{1-\mu}/(1-\mu)$, and a Cobb-Douglas production function, $f(k) = \phi k^{\alpha}$.

Under these assumptions, the policy function has to satisfy the following functional equation:

$$c(k) = c(k') \left\{ \frac{\tilde{\beta}}{\gamma} \left[\alpha\phi (k')^{\alpha - 1} + 1 - \delta \right] \right\}^{-\frac{1}{\mu}} \tag{3.91}$$

where:

$$k' = \frac{\phi k^{\alpha} + (1 - \delta) k - c(k)}{\gamma} \tag{3.92}$$

We again approximately solve (3.91) using *Chebyshev collocation* method, under the following benchmark parameterization: $\beta = 0.96$, $\mu = 2$, $\alpha = 0.4$, $\gamma = 1.016$, and $\delta = 0.1$.[10] The value of the scale parameter ϕ is chosen so to make the steady-state capital level equal to unity. Once an approximated policy function is at hand, we can recursively solve the system for the given initial condition.

In Figure 6.6 we plot the model's transitional dynamics for $k_0 = 0.8$. In other words, we describe the adjustment process of output and capital for a RCK economy starting at date 0 with a capital stock equal to 80% of its steady-state value. As we can see, the variables grow over time and smoothly converge to their steady-state values, as expected. Note however, that convergence is *extremely* rapid: half the way to the steady-state is covered in about 8 years.

[9] It also implies that $\lim_{t\to\infty} (c_t/k_t) = \zeta$. The necessary condition for endogenous growth requires that $A > \delta + \zeta$. It can be easily shown that this requirement boils down to $A > \delta + \rho$.

[10] See the Appendix for more details. Another possibility, theoretically simpler but numerically less efficient, is to use time iteration on the Euler equation, as again explained in the Appendix.

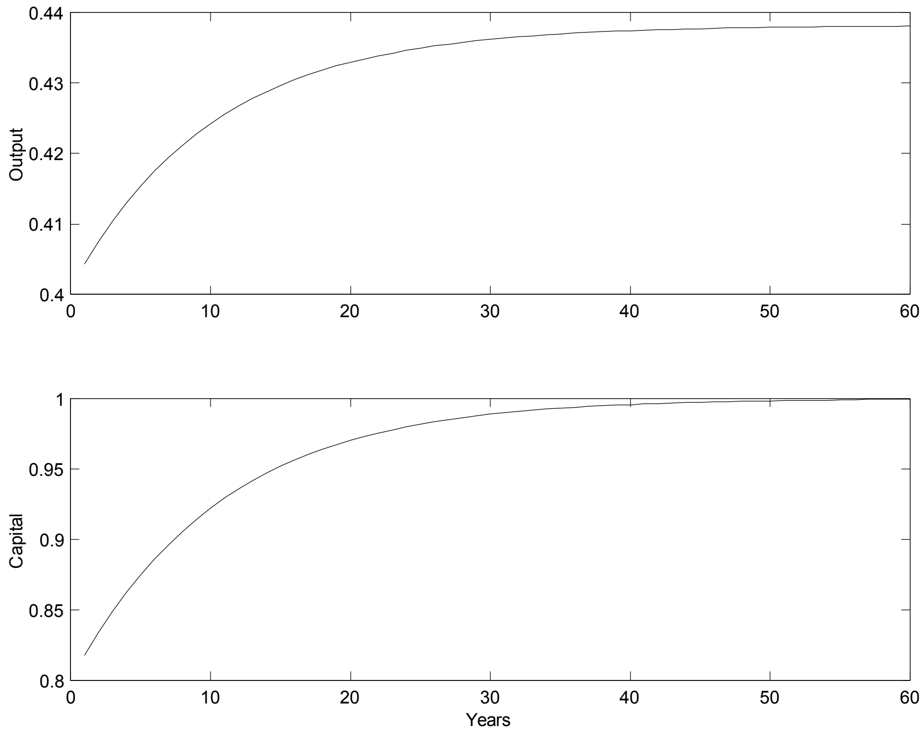

Figure 3.6: Transitional dynamics in the RCK model.

In Figure 6.7 we plot the corresponding growth rates (we are considering normalized variables, so these growth rates have to be added to the exogenous one, equal in this case to 1.6% a year). Our results imply that, at date zero, the growth rate of physical capital is about 2.2 points higher than in steady-state, but this difference practically disappears in less than twenty years.

3.4.1 Sensitivity analysis

Once we fully characterized the model's steady-state and transitional dynamics, it would be interesting to study how the model reacts to unexpected changes in the deep parameters. In particular, we may be interested in evaluating how and to which extent changes in the parameterization affect both the steady-state and the transitional dynamics. This kind of experiments are called *sensitivity analysis*.

Figure 3.8 shows the effect of a sudden increase in μ (from 2 to 3), the preference parameter inversely related to the elasticity of intertemporal substitution; we plot the percentage deviations from the initial steady-state values. The economic intuition is simple: if μ increases, the elasticity of intertemporal substitution decreases, and the representative individual is less willing to substitute current for future consumption. Current consumption increases on impact by nearly four percentage points, while investment decreases by almost ten points; during the transition to the new steady-state, consumption, investment, output and capital decrease, while the interest rate rises. In steady-state, the consumption level is nearly three points lower, simply because the induced decrease in investment implies a lower steady-state capital stock.

Figure 3.9, instead, shows the effect of a decrease in β (from 0.96 to 0.95), the intertemporal discount factor. Our representative individual becomes more impatient, and

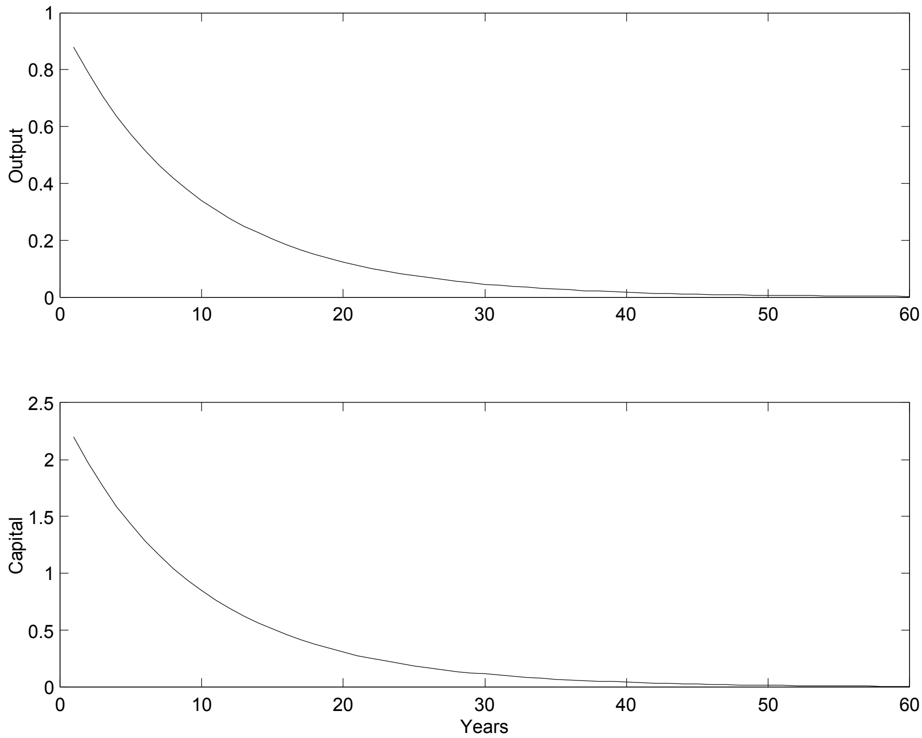

Figure 3.7: Growth rates during transition.

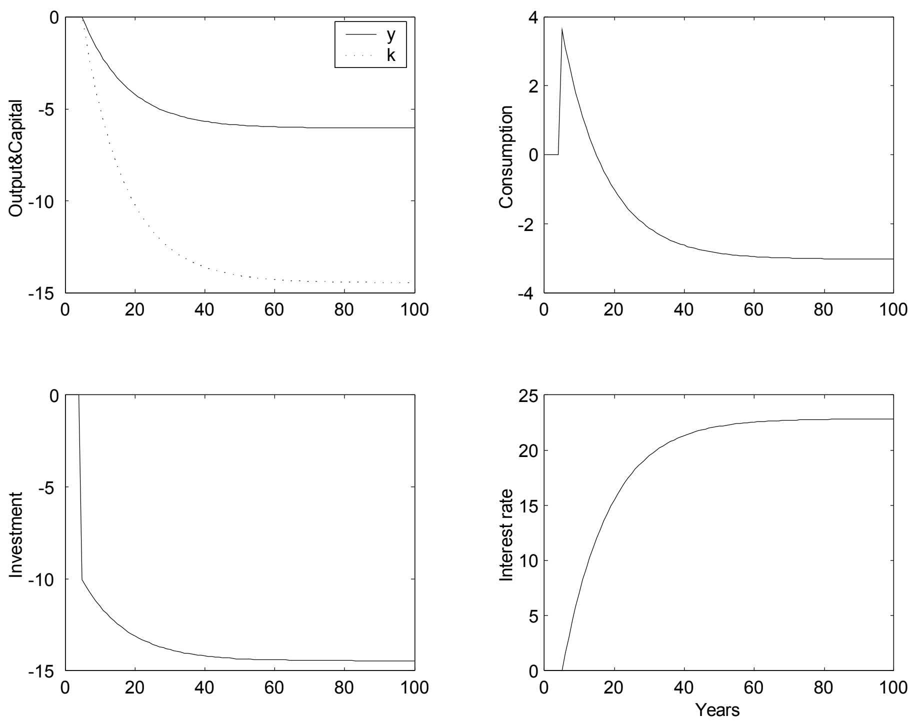

Figure 3.8: Increase in μ (from 2 to 3).

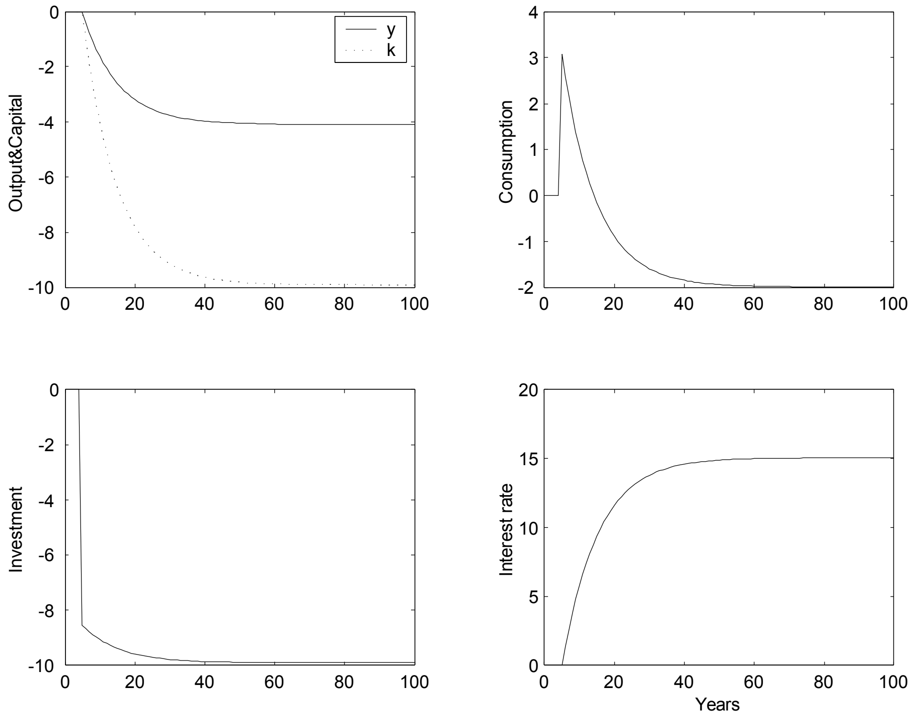

Figure 3.9: Decrease in β (from 0.96 to 0.95).

so current consumption increases on impact by three percentage points, while investment decreases by almost nine points; as before, during the transition to the new steady-state all variables but the interest rate decrease. In steady-state, the consumption level is nearly two points lower, again because the steady-state capital stock is ten points lower than before.

The effect of a decrease in δ (from 0.10 to 0.09), the depreciation rate, is summarized in Figure 3.10. If the depreciation rate decreases, a lower level of investment is needed to accumulate a given level of future capital, and so investment decreases on impact while consumption slightly increases. The sudden increase in the interest rate, however, quickly drives the investment level back to its previous steady-state level and even further up; this increases the capital stock, and consequently output and consumption, while the interest rate converges back to its steady-state value. As a result, the steady-state levels of capital, output, and consumption are higher than before.

Figure 3.11, finally, shows the effect of a decrease in γ (from 1.016 to 1.015), the exogenous growth factor. The decrease in γ has, *ceteris paribus*, three effects: first, it increases the "modified" intertemporal discount factor $\tilde{\beta} = \beta\gamma^{1-\mu}$ (note that $1 - \mu < 1$); second, it directly increases the slope of the consumption path, via the Euler equation (3.91); third, it decreases the level of future (normalized) capital stock implied by a given level of current investment, via the accumulation equation (3.92). As a result, current consumption decreases on impact, while investment increases. During the transition, all variable except the interest rate increase, and the (normalized) system reaches a new steady-state where capital, output, consumption and investment are *higher* than before.

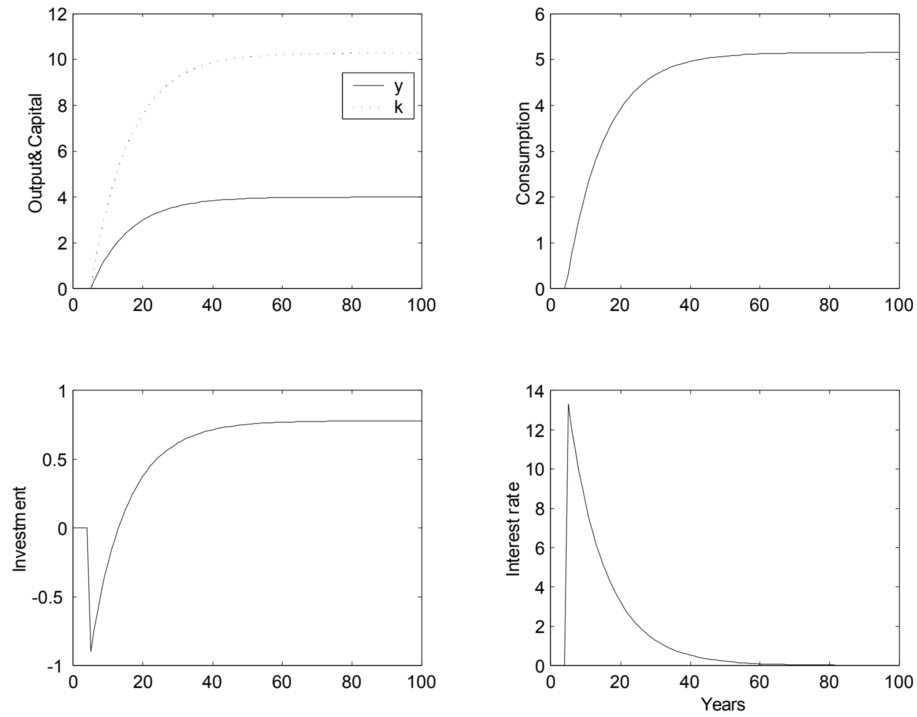

Figure 3.10: Decrease in δ (from 0.1 to 0.09).

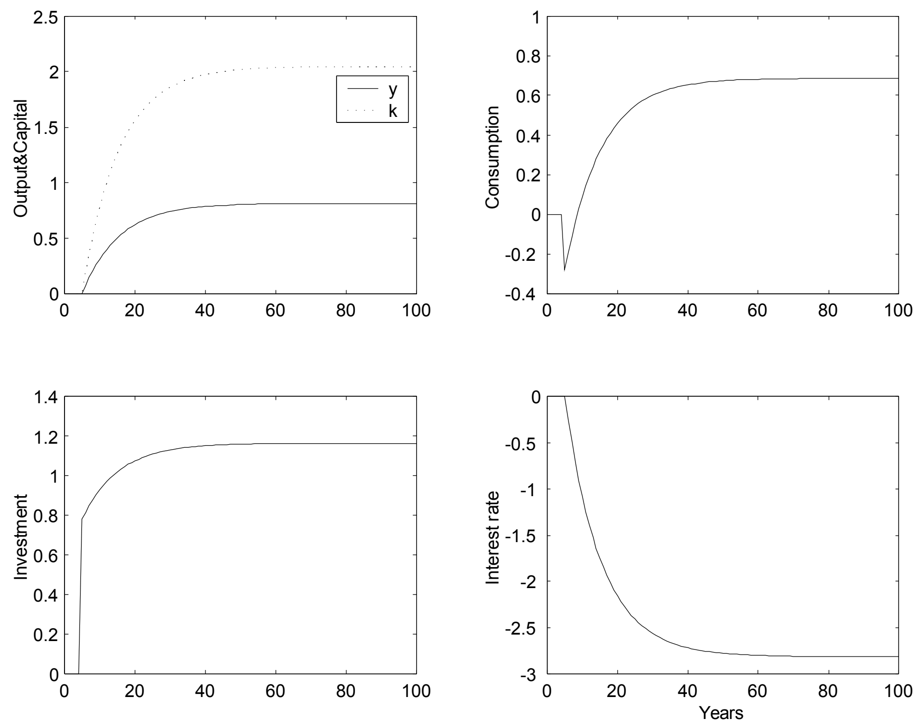

Figure 3.11: Decrease in γ (from 1.016 to 1.015).

3.4.2 Conditional convergence

All previous Sections were devoted to a pure *qualitative* analysis of the RCK model. We reached the conclusion that, if the initial capital level is lower than the steady-state one, consumption, capital, and output increase over time at a decreasing pace, converging towards the steady-state allocation in the long-run. Furthermore, it can be shown that the initial growth rates of output and capital are higher the lower the initial capital stock. In other words, the basic RCK models implies that "poor" countries should growth faster, *ceteris paribus*, than "rich" ones. Furthermore, it suggests that, once we rule differences in technology and preferences out, all countries should converge to the same steady-state.

The available empirical evidence shows that:

1. different countries grow at different rates; in particular, poor countries tend to grow faster, once the industrialization process takes off;

2. differences in growth rates tend to persist over time, even for decades.

The previously described RCK comparative dynamics seems consistent with these two stylized facts. These implications, summarized as cross-country *conditional convergence* (both in levels and growth rates) have been extensively tested in the literature, with mildly positive results.

However, nothing in our analysis guarantees that the qualitative behavior of the RCK model may also be *quantitatively* reasonable. In an influential paper, King and Rebelo (1993) actually challenged the profession's wisdom with a simple but surprising numerical experiment. To partially replicate their results, let us now identify our toy economy with the US, the "rich" country by definition. In the 50s, right after the end of WWII, per-capita output level in the US was more than five times the per-capita level in Japan; assuming identical technologies, the initial capital level in Japan, if the US one was 0.8, should have been 0.0126. Figure 6.10 plots the growth rates for our two countries: well, according to the RCK model, in the 50s the growth rate of per-capita output in Japan should have been nearly 40 times higher than in the US. Furthermore, this tremendous gap should have been covered in only ten years, more or less. There is not much to say about these prediction: they are simply wrong!

There is an even more important problem. Figure 6.11 plots the interest rates (rental rates minus depreciation) for our two countries. The RCK model predicts not only extremely high interest rates, even 140% in Japan, but implausible interest rate differentials: the interest rate in Japan should have been nearly 14 times higher than in the US at date zero, and a quantitatively significant difference lasts for nearly twenty years. These figures are hard to reconcile with the empirical evidence suggesting that capital is at least partially mobile across countries: why did individuals in the US not exploit these arbitrage opportunities, if they really existed?

We are presenting, of course, an extreme case, but the message is clear. Under our benchmark parameterization, the RCK model is unable to account for *sustained* differences in growth rates of per-capita output and capital. Furthermore, quantitatively relevant differences in initial growth rates translate into implausibly high interest rate differentials. King and Rebelo (1993) show that these outcomes are robust to many generalizations, and do not depend on the chosen parameterization.[11]

[11] Actually, by increasing the intertemporal elasticity of substitution we may generate sustained growth differentials, but produce even higher interest rate differentials.

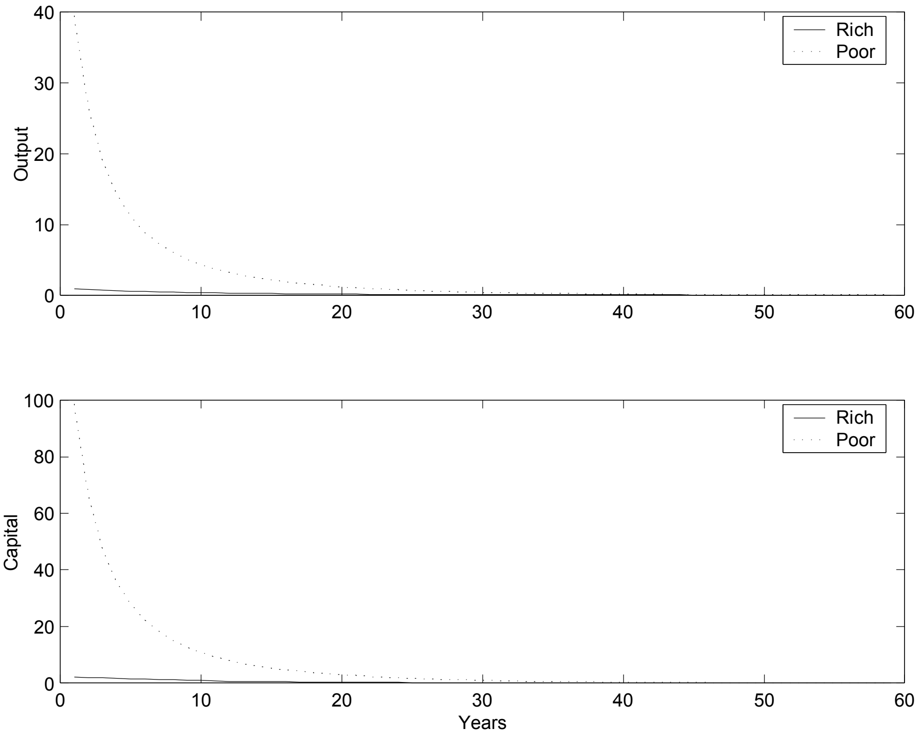

Figure 3.12: Conditional convergence.

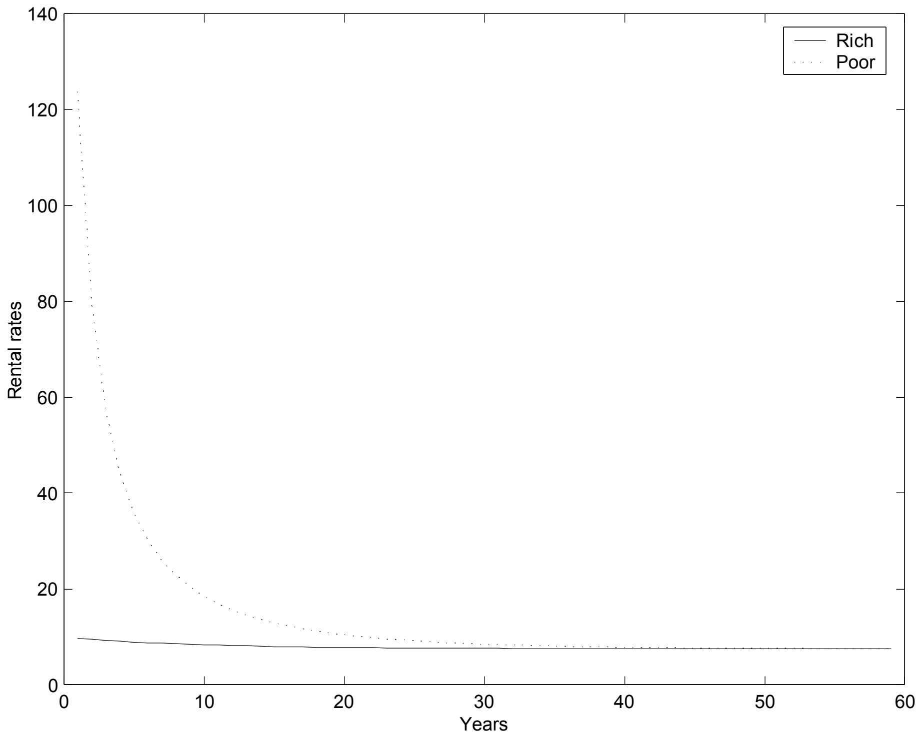

Figure 3.13: Interest rate differentials.

In other words, we may conclude that physical capital accumulation alone, in a neoclassical optimizing framework, is not sufficient to explain sustained cross-country differentials in growth rates, because the implications of diminishing returns are highly counterfactual. King and Rebelo (1993) conclude that their results may point to the use of "endogenous growth" models instead of "exogenous growth" ones.

As always, many *caveats* apply to our analysis. The most important one, at least in our option, is that the real world is not a collection of closed RCK economies, since countries trade in goods and to some extend in factors of production. International trade may have interesting effects on the transitional dynamics in a world of RCK economies, as suggested in Cuñat and Maffezzoli (2004).

3.5 Appendix: computational details

3.5.1 Chebyshev collocation

Following Judd (1992) again, we approximate the policy function $c(k)$ over an interval $D \equiv [\underline{k}, \overline{k}] \in R_+$ with a linear combination of Chebyshev polynomials:

$$\widehat{c}(k;\boldsymbol{\theta}) = \sum_{i=0}^{d} \theta_i \psi_i(k) \tag{3.93}$$

where:

$$\psi_i(k) \equiv T_i\left(2\frac{k-\underline{k}}{\overline{k}-\underline{k}} - 1\right) \tag{3.94}$$

The functional equation (3.91) becomes:

$$\hat{c}(k)\left(\frac{\tilde{\beta}}{\gamma}\right)^{\frac{1}{\mu}}\left[\alpha\phi\left[\frac{\phi k^{\alpha}+(1-\delta)k-\hat{c}(k)}{\gamma}\right]^{\alpha-1}+1-\delta\right]^{\frac{1}{\mu}} = \hat{c}\left[\frac{\phi k^{\alpha}+(1-\delta)k-\hat{c}(k)}{\gamma}\right] \tag{3.95}$$

We find n zeros of Chebyshev polynomials in $[-1,1]$, reverse the normalization and transform them into the corresponding values in $[\underline{k}, \overline{k}]$. Then, we numerically solve equation (3.95) at these points for the n parameters in $\boldsymbol{\theta}$. In other words, we approximate a functional equation with a system of non-linear equations, which can be easily solved with any standard algorithm. In our exercise, we choose $\underline{k} = 0.01$, $\overline{k} = 1.99$, and $d = 14$. The relative simplicity of the problem guarantees a high numerical accuracy.

3.5.2 Time iteration

An alternative approach is to approximate the policy function with cubic interpolation over a fixed grid, and use time iteration on the Euler equation to converge to the solution:

Algorithm 45 *Choose a suitable grid of points over an interval* $[\underline{k}, \overline{k}] \in R_+$*, say* $\mathbf{k} =$

$\{k_i\}_{i=1}^n$,[12] *and an initial guess for the optimal consumption levels at the nodes* k_j, *say* $\mathbf{c}_0 = \{c_{0i}\}_{i=1}^n$. *Then, for* $j \geq 0$:

1. *Given* $\mathbf{c}_j$, *compute:*
$$k'_{j,i} = \frac{\phi k_i^\alpha + (1-\delta)\,k_i - c_{j,i}}{\gamma} \tag{3.96}$$

2. *Given* $\mathbf{k}$, $\mathbf{c}_j$, *and* $\mathbf{k}'_j = \left\{k'_{j,i}\right\}_{i=1}^n$, *obtain* $\mathbf{c}'_j$ *via cubic interpolation (or extrapolation, if needed).*

3. *Given* $\mathbf{c}'_j$, *compute* $\hat{\mathbf{c}}_j$ *as:*
$$\hat{c}_{j,i} = c'_{j,i}\left\{\frac{\tilde{\beta}}{\gamma}\left[\alpha\phi\left(k'_{j,i}\right)^{\alpha-1} + 1 - \delta\right]\right\}^{-\frac{1}{\mu}}. \tag{3.97}$$

4. *Update the current guess:*
$$\mathbf{c}_{j+1} = \upsilon\hat{\mathbf{c}}_j + (1-\upsilon)\,\mathbf{c}_j \tag{3.98}$$
where $\upsilon \in (0,1)$, *and iterate on (1)-(4) until convergence.*

[12] The gird has not to be uniformly distributed over the interval: it's possible to concentrate a larger mass of points in the region where the policy function is particularly nonlinear.

Chapter 4

Dynamic Stochastic General Equilibrium

4.1 Equilibrium with complete markets

We shall now characterize the concept of dynamic competitive equilibrium in a pure exchange economy with infinitely living individuals and stochastic endowments.[1]

Consider a simple pure-exchange infinite-horizon economy, characterized by stochastic endowments. In each period $t \geq 0$, randomness is resolved with the realization of stochastic event $s_t \in S$. The full history of events up to date t is denoted $s^t = \{s_0, s_1, ..., s_t\}$ and is publicly observable by all agents: in other words, the vector s^t, by summarizing the sequence of actual realizations of s_t, fully represents the current "state of the world." The unconditional probability associated with a particular sequence s^t is given by the probability measure $\pi_t(s^t)$. All trading occurs after observing s_0, hence $\pi_0(s_0) = 1$.[2] Note that the introduction of s^t creates a wider commodity space in which economic goods are differentiated not only by date of delivery, but also by history, or "state of the world."

The economy is inhabited by a large but finite number of identical households, indexed by $i \in \{1, 2, ..., I\}$. Each household is approximately of *measure zero*,[3] hence price-taker on the consumption good and asset markets. Households can purchase history-dependent consumption streams of the form $c_i = \{c_{it}(s^t)\}_{t=0}^{\infty}$; a collection of consumption streams, one for each household, $\{\{c_{it}(s^t)\}_{t=0}^{\infty}\}_{i=1}^{I}$, is called an *allocation*. Households' preferences on these consumption streams can be represented by the standard intertemporal utility function:

$$U(c_i) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_{it}(s^t)\right] \pi_t(s^t) = E_0\left[\sum_{t=0}^{\infty} \beta^t u(c_{it})\right] \tag{4.1}$$

where $E_0(\cdot) \equiv E(\cdot|s_0)$. As in Section 1.2.1, p. 5, we assume that $u(\cdot)$ is C^2, strictly increasing, and strictly concave, with $\lim_{c \to 0} u'(c) = +\infty$.

In each period, the households receive a positive endowment of the single consumption good that depends on the state of the world, $y_{it}(s^t)$ and is therefore eminently stochastic.

[1] The interested reader may find more details in Mas-Colell *et al.* (1995, Ch. 19) and Ljungqvist and Sargent (2004, Ch. 8).

[2] For the sake of exposition, assume that $S = \{0, 1\}$ and $s_0 = 0$. In period 1, there will be two possible states of the world: $s_1^1 = \{0, 1\}$ or $s_2^1 = \{0, 0\}$. In period 2, the possible states of the world become four: $s_1^2 = \{0, 1, 1\}$, $s_2^2 = \{0, 1, 0\}$, $s_3^2 = \{0, 0, 1\}$, or $s_4^2 = \{0, 0, 0\}$. Quite evidently, the number of possible realizations of s^t increase dramatically with t.

[3] An agent of measure zero is infinitely small relatively to the size of the economy.

4.1.1 Pareto-efficient allocations

In order to identify the Pareto-efficient allocations in this economy, we set up a fictitious social planning problem. Assume that a benevolent social planner assigns to the households a set of nonnegative Pareto weights θ_i in order to define a social welfare function of the form:[4]

$$W \equiv \sum_i \theta_i U(c_i) \tag{4.2}$$

The planner then chooses the allocations that maximize social welfare under a feasibility constraint:

$$\begin{aligned} \max_{\{c_i\}_{i=1}^I} W &= \sum_i \theta_i U(c_i) \\ \text{s.t.} \quad & \sum_i c_{it}\left(s^t\right) = \sum_i y_{it}\left(s^t\right), \quad \forall t, s^t \end{aligned} \tag{4.3}$$

An allocation is Pareto-efficient if it solves the previous problem for a set of nonnegative Pareto weights.

The corresponding Lagrangian is:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \sum_i \left\{ \theta_i \beta^t u\left[c_{it}\left(s^t\right)\right] \pi_t\left(s^t\right) + \lambda_t\left(s^t\right)\left[y_{it}\left(s^t\right) - c_{it}\left(s^t\right)\right] \right\} \tag{4.4}$$

To obtain the first order conditions we derive L with respect to $c_{it}\left(s^t\right)$, the consumption level for each household, in each period, and for each state of the world:

$$\theta_i \beta^t u'\left[c_{it}\left(s^t\right)\right] \pi_t\left(s^t\right) = \lambda_t\left(s^t\right), \quad \forall i, t, s^t \tag{4.5}$$

Divide the first order condition (4.5) for a generic household i by the first order condition for household 1 to get:

$$u'\left[c_{it}\left(s^t\right)\right] = \frac{\theta_1}{\theta_i} u'\left[c_{1t}\left(s^t\right)\right] \tag{4.6}$$

and then solve for $c_{it}\left(s^t\right)$:

$$c_{it}\left(s^t\right) = \left(u'\right)^{-1} \left\{ \frac{\theta_1}{\theta_i} u'\left[c_{1t}\left(s^t\right)\right] \right\} \tag{4.7}$$

Substituting (4.7) into the feasibility constraint gets:

$$\sum_i \left(u'\right)^{-1} \left\{ \frac{\theta_1}{\theta_i} u'\left[c_{1t}\left(s^t\right)\right] \right\} = \sum_i y_{it}\left(s^t\right) \tag{4.8}$$

Remark 46 *Equation (4.8) has a very important implication. Note that the right-hand side corresponds to the realized aggregate endowment:* $c_{1t}\left(s^t\right)$ *is implicitly a function of the current realization of aggregate endowment only. The individual realized endowments* $y_{it}\left(s^t\right)$ *and the actual state of the world* s^t *that generated the aggregate outcome do not play an independent role.*

---

[4] The weights can be arbitrarily normalized; one convenient possibility is to impose $\sum_{i=1}^I \theta_i = 1$.

4.1.2 Arrow-Debreu complete markets

Following Arrow and Debreu, we assume that a complete set of competitive markets for so called *Arrow-Debreu securities* exist at date zero. In other words, households are allowed to trade a complete set of date- *and* state-contingent claims to consumption (claims to consumption can express a "title to receive" or a "duty to deliver"). All trades take place at date zero, after s_0 has been realized, but before the uncertainty as far as the future states of the world is resolved; from date one onwards, the individuals simply execute the contracts signed at date 0. A *price system* for this economy is a sequence of functions $\{q_t(s^t)\}_{t=0}^{\infty}$, where $q_t(s^t)$ represents the date-0 price of claims to consumption good units to be delivered in date t, contingent on history s^t, and expressed in terms of a numeraire that is left for the moment unspecified.

Since all trades occur at date 0, the households face the following single intertemporal budget constraint (that holds with probability one):

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right) c_{it}\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right) y_{it}\left(s^t\right) \tag{4.9}$$

Each household solves the following problem:

$$\begin{aligned} \max_{c_i} U_i &= E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_{it}\right)\right] && (4.10)\\ \text{s.t.} \quad & \sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right) c_{it}\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right) y_{it}\left(s^t\right) \end{aligned}$$

The corresponding Lagrangian becomes:

$$L_i = E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_{it}\right)\right] + \mu_i \sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right)\left[y_{it}\left(s^t\right) - c_{it}\left(s^t\right)\right] \tag{4.11}$$

The first order conditions are the following:

$$\beta^t u'\left[c_{it}\left(s^t\right)\right]\pi_t\left(s^t\right) = \mu_i q_t\left(s^t\right), \quad \forall t, s^t \tag{4.12}$$

Being the units of the price system arbitrary, we can normalize one of the prices to any positive value.[5]

Note that, for all pairs of households (i, j):

$$u'\left[c_{it}\left(s^t\right)\right] = \frac{\mu_i}{\mu_j} u'\left[c_{jt}\left(s^t\right)\right] \tag{4.13}$$

Remark 47 *The ratios of marginal utilities between pairs of households are constant across all dates and all states of the world.*

Example 48 *This equilibrium condition has some important implications. Assume that*

[5] A convenient choice is $q_0(s_0) = 1$: this takes the date-0 consumption good as the numeraire. In this case, equation (4.12) implies that $\mu_i = u'\left[c_{i0}(s_0)\right]$.

the Bernoulli utility function is of the Constant Relative Risk Aversion (CRRA) form:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{4.14}$$

for $\gamma > 0$*. Condition (4.13) implies that:*

$$\frac{c_{it}(s^t)}{c_{jt}(s^t)} = \left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}} \tag{4.15}$$

Remark 49 *If the utility function is of the CRRA form, in all periods and all states of the world the individual consumption levels are a constant fraction of one another. Hence, each household is assigned a constant fraction of the aggregate endowment, and consumption is perfectly smoothed across time and states of the world.*

Definition 50 *A* ***complete markets competitive equilibrium*** *is a feasible allocation and a price system such that, for the given price system, the allocation solves each household's problem.*

Hence, a competitive equilibrium is characterized by the following equilibrium conditions:

$$u'\left[c_{it}(s^t)\right] = \frac{\mu_i}{\mu_j} u'\left[c_{jt}(s^t)\right], \quad \forall i, j \tag{4.16}$$

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t)\left[c_{it}(s^t) - y_{it}(s^t)\right] = 0, \quad \forall i \tag{4.17}$$

$$\sum_i \left[c_{it}(s^t) - y_{it}(s^t)\right] = 0, \quad \forall t, s^t \tag{4.18}$$

Condition (4.16) implies that:

$$c_{it}(s^t) = (u')^{-1}\left\{\frac{\mu_i}{\mu_1} u'\left[c_{1t}(s^t)\right]\right\} \tag{4.19}$$

Substituting this result in (4.18) we get:

$$\sum_i (u')^{-1}\left\{\frac{\mu_i}{\mu_1} u'\left[c_{1t}(s^t)\right]\right\} = \sum_i y_{it}(s^t) \tag{4.20}$$

Equation (4.20) implies, together with (4.19), that $c_{it}(s^t)$ depends on the aggregate realized endowment only: again, neither the individual endowments nor the actual state of the world play an independent role.

Remark 51 *By comparing (4.8) and (4.20) we immediately realize that the equilibrium allocation is also Pareto-efficient, since it solves the Pareto problem for the following set of nonnegative weights:* $\theta_i = 1/\mu_i$ *for* $i = 1, 2, ..., I$*.*

Furthermore, comparing (4.5) and (4.12) suggests that, for the previously specified choice of Pareto weights, the Lagrange multipliers (shadow prices) $\lambda_t(s^t)$ for the Pareto problem are equal to the competitive prices $q_t(s^t)$.

Example 52 *Assume that s_t takes values over the unit interval $[0,1]$, and that there are just two types of households characterized by $y_{1t}(s^t) = s_t$ and $y_{2t}(s^t) = 1 - s_t$. The aggregate endowment is constant and equal to unity, $y_t = \sum_{i=1}^{2} y_{it}(s^t) = 1$. Hence there is no aggregate uncertainty. Condition (4.20) implies that in equilibrium $c_{it} = \bar{c}_i$ for all i, t, and s^t. From equation (4.12):*

$$q_t\left(s^t\right) = \beta^t \pi_t\left(s^t\right) \frac{u'(\bar{c}_i)}{\mu_i}, \quad \forall i, t, s^t \tag{4.21}$$

The household i's budget constraint implies:

$$\frac{u'(\bar{c}_i)}{\mu_i} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right)\left[\bar{c}_i - y_{it}\left(s^t\right)\right] = 0 \tag{4.22}$$

Solving for $\bar{c}_i$ takes us to:

$$\bar{c}_i = (1-\beta) E_0 \left(\sum_{t=0}^{\infty} \beta^t y_{it} \right) \tag{4.23}$$

If $\beta^{-1} = 1 + r$ where r is the risk-free rate of interest, then equation (4.23) can be rewritten as:

$$\bar{c}_i = \frac{r}{1+r} E_0 \left(\sum_{t=0}^{\infty} \frac{y_{it}}{(1+r)^t} \right) \tag{4.24}$$

Remark 53 *This is a version of Friedman's permanent income model: a household with zero financial assets consumes the annuity value of its human wealth defined as the expected discounted value of its labor (endowment) income.*

Implicit wealth dynamics

In the Arrow-Debreu complete market model, the household's *implicit financial wealth* in a given period t, contingent to the actually realized history s^t, corresponds to the current value of all household's purchased net claims to current and future consumption (in all possible states of the world) net of its outstanding liabilities. Note that only claims and liabilities contingent on the particular realization of s^t have to be taken into account: the others have to be discarded.

More formally, the household's implicit financial wealth, expressed in terms of date t, history s^t consumption good, is:[6]

$$\mathcal{W}_{it}\left(s^t\right) \equiv \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t\left(s^\tau\right)\left[c_{i\tau}\left(s^\tau\right) - y_{i\tau}\left(s^\tau\right)\right] \tag{4.25}$$

where:

$$q_\tau^t\left(s^\tau\right) \equiv \frac{q_\tau\left(s^\tau\right)}{q_t\left(s^t\right)} \tag{4.26}$$

By setting $c_{i\tau}(s^\tau) = 0$ for all $\tau \geq t$ and all s^τ we easily obtain the *natural borrowing*

[6] Note that, for the feasibility constraint, $\sum_{i=1}^{I} \mathcal{W}_{it}\left(s^t\right) = 0 \ \forall t, s^t$.

limit at date t and history s^t:

$$\mathcal{D}_{it}\left(s^t\right) \equiv \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t\left(s^\tau\right) y_{i\tau}\left(s^\tau\right), \quad \forall t, s^t \tag{4.27}$$

In other words, household i at date $t-1$ and history s^{t-1} cannot promise to pay more than the maximal value it can repay from date t onwards by setting consumption to zero, contingent on the realization of s_t.

The Negishi algorithm

1. Set one of the Lagrange multipliers, say μ_1, to an arbitrary positive value. Guess some positive values for the remaining multipliers μ_i, for $i = 2, 3, ..., I$.

2. Solve (4.20) for $c_{1t}\left(s^t\right)$ and use (4.19) to recover the remaining $c_{it}\left(s^t\right)$. This generates a candidate consumption allocation $\{\{c_{it}\left(s^t\right)\}_{t=0}^{\infty}\}_{i=1}^{I}$.

3. Solve (4.12) for the price system $\{q_t\left(s^t\right)\}_{t=0}^{\infty}$.

4. For all households, compute $\sum_{t=0}^{\infty} \sum_{s^t} q_t\left(s^t\right) c_{it}\left(s^t\right)$ and $\sum_{t=0}^{\infty} \sum_{s^t} q_t\left(s^t\right) y_{it}\left(s^t\right)$. For the households for which the present value of consumption exceeds the present value of their endowment, increase the corresponding μ_i, while for those households for which the reverse holds, decrease μ_i.

5. Iterate until convergence on steps $2-4$.

4.1.3 Arrow sequential markets

Arrow (1953) shows that the equilibrium allocation obtained under complete markets is equivalent to the one obtained under a recursive *sequential market structure* when individuals are endowed with *rational,* or *self-fulfilled, expectations.* In other words, a competitive economy where financial trades occur only in a complete set of one-period-ahead state-contingent claims to consumption reaches the same allocation as the complete market economy if: (i) one-period markets reopen each period; (ii) individuals correctly forecast future state-contingent market prices.

At each date $t \geq 0$, households are allowed to trade claims to $t+1$ consumption that are contingent on the realization of s_{t+1}. Denote $a_{it}\left(s^t\right)$ the total amount of net claims to date t consumption that household i inherits from the previous period, contingent on the realization of history s^t, and $Q_t\left(s_{t+1}|s^t\right)$ the price of one unit of consumption at date $t+1$, contingent on state s_{t+1}, if the current history is s^t, expressed in units of date t, history s^t consumption; the function Q_t is called *pricing kernel.*

The household faces a sequence of intratemporal budget constraints of the form:

$$c_{it}\left(s^t\right) + \sum_{s_{t+1}} Q_t\left(s_{t+1}|s^t\right) a_{i,t+1}\left(s_{t+1}|s^t\right) = a_{it}\left(s^t\right) + y_{it}\left(s^t\right) \tag{4.28}$$

where $a_{i,t+1}\left(s_{t+1}|s^t\right)$ represents the total amount of net claims to date $t+1$ consumption, contingent on the realization of s_{t+1}, if the current state of the world is s^t.

To avoid the possibility of Ponzi schemes, let us impose the following NPG condition:

$$\lim_{j\to\infty} q_{t+j}^{t}\left(s^{t+j}\right) a_{i,t+j}\left(s^{t+j}\right) \geq 0, \quad \forall s^{t+j} \tag{4.29}$$

where $q_{t+j}^{t}\left(s^{t+j}\right) \equiv q_{t+j}\left(s^{t+j}\right)/q_{t}\left(s^{t}\right)$ is the Arrow-Debreu date-t price of claims to consumption good units to be delivered in date $t+j$, contingent on history s^{t+j}.

Each household solves the following optimal control problem:

$$\begin{aligned} \max_{\left\{c_{it}(s^t),\{a_{i,t+1}(s_{t+1}|s^t)\}_{s_{t+1}}\right\}} \quad & U_i = E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_{it}\right)\right] & (4.30)\\ \text{s.t.} \quad & c_{it}\left(s^t\right) + \sum_{s_{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right) a_{i,t+1}\left(s_{t+1}|s^t\right) = a_{it}\left(s^t\right) + y_{it}\left(s^t\right) \\ & \lim_{j\to\infty} q_{t+j}^{t}\left(s^{t+j}\right) a_{i,t+j}\left(s^{t+j}\right) \geq 0, \quad \forall s^{t+j} \\ & a_{i0}\left(s_0\right) \quad \text{given} \end{aligned}$$

The corresponding Lagrangian is:

$$\begin{aligned} L_i = & \sum_{t=0}^{\infty}\sum_{s^t}\left\{\beta^t u\left[c_{it}\left(s^t\right)\right]\pi_t\left(s^t\right) + \right. & (4.31)\\ & \left.\eta_{it}\left(s^t\right)\left[a_{it}\left(s^t\right) + y_{it}\left(s^t\right) - c_{it}\left(s^t\right) - \sum_{s_{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right) a_{i,t+1}\left(s_{t+1}|s^t\right)\right]\right\} \end{aligned}$$

To obtain the first order conditions, derive the Lagrangian with respect to $c_{it}\left(s^t\right)$ and $\left\{a_{i,t+1}\left(s_{t+1}|s^t\right)\right\}_{s_{t+1}}$:

$$\beta^t u'\left[c_{it}\left(s^t\right)\right]\pi_t\left(s^t\right) = \eta_{it}\left(s^t\right) \tag{4.32}$$

$$\eta_{it}\left(s^t\right) Q_t\left(s_{t+1}|s^t\right) = \eta_{it+1}\left(s^{t+1}\right) \tag{4.33}$$

Combining the previous equations takes to:

$$Q_t\left(s_{t+1}|s^t\right) = \beta\frac{u'\left[c_{it+1}\left(s^{t+1}\right)\right]}{u'\left[c_{it}\left(s^t\right)\right]}\pi_{t+1}\left(s^{t+1}|s^t\right) \tag{4.34}$$

since $\pi_{t+1}\left(s^{t+1}|s^t\right) = \pi_{t+1}\left(s^{t+1}\right)/\pi_t\left(s^t\right)$.

The sequence of pricing kernels $\left\{Q_\tau\left(s_{\tau+1} \mid s^\tau\right)\right\}_{\tau=t+1}^{\infty}$ taken as given by our individuals in problem (4.30) should be more precisely defined as the sequence of future *expected*, or *perceived*, pricing kernels: recall that the markets for future one-period-ahead consumption claims are closed. Nothing in the model guarantees *a priori* that the sequence of perceived state-contingent prices will eventually correspond to the actual sequence of equilibrium prices. As already mentioned, we impose this outcome by assuming rational expectations, *i.e.* that the expected state-contingent prices do actually clear the markets once the corresponding date has arrived and the state of the world is revealed.

Definition 54 *A* ***sequential trading competitive equilibrium*** *is an initial distribution of wealth* $\left\{a_{i0}\left(s_0\right)\right\}_{i=1}^{I}$*, an allocation* $\left\{c_i\right\}_{i=1}^{I}$*, and a set of pricing kernels* $\left\{Q_t\left(s_{t+1}|s^t\right)\right\}_{t=0}^{\infty}$ *such that:*

1. *For all i, the consumption allocation c_i solves household i's problem, given $a_{i0}(s_0)$ and the pricing kernels;*

2. *The households' consumption allocation and the implied asset allocation satisfy the following feasibility constraints:*

$$\sum_i \left[c_{it}\left(s^t\right) - y_{it}\left(s^t\right)\right] = 0 \tag{4.35}$$

$$\sum_i a_{it+1}\left(s_{t+1}|s^t\right) = 0 \quad \forall s_{t+1} \tag{4.36}$$

for all realizations of $\{s^t\}_{t=0}^{\infty}$.

Equivalence of allocations

Remark 55 *We shall now show that the equilibrium allocation of the complete markets model is also an equilibrium allocation for the sequential trading model if:*

1. $\{a_{i0}(s_0)\}_{i=1}^{I} = \mathbf{0}$, *i.e. each household must exclusively rely on its own endowment flow to finance consumption;*

2. $Q_t(s_{t+1}|s^t) = \frac{q_{t+1}(s^{t+1})}{q_t(s^t)}$, *i.e. the pricing kernel corresponds to the Arrow-Debreu price at date $t+1$ contingent on s^{t+1} expressed in terms of date t, history s^t consumption good.*

The first step is quite straightforward. Take the first order conditions for the Arrow-Debreu model, equation (4.12), and divide the condition in period $t+1$ by the condition in period t:

$$\frac{\beta u'\left[c_{it+1}\left(s^{t+1}\right)\right]\pi_{t+1}\left(s^{t+1}\right)}{u'\left[c_{it}\left(s^t\right)\right]\pi_t\left(s^t\right)} = \frac{q_{t+1}\left(s^{t+1}\right)}{q_t\left(s^t\right)} = Q_t\left(s_{t+1}|s^t\right) \tag{4.37}$$

Remark 56 *Equation (4.37) is actually identical to the equilibrium condition (4.34): hence, if the pricing kernel is properly defined, the equilibrium allocations in the complete markets and sequential trade models follow the same time path.*

However, it remains to be shown that the two equilibrium allocations share not only the time path but also the initial consumption level. In the sequential trading framework, the variable $a_{it}(s^t)$ represents household i's financial wealth in date t contingent on the realization of s^t. In the complete market framework, a comparable concept of financial wealth was defined as $\mathcal{W}_{it}(s^t)$ in (4.25).[7]

Consider the intratemporal budget constraint (4.28) evaluated at date 0 under the assumption that $a_{i0}(s_0) = 0$:

$$c_{i0}(s_0) + \sum_{s_1} Q_0(s_1|s_0)\, a_{i1}(s_1|s_0) = y_{i0}(s_0) \tag{4.38}$$

[7] Note that the NPG condition (4.29) imposes the *natural debt limit* at date t and history s^t: $-a_{it}(s^t) \leq \mathcal{D}_{it}(s^t)\ \forall s^t$.

Being $a_{i1}(s_1|s_0) = \mathcal{W}_{i1}(s^1)$ by construction, we have that:

$$\begin{aligned}\sum_{s_1} Q_0(s_1|s_0)\, a_{i1}(s_1|s_0) &= \sum_{s^1|s_0} q_1^0(s^1)\, \mathcal{W}_{i1}(s^1) = \\ &\sum_{\tau=1}^{\infty} \sum_{s^\tau|s_0} q_\tau^0(s^\tau)\left[c_{i\tau}(s^\tau) - y_{i\tau}(s^\tau)\right]\end{aligned} \tag{4.39}$$

since:

$$q_\tau^1(s^\tau)\, q_1^0(s^1) = \frac{q_\tau(s^\tau)}{q_1(s^1)} \frac{q_1(s^1)}{q_0(s_0)} = q_\tau^0(s^\tau) \tag{4.40}$$

Equation (4.38) can therefore be expressed as:

$$q_0(s_0)\, c_{i0}(s_0) + \sum_{t=1}^{\infty} \sum_{s^t} q_t(s^t)\left[c_{it}(s^t) - y_{it}(s^t)\right] = q_0(s_0)\, y_{i0}(s_0) \tag{4.41}$$

Remark 57 *Note that equation (4.41) turns out to perfectly reproduce the intertemporal budget constraint for the complete market model (4.9). This implies that $c_{i0}(s_0)$ will be identical in the two frameworks. Hence, the equilibrium allocations in the sequential trading model and the complete markets model are identical, since they share the same starting point and the subsequent dynamics.*

4.2 Recursive competitive equilibrium

We shall now restrict our approach by imposing a particularly convenient form for the exogenous stochastic forcing process, in order to allow a recursive formulation of the sequential trading model.

Assume that the forcing process s_t follows a discrete-state Markov chain characterized by:

- a state space S;
- a transition density $\pi(s'|s)$ such that $\pi(s'|s) \geq 0$ and $\sum_{s'} \pi(s'|s)\, ds = 1$, $\forall s \in S$;
- an initial density $\pi_0(s)$ such that $\sum_s \pi_0(s)\, ds = 1$.

Hence:

$$\begin{aligned}\pi(s'|s) &= \text{Prob}(s_{t+1} = s'|s_t = s) && (4.42)\\ \pi_0(s) &= \text{Prob}(s_0 = s) && (4.43)\end{aligned}$$

Note that the density over the history $s^t = \{s_t, s_{t-1}, ..., s_0\}$ is:

$$\pi(s^t) = \pi(s_t|s_{t-1})\, \pi(s_{t-1}|s_{t-2}) \cdots \pi(s_1|s_0)\, \pi_0(s_0) \tag{4.44}$$

If trading occurs after s_0 has been observed, then $\pi_0(s_0) = 1$ for the given s_0.

Remark 58 *The Markov property guarantees that:*

$$\pi(s^t|s^{t-1}) = \pi(s_t|s_{t-1}) \tag{4.45}$$

Assume furthermore that the households' endowments are time-invariant measurable functions of s_t:

$$y_{it}\left(s^t\right) = y_i\left(s_t\right) \tag{4.46}$$

4.2.1 Recursivity

Consider (4.20) again:

$$\sum_i \left(u'\right)^{-1}\left\{\frac{\mu_i}{\mu_1}u'\left[c_{1t}\left(s^t\right)\right]\right\} = \sum_i y_i\left(s_t\right) \tag{4.47}$$

Quite evidently, $c_{1t}\left(s^t\right) = c_1\left(s_t\right)$. Equation (4.19) then implies that $c_{it}\left(s^t\right) = c_i\left(s_t\right)$.

Remark 59 *In a recursive framework, individual consumption in a complete market competitive equilibrium is a time-invariant function of the current realization of s_t alone.*

The first order conditions (4.12):

$$\beta^t u'\left[c_i\left(s_t\right)\right]\pi\left(s^t\right) = \mu_i q\left(s_t\right), \quad \forall t, s_t \tag{4.48}$$

imply that:

$$\frac{q\left(s_{t+1}\right)}{q\left(s_t\right)} = \frac{\beta u'\left[c_i\left(s_{t+1}\right)\right]\pi\left(s_{t+1}|s_t\right)}{u'\left[c_i\left(s_t\right)\right]} \tag{4.49}$$

Remark 60 *In a recursive framework, the pricing kernel in the sequential trading equilibrium boils down to a time-invariant function of the current s_t only:*

$$Q_t\left(s_{t+1}|s^t\right) = \frac{q\left(s_{t+1}\right)}{q\left(s_t\right)} = Q\left(s_{t+1}|s_t\right) \tag{4.50}$$

4.2.2 The Bellman equation

Under the previous assumptions on the stochastic properties of the forcing process s_t, the sequential trading model can be easily restated in recursive form, using the dynamic programming approach. For the generic household, the state of the economy is summarized by a vector of two state variables, the exogenous aggregate state s_t and the endogenous, individual state financial wealth a. Hence, the solution to the household's dynamic problem will be summarized by a pair of policy functions, one for consumption, $\hat{c} = c\left(a, s\right)$, and one for next-period wealth, $\hat{a}\left(s'\right) = a'\left(a, s, s'\right)$.

$\hat{c} = c\left(a, s\right), \quad \hat{a}\left(s'\right) = a'\left(a, s, s'\right)$

The Bellman equation for a generic household is the following:

$$\begin{aligned} v\left(a, s\right) &= \max_{\{\hat{c}, \hat{a}(s')\}}\left\{u\left(\hat{c}\right) + \beta\sum_{s'} v\left[\hat{a}\left(s'\right), s'\right]\pi\left(s'|s\right)\right\} \\ &\text{s.t.}\ \ \hat{c} + \sum_{s'|s}\hat{a}\left(s'\right)Q\left(s'|s\right) = a + y\left(s\right) \\ &\hat{a}\left(s'\right) \geq -\mathcal{D}\left(s'\right), \quad \forall s' \end{aligned} \tag{4.51}$$

where:

$$\mathcal{D}(s) \equiv \sum_{t=0}^{\infty} \sum_{s_t|s} \frac{q(s_t)}{q(s)} y(s) \tag{4.52}$$

The Lagrangian for the maximization problem in (4.51) is:

$$L = u(\hat{c}) + \beta \sum_{s'} v[\hat{a}(s'), s'] \pi(s'|s) + \lambda \left[a + y(s) - \hat{c} - \sum_{s'|s} \hat{a}(s') Q(s'|s) \right] \tag{4.53}$$

The first order conditions:

$$u'(\hat{c}) = \lambda \tag{4.54}$$

$$\beta \frac{\partial v[\hat{a}(s'), s']}{\partial a} \pi(s'|s) = \lambda Q(s'|s) \tag{4.55}$$

The Benveniste-Scheinkman formula implies that:

$$\frac{\partial v(a, s)}{\partial a} = u'(\hat{c}) \tag{4.56}$$

Hence:

$$Q(s'|s) = \frac{\beta u'(\hat{c}') \pi(s'|s)}{u'(\hat{c})} \tag{4.57}$$

Definition 61 *A* ***recursive competitive equilibrium*** *is an initial distribution of wealth* $\{a_{i0}\}_{i=1}^{I}$*, a set of policy functions* $\{\hat{c}_i, \hat{a}_i\}_{i=1}^{I}$*, a set of value functions* $\{v_i\}_{i=1}^{I}$*, and a pricing kernel* $Q(s'|s)$ *such that:*

1. *For all* i*, the policy functions solve household* i*'s problem, given* a_{i0} *and the pricing kernel;*

2. *The households' consumption and asset allocations implied by the policy functions satisfy the following feasibility constraints:*

$$\sum_i [\hat{c}_i(s) - y_i(s)] = 0, \quad \forall s \tag{4.58}$$

$$\sum_i \hat{a}_i(s') = 0 \quad \forall s' \tag{4.59}$$

4.3 The stochastic growth model

The economy is inhabited by a *continuum* of infinitely-living identical agents that can be aggregated into a single representative agent, whose preferences on infinite history-dependent consumption streams $c = \{c_t(s^t)\}_{t=0}^{\infty}$ may be represented by the intertemporal utility function defined in (4.1):

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t\left(s^t\right)\right] \pi_t\left(s^t\right) = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{4.60}$$

The final consumption good is produced using a CRS technology summarized by the following production function:

$$y_t\left(s^t\right) = A_t\left(s^t\right) F\left[k_t\left(s^{t-1}\right), n_t\left(s^t\right) L\right] \tag{4.61}$$

where $y_t\left(s^t\right)$ is the per-capita output level at date t and history s^t, $k_t\left(s^{t-1}\right)$ the per-capita contingent physical capital stock, $n_t\left(s^t\right) \in [0,1]$ the contingent time share devoted to labor, $A_t\left(s^t\right)$ the so called *Total Factor Productivity* (TFP), and L the fixed time endowment (from now on, for the sake of simplicity, $L = 1$). The production function enjoys all the properties listed in Section 2, p. 32. Being leisure not valued in the utility function, can safely assume that $n_t\left(s^t\right) = 1\ \forall t, s^t$. This allows us to define the *intensive production function* as $f(k) \equiv F(k,1)$; note that $f(0) = 0$, $f' > 0$, $f'' < 0$, $\lim_{k\to 0} f(k) = \infty$, and $\lim_{k\to\infty} f(k) = 0$.

Physical capital is the only durable good in the economy. An aggregate intratemporal resource constraint holds in each period:

$$k_{t+1}\left(s^t\right) = (1-\delta) k_t\left(s^{t-1}\right) + y_t\left(s^t\right) - c_t\left(s^t\right) \tag{4.62}$$

where $\delta \in [0,1]$ is the depreciation rate. We assume the existence of a strictly positive initial capital stock k_0. The consumption good can be freely transformed into physical capital, and vice-versa, physical capital can be freely transformed into the consumption good (hence, investments may be negative).

4.3.1 The planning problem

In order to identify the Pareto-efficient allocation, se set up the usual fictitious social planner problem:

$$\max_{\{c_t(s^t),k_{t+1}(s^t)\}_{t=0}^{\infty}} \quad U(c) = E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right] \tag{4.63}$$
$$\text{s.t. } k_{t+1}\left(s^t\right) = (1-\delta) k_t\left(s^{t-1}\right) + A_t\left(s^t\right) f\left[k_t\left(s^{t-1}\right)\right] - c_t\left(s^t\right)$$
$$k_0 > 0 \text{ given}$$

The usual Lagrangian is:

$$\begin{aligned} L = {} & \sum_{t=0}^{\infty}\sum_{s^t}\left[\beta^t u\left[c_t\left(s^t\right)\right]\pi_t\left(s^t\right) + \right. \\ & \left.\lambda_t\left(s^t\right)\left\{(1-\delta) k_t\left(s^{t-1}\right) + A_t\left(s^t\right) f\left[k_t\left(s^{t-1}\right)\right] - c_{it}\left(s^t\right) - k_{t+1}\left(s^t\right)\right\}\right] \end{aligned} \tag{4.64}$$

To obtain the first order conditions derive the Lagrangian with respect to $c_t\left(s^t\right)$ and $k_{t+1}\left(s^t\right)$:

$$u'\left[c_t\left(s^t\right)\right]\pi_t\left(s^t\right) = \lambda_t\left(s^t\right) \tag{4.65}$$

$$u'\left[c_t\left(s^t\right)\right]\pi_t\left(s^t\right) = \tag{4.66}$$
$$\beta\sum_{s^{t+1}} u'\left[c_{t+1}\left(s^{t+1}\right)\right]\pi_{t+1}\left(s^{t+1}\right)\left\{A_{t+1}\left(s^{t+1}\right) f\left[k_{t+1}\left(s^t\right)\right] + 1 - \delta\right\}$$

Note that (4.66) can be rewritten as:

$$u'\left[c_t\left(s^t\right)\right] = \beta \sum_{s^{t+1}|s^t} u'\left[c_{t+1}\left(s^{t+1}\right)\right]\pi_{t+1}\left(s^{t+1}|s^t\right)\left\{A_{t+1}\left(s^{t+1}\right)f\left[k_{t+1}\left(s^t\right)\right]+1-\delta\right\} \tag{4.67}$$

or as:

$$u'\left(c_t\right) = \beta E_t\left\{u'\left(c_{t+1}\right)\left[A_{t+1}f\left(k_{t+1}\right)+1-\delta\right]\right\} \tag{4.68}$$

4.3.2 Arrow-Debreu complete markets

Assume that the representative household owns all factors of production, makes consumption/investment decisions under perfect foresight, and rents capital and labor to a representative firm. The latter uses the services of capital and labor to produce the final consumption (investment) good. All markets are competitive, and all trades occur at date 0. For this economy, a price system is a sequence of history-dependent prices $\left\{q_t\left(s^t\right), r_t\left(s^t\right), w_t\left(s^t\right)\right\}_{t=0}^{\infty}$ where (i) $q_t\left(s^t\right)$ is the date-0 price of one unit of consumption at date t and history s^t; (ii) r_t is the date-0 price of one unit of capital rented at date t and history s^t; (iii) w_t is the date-0 price of one unit of labor rented at date t and history s^t.

Households

The representative household faces the following intratemporal budget constraint:

$$q_t\left(s^t\right)\left[c_t\left(s^t\right)+k_{t+1}\left(s^t\right)-\left(1-\delta\right)k_t\left(s^{t-1}\right)\right] = r_t\left(s^t\right)k_t\left(s^{t-1}\right)+w_t\left(s^t\right) \tag{4.69}$$

Iterating on (4.69), we get the present-value intertemporal budget constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right)c_t\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t}\left\{w_t\left(s^t\right)+\left[q_t\left(s^t\right)\left(1-\delta\right)+r_t\left(s^t\right)-q_{t-1}\left(s^{t-1}\right)\right]k_t\left(s^{t-1}\right)\right\}+ - \lim_{t\to\infty}\sum_{s^t} q_t\left(s^t\right)k_{t+1}\left(s^t\right)$$

where $q_{-1}\left(s^{-1}\right)=0$. Note that, if $u'\left(\cdot\right)>0$, then it will never be optimal to held a strictly positive capital stock in the limit, because this would decrease the amount of resources available for consumption.

Remark 62 *Optimality requires the following terminal condition:*

$$\lim_{t\to\infty}\sum_{s^t} q_t\left(s^t\right)k_{t+1}\left(s^t\right) = 0 \tag{4.70}$$

This implies that the household's problem can be reformulated as:

$$\max_{\{c_t(s^t),k_{t+1}(s^t)\}_{t=0}^{\infty}} \quad U = E_t\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] \tag{4.71}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty}\sum_{s^t} q_t\left(s^t\right)\left[c_t\left(s^t\right)+k_{t+1}\left(s^t\right)-(1-\delta)\,k_t\left(s^{t-1}\right)\right] = \sum_{t=0}^{\infty}\sum_{s^t}\left[r_t\left(s^t\right)k_t\left(s^{t-1}\right)+w_t\left(s^t\right)\right]$$

where the sequence of prices $\{q_t(s^t), r_t(s^t), w_t(s^t)\}_{s^t}$ and the initial condition $k_0 > 0$ are taken as given.

The corresponding present-value Lagrangian can be written as:

$$L = E_t\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] + \tag{4.72}$$

$$\mu\sum_{t=0}^{\infty}\sum_{s^t}\left\{\left[r_t\left(s^t\right)k_t\left(s^{t-1}\right)+w_t\left(s^t\right)\right]+\right. \tag{4.73}$$

$$\left.-q_t\left(s^t\right)\left[c_t\left(s^t\right)+k_{t+1}\left(s^t\right)-(1-\delta)\,k_t\left(s^{t-1}\right)\right]\right\} \tag{4.74}$$

Deriving the Lagrangian with respect to $c_t(s^t)$ and $k_{t+1}(s^t)$ we obtain the first order conditions:

$$\beta^t u'\left[c_t\left(s^t\right)\right]\pi_t\left(s^t\right) = q_t\left(s^t\right)\mu \tag{4.75}$$

$$q_t\left(s^t\right) = \sum_{s^{t+1}|s^t}\left[r_{t+1}\left(s^{t+1}\right)+q_{t+1}\left(s^{t+1}\right)(1-\delta)\right] \tag{4.76}$$

Firms

The representative firm maximizes the present value of economic profits, given by:

$$\Pi = \sum_{t=0}^{\infty}\sum_{s^t}\left\{q_t\left(s^t\right)A_t\left(s^t\right)f\left[k_t\left(s^{t-1}\right)\right]-w_t\left(s^t\right)-r_t\left(s^t\right)k_t\left(s^{t-1}\right)\right\} \tag{4.77}$$

The first order conditions:

$$w_t\left(s^t\right) = q_t\left(s^t\right)A_t\left(s^t\right)\left\{f\left[k_t\left(s^{t-1}\right)\right]-f_k\left[k_t\left(s^{t-1}\right)\right]k_t\left(s^{t-1}\right)\right\} \tag{4.78}$$

$$r_t\left(s^t\right) = q_t\left(s^t\right)A_t\left(s^t\right)f_k\left[k_t\left(s^{t-1}\right)\right] \tag{4.79}$$

Equilibrium

In equilibrium, we have:

$$u'\left(c_t\right) = \beta E_t\left\{u'\left(c_{t+1}\right)\left[A_{t+1}f_k\left(k_{t+1}\right)+(1-\delta)\right]\right\} \tag{4.80}$$

Implicit wealth dynamics

In this setting the household's implicit financial (non-human) wealth in a given period t, contingent to the actually realized history s^t, corresponds to the current net value of all household's purchased claims to current and future consumption (in all possible states of the world) net of its labor income. Hence, the value off all household's current and future consumption claims, net of labor income and expressed in terms of date t , history s^t consumption good is:

$$\mathcal{W}_t\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \left[q_\tau^t\left(s^\tau\right) c_\tau\left(s^\tau\right) - w_\tau^t\left(s^\tau\right)\right] \tag{4.81}$$

where:

$$w_t^t\left(s^t\right) \equiv \frac{w_t\left(s^t\right)}{q_t\left(s^t\right)} \tag{4.82}$$

Note that, after a slightly cumbersome and tedious derivation, (4.81) boils down to:

$$\mathcal{W}_t\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \left[q_\tau^t\left(s^\tau\right) c_\tau\left(s^\tau\right) - w_\tau^t\left(s^\tau\right)\right] = \tag{4.83}$$

$$\sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \left[q_\tau^t\left(s^\tau\right)\left\{\underbrace{A_\tau\left(s^\tau\right) f\left[k_\tau\left(s^{\tau-1}\right)\right]}_{r_\tau(s^\tau)k_\tau(s^{\tau-1})+w_\tau(s^\tau)} + (1-\delta) k_\tau\left(s^{\tau-1}\right) - k_{\tau+1}\left(s^\tau\right)\right\} - w_\tau^t\left(s^\tau\right)\right] = \tag{4.84}$$

$$\sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \left\{q_\tau^t\left(s^\tau\right)\left[(1-\delta) k_\tau\left(s^{\tau-1}\right) - k_{\tau+1}\left(s^\tau\right)\right] + r_\tau^t\left(s^\tau\right) k_\tau\left(s^{\tau-1}\right)\right\} = \tag{4.85}$$

$$\left[r_t^t\left(s^t\right) + (1-\delta)\right] k_t\left(s^{t-1}\right) + \tag{4.86}$$

$$\sum_{\tau=t+1}^{\infty} \sum_{s^{\tau-1}|s^t} \left\{\underbrace{\sum_{s^\tau|s^{\tau-1}} \left[r_\tau^t\left(s^\tau\right) + q_\tau^t\left(s^\tau\right)(1-\delta)\right] - q_{\tau-1}^t\left(s^{\tau-1}\right)}_{=0}\right\} k_\tau\left(s^{\tau-1}\right) = \tag{4.87}$$

$$\left[r_t^t\left(s^t\right) + 1 - \delta\right] k_t\left(s^{t-1}\right) \tag{4.88}$$

where:

$$r_t^t\left(s^t\right) \equiv \frac{r_t\left(s^t\right)}{q_t\left(s^t\right)} \tag{4.89}$$

4.3.3 Arrow sequential markets

At each date $t \geq 0$, the representative households is allowed to trade claims to date $t+1$ consumption that are contingent on the realization of s_{t+1}. Denote $a_t\left(s^t\right)$ the total amount of net claims to date t consumption that the household inherits from the previous period, contingent on the realization of history s^t, and $Q_t\left(s_{t+1}|s^t\right)$ the price of one unit of consumption at date $t+1$, contingent on state s_{t+1}, if the current history is s^t. The household faces a sequence of intratemporal budget constraints of the form:

$$c_t\left(s^t\right) + \sum_{s_{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right) a_{t+1}\left(s_{t+1}|s^t\right) + k_{t+1}\left(s^t\right) = a_t\left(s^t\right) + \left[r_t\left(s^t\right) + 1 - \delta\right] k_t\left(s^{t-1}\right) + w_t\left(s^t\right)$$

where $a_{t+1}\left(s_{t+1}|s^t\right)$ represents the total amount of net claims to date $t+1$ consumption, contingent on the realization of s_{t+1}, if the current state of the world is s^t.

To avoid the possibility of Ponzi schemes, let us impose the following NPG condition:

$$\lim_{j\to\infty} q_{t+j}^t\left(s^{t+j}\right) a_{t+j}\left(s^{t+j}\right) \geq 0, \quad \forall s^{t+j} \tag{4.90}$$

where $q_{t+j}^t\left(s^{t+j}\right) \equiv q_{t+j}\left(s^{t+j}\right)/q_t\left(s^t\right)$ is the Arrow-Debreu date-t price of claims to consumption good units to be delivered in date $t+j$, contingent on history s^{t+j}.

The household solves the following optimal control problem:

$$\begin{aligned}
\max_{\left\{c_t(s^t), k_{t+1}(s^t), \{a_{t+1}(s_{t+1}|s^t)\}_{s_{t+1}}\right\}} U &= E_0\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right] && (4.91)\\
\text{s.t.} \quad & c_t\left(s^t\right) + \sum_{s_{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right) a_{t+1}\left(s_{t+1}|s^t\right) + k_{t+1}\left(s^t\right) = \\
& a_t\left(s^t\right) + \left[r_t\left(s^t\right) + 1 - \delta\right] k_t\left(s^{t-1}\right) + w_t\left(s^t\right) \\
& \lim_{j\to\infty} q_{t+j}^t\left(s^{t+j}\right) a_{t+j}\left(s^{t+j}\right) \geq 0, \quad \forall s^{t+j} \\
& a\left(s_0\right) \quad \text{given}
\end{aligned}$$

The corresponding Lagrangian is:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left[\beta^t u\left[c_t\left(s^t\right)\right] \pi_t\left(s^t\right) + \eta_t\left(s^t\right)\left\{a_t\left(s^t\right) + \left[r_t\left(s^t\right) + 1 - \delta\right] k_t\left(s^{t-1}\right) + w_t\left(s^t\right) + -c_t\left(s^t\right) - \sum_{s_{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right) a_{t+1}\left(s_{t+1}|s^t\right) - k_{t+1}\left(s^t\right)\right\}\right]$$

To obtain the first order conditions, derive the Lagrangian with respect to $c_t\left(s^t\right)$, $k_{t+1}\left(s^t\right)$, and $\left\{a_{t+1}\left(s_{t+1}|s^t\right)\right\}_{s_{t+1}}$:

$$\beta^t u'\left[c_t\left(s^t\right)\right] \pi_t\left(s^t\right) = \eta_t\left(s^t\right) \tag{4.92}$$

$$\sum_{s^{t+1}|s^t} \eta_{t+1}\left(s^{t+1}\right)\left[\left(r_{t+1}\left(s^{t+1}\right) + 1 - \delta\right)\right] = \eta_t\left(s^t\right) \tag{4.93}$$

$$\eta_t\left(s^t\right) Q_t\left(s_{t+1}|s^t\right) = \eta_{t+1}\left(s^{t+1}\right) \tag{4.94}$$

Combining the previous equations takes to:

$$1 = \sum_{s^{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right)\left[\left(r_{t+1}\left(s^{t+1}\right)+1-\delta\right)\right] \quad (4.95)$$

$$Q_t\left(s_{t+1}|s^t\right) = \beta\frac{u'\left[c_{t+1}\left(s^{t+1}\right)\right]}{u'\left[c_t\left(s^t\right)\right]}\pi_{t+1}\left(s^{t+1}|s^t\right) \quad (4.96)$$

Note that, if $Q_t\left(s_{t+1}|s^t\right) = q^t_{t+1}\left(s^{t+1}\right)$, then:

$$\sum_{s_{t+1}} Q_t\left(s_{t+1}|s^t\right)a_{t+1}\left(s_{t+1}|s^t\right)+k_{t+1}\left(s^t\right) = \sum_{s^{t+1}|s^t} q^t_{t+1}\left(s^{t+1}\right)\mathcal{W}_{t+1}\left(s^{t+1}\right) =$$
$$\sum_{s^{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right)\underbrace{\left[\left(r_{t+1}\left(s^{t+1}\right)+1-\delta\right)\right]k_{t+1}\left(s^t\right)}_{\mathcal{W}_{t+1}\left(s^{t+1}\right)}$$

Is to say:

$$\sum_{s_{t+1}} Q_t\left(s_{t+1}|s^t\right)a_{t+1}\left(s_{t+1}|s^t\right)+k_{t+1}\left(s^t\right) =$$
$$k_{t+1}\left(s^t\right)\underbrace{\sum_{s^{t+1}|s^t} Q_t\left(s_{t+1}|s^t\right)\left[\left(r_{t+1}\left(s^{t+1}\right)+1-\delta\right)\right]}_{=1} \quad (4.97)$$

or:

$$\sum_{s_{t+1}} Q_t\left(s_{t+1}|s^t\right)a_{t+1}\left(s_{t+1}|s^t\right) = 0 \quad (4.98)$$

4.3.4 Recursive formulation

Assume again that the forcing process s_t follows a discrete-state Markov chain characterized by a state space S, a transition density $\pi\left(s'|s\right)$, and an initial density $\pi_0\left(s\right)$, so that:

$$\pi\left(s^t\right) = \pi\left(s_t|s_{t-1}\right)\pi\left(s_{t-1}|s_{t-2}\right)\cdots\pi\left(s_1|s_0\right)\pi_0\left(s_0\right) \quad (4.99)$$

Assume that the aggregate productivity level $A_t\left(s^t\right)$ is a time-invariant measurable function of its own past level and the current state s_t:

$$A_t\left(s^t\right) = A\left[A_{t-1}\left(s^{t-1}\right), s\right] \quad (4.100)$$

For simplicity, assume that:

$$A_t\left(s^t\right) = s_tA_{t-1}\left(s^{t-1}\right) = \prod_{j=0}^{t} s_jA_{-1} \quad (4.101)$$

for a given initial value of A_{-1}.

Pareto-efficient allocations

The Bellman equation for the planning problem is:

$$\begin{aligned} V(K,A,s) &= \max_C u(C) + \beta \sum_{s'} V(K',A',s')\,\pi(s'|s) \\ \text{s.t.} \quad & K' = (1-\delta)K + Asf(K) - C \\ & A' = As \end{aligned} \tag{4.102}$$

The solution to this problem can be represented as a policy function $C = \mathcal{C}(X)$, where $X \equiv \{K, A, s\}$ is the vector of state variables. The first order condition for the maximization problem on the right-hand side of problem (4.102) is:

$$u_C(C) = \beta \sum_{s'} V_K(X')\,\pi(s'|s) \tag{4.103}$$

The envelope condition:

$$V_K(X) = u_C(C)\left[1 - \delta + Asf_K(K)\right] \tag{4.104}$$

Hence, the policy functions solve the following functional equation:

$$u_C[\mathcal{C}(X)] = \beta \sum_{s'} u_C[\mathcal{C}(X')]\left[1-\delta + A's'f_K(K')\right]\pi(s'|s) = \beta E\left\{u_C[\mathcal{C}(X')]\left[1-\delta + A's'f_K(K')\right]|s\right\}$$

where:

$$K' = (1-\delta)K + Asf(K) - \mathcal{C}(X) \tag{4.105}$$

$$A' = As \tag{4.106}$$

Sequential trading

The vector $X = \{K, A, s\}$ is a complete description of the aggregate state of the economy, in terms of which the one-period contingent payoffs are defined. In a decentralized economy with sequential trading the spot prices will depend on X. Hence, we are allowed to define a set of price functions, $r(X)$, the rental rate of capital, and $w(X)$, the wage rate, both measured in units of consumption good to be delivered in the current period. Note that while the aggregate value of K, the endogenous element of X, is determined by the households' decisions on consumption and investment, each single household, being a price taker, considers the aggregate capital stock K as beyond its control. Households know that only the individual capital stock, k, is under their direct control.

As already described in (4.102), the aggregate capital stock and the aggregate productivity level evolve according to the following laws of motion:

$$K' = (1-\delta)K + Asf(K) - C \tag{4.107}$$

$$A' = As \tag{4.108}$$

The representative household forms believes regarding the price functions, the aggregate laws of motion for K and A, and the transition probability $\pi(s'|s)$, formally denoted

$\hat{r}(X)$, $\hat{w}(X)$, $K' = \mathcal{K}(X)$, $A' = \mathcal{A}(X)$, and $\hat{\pi}(s'|s)$. Given these believes, the households solve the following dynamic programming problem:

$$\begin{aligned} v(k,X) &= \max_{\{c,k'\}} u(c) + \beta \sum_{s'} v(k',X')\hat{\pi}(s'|s) \\ \text{s.t.} \quad & k' = [1-\delta+\hat{r}(X)]k + \hat{w}(X) - c \\ & K' = \mathcal{K}(X) \\ & A' = \mathcal{A}(X) \end{aligned} \tag{4.109}$$

The first order condition:

$$u_c(c) = \beta \sum_{s'} v_k(k',X')\hat{\pi}(s'|s) \tag{4.110}$$

The envelope condition:

$$v_k(k,X) = u_c(c)[1-\delta+\hat{r}(X)] \tag{4.111}$$

The Euler equation:

$$u_c(c) = \beta \sum_{s'} u_c(c')[1-\delta+\hat{r}(X')]\hat{\pi}(s'|s) = \beta E\{u_c(c')[1-\delta+\hat{r}(X')]|s\} \tag{4.112}$$

The representative firm forms believes on the price functions, $\check{r}(X)$ and $\check{w}(X)$, and solves the following static profit maximization problem:

$$\max_k Asf(k) - \check{r}(X)k - \check{w}(X) \tag{4.113}$$

The first order conditions:

$$\check{r}(X) = Asf_k(k) \tag{4.114}$$

$$\check{w}(X) = Asf(k) - Asf_k(k)k \tag{4.115}$$

In equilibrium, under *rational expectations*:

$$\begin{aligned} k &= K \\ c(K,X) &= C(X) \\ \hat{r}(X) &= \check{r}(X) = r(X) = Asf_K(K) \\ \hat{w}(X) &= \check{w}(X) = w(X) = Asf(K) - r(K)K \\ \mathcal{K}(X) &= (1-\delta)K + Asf(K) - C(X) \\ \mathcal{A}(X) &= As \\ \hat{\pi}(s'|s) &= \pi(s'|s) \end{aligned}$$

More formally:

Definition 63 *A* ***recursive competitive equilibrium*** *is a value function $v(k,X)$ and a policy function $c(k,X)$ for the representative household, an aggregate per capita policy function $C(X)$, factor price functions $r(X)$ and $w(X)$ such that these functions satisfy:*

1. *the representative household's Bellman equation (4.109);*

2. *the necessary and sufficient first order conditions for profit maximization (4.114)-(4.115);*

3. *the consistency between individual and aggregate decisions,* $c(K, X) = C(X)$*;*

4. *the aggregate resource constraint (4.107).*

Note that $C(X) = \mathcal{C}(X)$, since they solve the same dynamic programming problem. In other we showed once again that a recursive competitive equilibrium is Pareto-efficient.

Chapter 5

Asset pricing

5.1 The consumption CAPM

In our first Chapter we analyzed the intertemporal consumption/saving problem under the assumption that households had the possibility to accumulate assets, focusing however on the characteristics of the optimal consumption path. We will now discuss the role of assets in greater detail, and, in particular, how different assets (shares, bonds, securities, and so on) can be competitively priced in our dynamic and recursive macroeconomic framework.

5.1.1 Equity shares and the equity premium

As a first step in this direction, we slightly modify the previously developed model by assuming that assets may be held as *equity shares*, *i.e.* as property claims on income flows, and as risk free one-period *financial bonds*.[1] Denote respectively q_t and b_t the number of shares and bonds held by the household. We furthermore assume that:

- During each period, shares are traded on a competitive market. The state-contingent price of a share, measured in consumption good units, is denoted p_t.
- At the beginning of each period, shares pay a non-negative stochastic dividend d_t, measured in consumption good units too; dividends are governed by an exogenous stochastic Markov process.[2]
- Bonds purchased during period t pay with probability one an interest rate $\bar{r}_{t+1}$ at the beginning of period $t+1$.
- Finally, for the sake of simplicity, non-asset income is zero, *i.e.* $y_t = 0\ \forall t$.

The household solves the following (fully recursive) stochastic optimal control problem taking the sequence of state-contingent prices $\{p_t, \bar{r}_{t+1}\}_{t=0}^{\infty}$ and the stochastic process

[1] One-period financial bonds pay both the capital and the interest at the end of the period.

[2] As usual, we assume that $\{d_s\}_{s=t}^{\infty}$ is of mean exponential order less than β^{-1}.

driving d_t as given:

$$\max_{\{c_t, s_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \quad U_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{5.1}$$

$$\text{s.t.} \quad p_t q_{t+1} + b_{t+1} = (p_t + d_t)\, q_t + (1 + \bar{r}_t)\, b_t - c_t$$

$$\{q_0, b_0\} \text{ given} \tag{5.2}$$

By deriving the Lagrangian with respect to c_t, b_{t+1}, and q_{t+1}, and reorganizing the first order conditions we obtain the following standard Euler equations:[3]

$$\beta E_t \left[u'(\hat{c}_{t+1}) \right] (1 + \bar{r}_{t+1}) = u'(\hat{c}_t) \tag{5.3}$$

$$\beta E_t \left[u'(\hat{c}_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right] = u'(\hat{c}_t) \tag{5.4}$$

As usual, we impose the transversality conditions:

$$\lim_{j \to \infty} \beta^j E_0 \left(\hat{\lambda}_j \hat{b}_{j+1} \right) = 0 \tag{5.5}$$

$$\lim_{j \to \infty} \beta^j E_0 \left(\hat{\lambda}_j p_j \hat{q}_{j+1} \right) = 0 \tag{5.6}$$

Risk neutrality and bubbles

Consider now equation (5.4), and rewrite it as:

$$p_t = E_t \left[\tilde{\beta}_{t,t+1} (p_{t+1} + d_{t+1}) \right] \tag{5.7}$$

where:

$$\tilde{\beta}_{t,t+j} \equiv \frac{\beta^j u'(\hat{c}_{t+j})}{u'(\hat{c}_t)} \tag{5.8}$$

is the intertemporal marginal rate of substitution between date t and $t+j$, better known as the *stochastic discount factor*.

Given that $E_t(xy) = E_t(x)\, E_t(y) + Cov_t(x, y)$, we can rewrite (5.7) as:

$$p_t = E_t \left(\tilde{\beta}_{t,t+1} \right) E_t (p_{t+1} + d_{t+1}) + Cov_t \left(\tilde{\beta}_{t,t+1}, p_{t+1} + d_{t+1} \right) \tag{5.9}$$

Under risk neutrality, *i.e.* if the Bernoulli utility function is linear and the marginal utility of consumption constant, (5.9) would collapse to:

$$p_t = \beta E_t (p_{t+1} + d_{t+1}) \tag{5.10}$$

Note that (5.10) *does not* endogenously determine the sequence of state-contingent prices that satisfy the individual's Euler equation; we are still in a partial equilibrium framework, and the sequence of state-contingent prices is exogenous. Equation (5.10) simply describes how a sequence of prices should behave to make the Euler equation hold.

Remark 64 *Under risk neutrality (a very restrictive assumption, stronger than certainty*

[3] Note that the risk-free interest rate will be paid in the future but is currently known with certainty; the dividend on shares, being stochastic, is instead currently unknown.

equivalence) the current share price should depend on its conditionally expected future value and the conditionally expected future dividend only.

In other words, once discounting and the future dividends have been accounted for, no other aggregate variable should *Granger-cause* the share price.

Equation (5.10) is a linear stochastic difference equation and therefore admits the following general class of solutions (check!):

$$p_t = E_t\left(\sum_{j=1}^{\infty}\beta^j d_{t+j}\right) + \frac{\xi_t}{\beta^t} \tag{5.11}$$

where ξ_t is a martingale, *i.e.* $E_t(\xi_{t+1}) = \xi_t$.

Remark 65 *Equation (5.11) is a forward-looking pricing rule relating the current share price to: (i) the expected discounted value of future dividends; (ii) a stochastic exogenous term following a martingale. The latter is called a "bubble", since the corresponding stochastic process is completely unrelated to any fundamentals.*

The bubble is a quite disturbing component of our pricing function, but unfortunately cannot be ruled out in the current partial equilibrium framework.

The equity premium

Define the rate of return on shares as $r_{t+1} \equiv (p_{t+1} + d_{t+1})/p_t - 1$, and rewrite (5.9) as:

$$E_t\left(\tilde{\beta}_{t,t+1}\right)[1 + E_t(r_{t+1})] = 1 - Cov_t\left(\tilde{\beta}_{t,t+1}, r_{t+1}\right) \tag{5.12}$$

Remark 66 *Equation (5.12) tells us some interesting things:*

- *Neither the variance of the stochastic discount factor, nor the variance of the rate of return play an explicit role in the Euler equation (of course, uncertainty plays an indirect role via precautionary saving).*
- *The covariance between the stochastic discount factor and the rate of return, instead, plays an explicit, and essential, role. Note, in particular, that there is a relationship between covariance and savings for a given value of $E_t(r_{t+1})$ (make sure you see it! Hint: note that $\tilde{\beta}_{t,t+1}$ depends on the slope of the consumption path).*

When the return to any asset covaries positively with non-asset income, the corresponding asset income will be high (low) when non-asset income is high (low). Evidently, a risk averse individual would prefer a high asset income in "bad" states, and a low asset income in "good" states, in order to smooth consumption over states of the world. A *risky asset*, i.e an asset whose returns covary positively with consumption, does not help the owner to hedge the risk associated with the volatility of income.

If the asset return covaries positively with consumption, then it covaries negatively with the marginal utility of consumption. The return to a risky asset, then, covaries *negatively* with the stochastic discount factor. Combining (5.3) and (5.12) we obtain:

$$E_t(r_{t+1}) - \bar{r}_{t+1} = -Cov_t\left(\tilde{\beta}_{t,t+1}, r_{t+1}\right)(1 + \bar{r}_{t+1}) \tag{5.13}$$

When equation (5.13) holds, the individual is indifferent between investing in shares or bonds.

Remark 67 *The left-hand side of (5.13) is called* ***conditional equity premium****, while the whole equation is also known as the* ***Consumption-based Capital Asset Pricing Model*** *(CCAPM).*

As we can see, the equity premium is proportional to minus the covariance between the stochastic discount factor and the rate of return. Note that:

- If the covariance in (5.13) is zero, then the expected rate of return on shares has to equal the interest rate on risk free bonds to make the individual indifferent between shares and bonds.
- If the covariance is negative, *i.e.* if shares are risky, then the individual demands a higher expected rate of return on shares, in order to compensate their riskiness.

The economic motivation behind the individual's behavior is always the same: consumption smoothing, not only across time but also across states of the world.

The empirical puzzles

We reconsider now equation (5.7), and assume an isoelastic Bernoulli utility function; taking the *unconditional* expectation of both sides, we obtain[4]:

$$\beta E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\zeta}(1+r_{t+1})\right]=1 \tag{5.14}$$

where ζ is the curvature parameter in the Bernoulli function, equal to the reciprocal of the intertemporal elasticity of substitution.

Furthermore, we assume that both the consumption growth and the rate of return on shares are jointly *log-normally* distributed:

$$\gamma_t \equiv \frac{c_{t+1}}{c_t} = \gamma e^{\epsilon_{t+1}-\frac{1}{2}\sigma_\epsilon^2} \tag{5.15}$$

$$1+r_{t+1} = (1+r)\, e^{\varepsilon_{t+1}-\frac{1}{2}\sigma_\varepsilon^2} \tag{5.16}$$

where $\{\epsilon_t, \varepsilon_t\}$ is a vector of *iid* innovations, jointly distributed as a multivariate normal with zero means, variances $\{\sigma_\epsilon^2, \sigma_\varepsilon^2\}$, and covariance $\sigma_{\epsilon\varepsilon}$.

Substituting (5.15) and (5.16) into (5.14) and simplify, we get:

$$(1+r)\,\beta E\left(e^{-\zeta\epsilon_{t+1}+\frac{1}{2}\xi\sigma_\epsilon^2} e^{\varepsilon_{t+1}-\frac{1}{2}\sigma_\varepsilon^2}\right)=\gamma^\zeta \tag{5.17}$$

Equation (5.17) can be rewritten as:

$$(1+r)\,\beta E\left(e^{-\zeta\epsilon_{t+1}+\frac{1}{2}\xi\sigma_\epsilon^2+\varepsilon_{t+1}-\frac{1}{2}\sigma_\varepsilon^2}\right)=\gamma^\zeta \tag{5.18}$$

It can be shown that, if x and y are normally distributed variables with means μ_x and μ_y, variances σ_x^2 and σ_y^2, and covariance σ_{xy}, then:

$$E\left[\exp(x+y)\right]=\exp\left(\mu_x+\frac{\sigma_x^2}{2}+\mu_y+\frac{\sigma_y^2}{2}+\sigma_{xy}\right) \tag{5.19}$$

[4] A straightforward extension of the Law of Iterated Expectations states that $E\left[E_t(x_t)\right]=E(x_t)$.

Taking this result into account, equation (5.18) can be rewritten as:

$$(1+r)\,\beta e^{(1+\zeta)\xi\frac{1}{2}\sigma_\epsilon^2-\xi\sigma_{\epsilon\varepsilon}}=\gamma^\zeta \tag{5.20}$$

Taking logs of (5.20) we obtain:

$$\ln(1+r)=-\ln(\beta)+\zeta\ln(\gamma)-(1+\zeta)\,\zeta\frac{1}{2}\sigma_\epsilon^2+\xi\sigma_{\epsilon\varepsilon} \tag{5.21}$$

Consider now equation (5.3). The same procedure leads us to:

$$\ln(1+\bar{r})=-\ln(\beta)+\zeta\ln(\gamma)-(1+\zeta)\,\zeta\frac{1}{2}\sigma_\epsilon^2 \tag{5.22}$$

Combining (5.21) and (5.22) and taking into account that $\ln(1+x)\approx x$, we obtain an expression for the long-run average equity premium:

$$r-\bar{r}\approx\xi\sigma_{\epsilon\varepsilon} \tag{5.23}$$

Now, consider that the historical average values of r, $\bar{r}$, and $\sigma_{\epsilon\varepsilon}$ for the US[5] are respectively equal to 0.01, 0.07, and 0.00219. Substituting these values in (5.23), we obtain a curvature parameter equal to $\zeta = 27.39$. Unfortunately, a vast empirical literature suggests that a value of ζ higher than 5 is at odds with available evidence on consumption behavior. This incongruence between theory and data is known as the *equity premium puzzle*.

We may however decide that there is something wrong with these empirical estimates, and accept that the curvature parameter is large[6]. In this case, another puzzle shows up. Consider equation (5.22). The average consumption growth factor γ is equal to 0.018, while its variance σ_ϵ^2 is equal to 0.00127. Given these numbers, equation (5.22) may hold only if the intertemporal discount factor is very close to one. Furthermore, if we use the summary statistics from post-WWII US data, which imply a lower variance of consumption growth, the same equation is compatible only with a discount factor greater than one. This second incongruence is known as the *risk-free rate puzzle*.

Finally, assume that we could be satisfied with both a large curvature parameter and a discount factor greater than one. Equation (5.22) would imply an extremely high sensitivity of the risk-free interest rate to changes in the average growth rate of consumption. Unfortunately, the available empirical evidence, both across time and across countries, supports exactly the opposite view.

5.1.2 Financial bonds and the term structure

Assumes that n competitive markets for risk-free n-period financial bonds exist. Denote $b_{t,t+j}$ the amount of j-period bonds purchased by the representative individual at date t. The interest rate on a j-period bond purchased at date t, denoted $\bar{r}_{t,t+j}$, is known

[5] Summary statistics for US annual data from 1889 to 1978. See Kotcherlakota (1996, Table 1), who uses the same data as Mehra and Prescott (1985).

[6] Campbell and Cochrane (1999, pp. 243-245) discuss the issue, suggesting that the high risk aversion implied by this class of models is not implausible as it seemed, and that microeconomic empirical evidence is not completely at odds with their conclusions. Furthermore, they argue that high risk aversion may be an inescapable feature of identical-agent models that want to be consistent with the equity premium facts in both the short and the long-run.

with certainty at the beginning of period t. For the sake fo simplicity, let us rule out the existence of secondary bond markets, i.e. once the representative individual has purchased a bond, she cannot resell it before the maturity date. The exogenous non-asset income y_t follows a stochastic Markov process.

The individual solves the following stochastic optimal control problem, taking the sequence of state-contingent prices $\left\{\{\bar{r}_{t,t+j}\}_{j=1}^{n}\right\}_{t=0}^{\infty}$ and the stochastic process driving y_t as given:

$$\max_{\left\{c_t,\{b_{t,t+j}\}_j\right\}_{t=0}^{\infty}} \quad U_0 = E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] \tag{5.24}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} b_{t,t+j} = \sum_{j=1}^{n}\left(1+\bar{r}_{t-j,t}\right)b_{t-j,t} + y_t - c_t$$

$$\left\{b_{0,0+j}\right\}_j \text{ given} \tag{5.25}$$

Deriving the Lagrangian with respect to c_t and $\{b_{t,t+j}\}_{j=1}^{n}$, and reorganizing the first order conditions, we obtain the following n Euler equations:

$$\beta^j E_t\left[u'\left(c_{t+j}\right)\right]\left(1+\bar{r}_{t,t+j}\right) = u'\left(c_t\right), \quad \forall j \tag{5.26}$$

We may then rewrite (5.26) as:

$$R_{t,t+j} = E_t\left(\tilde{\beta}_{t,t+j}\right) \tag{5.27}$$

where $R_{t,t+j} \equiv \left(1+\bar{r}_{t,t+j}\right)^{-1}$, or as:

$$R_{t,t+j} = E_t\left(\prod_{s=t}^{t+j-1}\tilde{\beta}_{s,s+1}\right) \tag{5.28}$$

We apply the *Law of Iterate Expectations* to obtain:

$$R_{t,t+j} = E_t\left[E_{t+j-1}\left(\prod_{s=t}^{t+j-1}\tilde{\beta}_{s,s+1}\right)\right] \tag{5.29}$$

At date $t+j-1$, all factors $\tilde{\beta}_{s,s+1}$ such that $s \leq t+j-2$ are known. Therefore:

$$R_{t,t+j} = E_t\left[\prod_{s=t}^{t+j-2}\tilde{\beta}_{s,s+1}E_{t+j-1}\left(\tilde{\beta}_{t+j-1,t+j}\right)\right] \tag{5.30}$$

Rewrite (5.30) as:

$$R_{t,t+j} = E_t\left[\tilde{\beta}_{t,t+j-1}E_{t+j-1}\left(\tilde{\beta}_{t+j-1,t+j}\right)\right] \tag{5.31}$$

Substitute now (5.27) into (5.31):

$$R_{t,t+j} = E_t\left(\tilde{\beta}_{t,t+j-1}R_{t+j-1,t+j}\right) \tag{5.32}$$

A by now familiar result tells us that:

$$R_{t,t+j} = E_t\left(\tilde{\beta}_{t,t+j-1}\right) E_t\left(R_{t+j-1,t+j}\right) + Cov_t\left(\tilde{\beta}_{t,t+j-1}, R_{t+j-1,t+j}\right) \tag{5.33}$$

Finally, we can substitute (5.27) into (5.33):

$$R_{t,t+j} = R_{t,t+j-1} E_t\left(R_{t+j-1,t+j}\right) + Cov_t\left(\tilde{\beta}_{t,t+j-1}, R_{t+j-1,t+j}\right) \tag{5.34}$$

Under the following strong assumption:

$$Cov_t\left(\tilde{\beta}_{t,t+j-1}, R_{t+j-1,t+j}\right) = 0, \quad \forall j \geq 2, \tag{5.35}$$

equation (5.34) boils down to a version of the well-known *Pure Expectation Theory of the term structure of interest rates*:

$$R_{t,t+j} = R_{t,t+1} \prod_{s=2}^{j} E_t\left(R_{t+s-1,t+s}\right) \tag{5.36}$$

Remark 68 *As implied by (5.36), the pure expectation theory states that the term structure is downward-sloping (upward-sloping) only if the individual expects a decreasing (increasing) sequence of future one-period interest rates.*

Equation (5.34) generalizes the approach by introducing of a risk premium component that depends on the covariance between the stochastic discount rate and the future one-period interest rate.

5.1.3 Discount bonds and the term premium

Consider now a slight modification of the previous setting. Each period, a set of risk-free discount bonds[7] with maturity length going from one to n periods are issued. We assume that competitive markets for bonds with maturity length from one to n exist in all periods, i.e. that secondary markets for issued bonds are always open. Note that bonds issued at date t with a maturity length of j periods are therefore materially indistinguishable form bonds issued at any previous date with the same maturity length left. In other words, bonds issued in the past with a current maturity length of j periods have to be traded at the same price of bonds issued in the current period with the same maturity length: bonds are characterized by their *current* maturity length only. Denote $b_{t,j}$ the number of j-period bonds held at the beginning of period t by the household. The price of a j-period bond, denoted $R_{t,j}$, is known at the beginning of period t (note that $R_{t,0} \equiv 1$). As before, exogenous income y_t follows a stochastic Markov process.

The individual solves the following stochastic optimal control problem, taking the sequence of state-contingent prices $\left\{\{R_{t,j}\}_{j=1}^{n}\right\}_{t=0}^{\infty}$ and the stochastic process driving y_t

[7] A j-periods discount bond issued at date t delivers one unit of consumption good at date $t+j$.

as given:

$$\max_{\{c_t,\{b_{t+1,j}\}_j\}_{t=0}^{\infty}} \quad U_0 = E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_s)\right] \tag{5.37}$$

$$s.t. \quad \sum_{j=1}^{n} R_{t,j}b_{t+1,j} = \sum_{j=1}^{n} R_{t,j-1}b_{t,j-1} + y_t - c_t$$

Note that in the previous budget constraint $b_{t+1,j}$ represents the stock of bonds with *current* maturity j that will be available to the household at the beginning of next period, when their maturity will clearly be $j-1$. Deriving the Lagrangian with respect to c_t and $\{b_{t+1,j}\}_{j=1}^{n}$, and reorganizing the first order conditions, we obtain the following n Euler equations:

$$\beta E_t\left[u'(c_{t+1})R_{t+1,j-1}\right] = u'(c_t)R_{t,j} \quad \forall j \tag{5.38}$$

We may then rewrite (5.38) as:

$$R_{t,j} = E_t\left(\tilde{\beta}_{t,t+1}R_{t+1,j-1}\right) \tag{5.39}$$

Iterating on (5.39) gets:

$$R_{t,j} = E_t\left(\tilde{\beta}_{t,t+j}\right) \tag{5.40}$$

Compare (5.27) and (5.40). Evidently, the price of a risk free j-periods discount bond has to be equal to the reciprocal of the gross interest rate on a j-period risk-free bond.

Applying our well known result to (5.40), we get:

$$R_{t,j} = E_t\left(\tilde{\beta}_{t,t+1}\right)E_t\left(R_{t+1,j-1}\right) + Cov_t\left(\tilde{\beta}_{t,t+1}, R_{t+1,j-1}\right) \tag{5.41}$$

Divide both sides of (5.41) by $R_{t,j}$, and define $h_{t+1,j} \equiv R_{t+1,j-1}/R_{t,j} - 1$ as the *one period holding return*, i.e. the rate of return to holding a j-period bond for one period and then sell it on the secondary market.

Remark 69 *Even if our discount bonds are intrinsically risk-free assets, the fact that future prices on secondary markets are currently unknown with certainty transforms the risk-free multi-period assets into possibly risky one-period assets.*

Substituting (5.27) into (5.41) we obtain:

$$E_t\left(h_{t+1,j}\right) - \bar{r}_{t,1} = -Cov_t\left(\tilde{\beta}_{t,t+1}, h_{t+1,j}\right)\left(1+\bar{r}_{t,1}\right) \tag{5.42}$$

Remark 70 *The left side of (5.42) is known as the* ***term risk premium****. When the j-period bond is risky, i.e. when its return is positively correlated with non-asset income, the representative individual requires a higher expected holding return to be indifferent between the risky bond and a risk free one-period bond.*

Exercise 71 *Prove that* $E_t\left(h_{t+s,j}\right) - \bar{r}_{t,s} = -Cov_t\left(\tilde{\beta}_{t,t+s}, h_{t+s,j}\right)\left(1+\bar{r}_{t,s}\right)$*, where* $h_{t+s,j}$ *is the rate of return to holding a j-period bond for s periods.*

5.2 The Lucas model

5.2.1 The basic framework

We present now the simplest general equilibrium asset pricing model, developed by Lucas (1978) and generally known as the "tree model". The model is based on the following assumptions:

- The economy is populated by a large number of identical individuals, whose preferences satisfy the usual regularity conditions.
- "Trees" are the only durable good in the economy. Each individual is endowed with one and only one tree at date 0. In each period, then, there are as many trees as individuals; we denote q_t the amount of trees held at the beginning of each period;
- All trees yield d_t non-storable fruits, or dividends, measured in consumption good units at the beginning of each period. Dividends are the only source of income, and they follow an exogenous stochastic Markov process.
- During each period, competitive markets for consumption goods and trees exist. As before, we denote p_t the price of a tree measured in consumption good units, and we normalize to unity the price of current consumption.[8]

Since all individuals are alike, we can summarize them in a representative individual, who solves the following simplified stochastic optimal control problem, taking the sequence of state contingent prices $\{p_t\}_{t=0}^{\infty}$ and the stochastic process driving d_t as given:

$$\max_{\{c_t, q_{t+1}\}_{t=0}^{\infty}} \quad U_0 = E_0\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right] \qquad (5.43)$$

$$\text{s.t.} \quad p_t q_{t+1} = \left(p_t + d_t\right) q_t - c_t$$

We easily obtain the corresponding Euler equation:

$$\beta E_t\left[u'\left(\hat{c}_{t+1}\right) \frac{p_{t+1} + d_{t+1}}{p_t}\right] = u'\left(\hat{c}_t\right) \qquad (5.44)$$

and the transversality condition:

$$\lim_{j \to \infty} \beta^j E_0\left[u'\left(\hat{c}_j\right) p_j q_{j+1}\right] = 0 \qquad (5.45)$$

The perceived sequence of state-contingent asset prices $\{p_s\}_{s=t}^{\infty}$ taken as given by the representative individual may a priori be different from the actual sequence of state-contingent equilibrium prices. Under rational expectations, however, these two sequences coincide. We therefore impose rational expectations, and interpret the asset prices in (5.44)-(5.45) as the actual equilibrium prices.

We can now push further our analysis by imposing a set of equilibrium conditions. Note that:

[8] An alternative but perfectly equivalent framework would be the following: each individual is endowed with a tree; equity shares, representing property claims to flows of dividends, are traded on a competitive market; p_t represents the price of shares measured in consumption goods.

- since all individuals are identical, the only possible equilibrium outcome is that each individual owns one and only one tree, *i.e.* $q_t = 1\ \forall t$ (if one individual wants to buy or sell a tree, all other individuals want to do the same; the only possible equilibrium outcome is that everybody keeps its tree);

- as long as $\lim_{c\to 0} u'(c) = +\infty$, it would never be optimal to waste resources.

Note that, to analytically solve the model, we have to *anticipate* these equilibrium outcomes and impose them on the first order conditions from the very beginning. Together, they imply that, in equilibrium:

$$c_t = d_t, \quad \forall t \tag{5.46}$$

The dynamics of consumption, as described by (5.44), depends on the current asset price, the expected future asset price, and the expected future dividend. The asset price is determined on the competitive market for trees, by equating aggregate demand and supply. The aggregate supply of trees is fixed and exogenous, while the aggregate demand depends on the saving behavior of the representative individual. However, we concluded that in equilibrium $c_t = d_t$, where d_t is an exogenous stochastic process. In some sense, then, in equilibrium consumption and savings are exogenous too (note that in equilibrium savings are zero all the way long). We have to find a sequence of state-contingent asset prices that equate the exogenous aggregate supply of trees to the corresponding aggregate demand determined by "exogenous" savings. Those prices, once announced by the "auctioneer," make the households happy with their current asset stocks, and therefore inhibit actual trade on the asset market. In other words, the asset market never sees a single transaction taking place. This does not mean, however, that it can be considered just a "virtual market": the model is based on the assumption that the *institution* called "asset market" exists and works properly in all periods, *i.e.* that a "Walrasian auctioneer" is always ready to announced equilibrium prices, independently of the actual trade volume.

To find the sequence of equilibrium prices, we have to solve equation (5.44) once the equilibrium condition $c_t = d_t$ has been imposed:

$$p_t = E_t\left[\tilde{\beta}_{t,t+1}\left(p_{t+1} + d_{t+1}\right)\right] \tag{5.47}$$

where:

$$\tilde{\beta}_{t,t+j} \equiv \frac{\beta^j u'(d_{t+j})}{u'(d_t)} \tag{5.48}$$

To solve (5.47) for p_t, we evaluate it at date $t+1$, substitute the result in (5.47) again, and reorganize:

$$\begin{aligned} p_t &= E_t\left[E_{t+1}\left(\tilde{\beta}_{t,t+2}p_{t+2}\right)\right] + \\ &\quad + E_t\left[E_{t+1}\left(\tilde{\beta}_{t,t+2}d_{t+2}\right)\right] + E_t\left(\tilde{\beta}_{t,t+1}d_{t+1}\right) \end{aligned} \tag{5.49}$$

Using the Law of Iterated Expectations and iterating, we obtain:

$$p_t = E_t\left(\sum_{j=1}^{\infty}\tilde{\beta}_{t,t+j}d_{t+j}\right) + \lim_{j\to\infty} E_t\left(\tilde{\beta}_{t,t+j}p_{t+j}\right) \tag{5.50}$$

Now, recall that in equilibrium $s_t = 1$; we can divide (5.45) by $u'(d_t) > 0$ and rewrite it as:

$$\lim_{j\to\infty} E_t\left(\tilde{\beta}_{t,t+j}p_{t+j}\right) = 0 \tag{5.51}$$

Note that, by combining the TVC with an equilibrium condition, we are now able to rule out asset price bubbles.

This takes us to:

$$p_t = E_t\left(\sum_{j=1}^{\infty}\tilde{\beta}_{t,t+j}d_{t+j}\right) \tag{5.52}$$

Remark 72 *Equation (5.52) shows that the value of a tree is simply equal to its expected discounted stream of dividends. Note that the tree's past performances are completely irrelevant. Only the future matters!*

Exercise 73 *Assume that the Bernoulli utility function is isoelastic, and that* $d_{t+1} = d_t\epsilon_{t+1}$*, where* ϵ_t *is a iid log-normal innovation, with* $\ln(\epsilon_t) \sim N(0,\sigma^2)$*. Show that* $\tilde{\beta}_t$ *and* r_t *are themselves iid and jointly log-normally distributed (not so easy ...).*

The stochastic discount factor revisited

The previous exposition of the Lucas model was silent about the possibility of trading state-contingent consumption claims. It should be by now clear that the complete-market or sequential-trading Arrow-Debreu machinery would be redundant in this case,[9] since in equilibrium, being all households identical, state-contingent consumption claims, that have to be in aggregate zero net supply, would not be traded. However, casting the Lucas model in terms of the Arrow sequential trading framework discussed in Section 4.1.3, p. 72, may help us to gain some insights on the very nature of the stochastic discount factor in general equilibrium.

Assume that the forcing process s_t follows a discrete-state Markov chain characterized by a state space S, a transition density $\pi(s'|s)$, and an initial density $\pi_0(s)$, so that $\pi(s^t) = \pi(s_t|s_{t-1})\cdots\pi(s_1|s_0)\pi_0(s_0)$. At each date $t \geq 0$, the representative households is allowed to trade a full set of j-period-ahead contingent consumption claims. Denote $a_{t,j}(s_{t+j})$ the date t end-of-period holdings of contingent claims to one unit of consumption j periods ahead at date $t+j$, contingent on the state at date $t+j$ being s_{t+j}. Furthermore, let the *j-period-ahead pricing kernel* $Q_j(s'|s)$ denote the price of one unit of consumption j periods ahead, contingent on the future, j-period-ahead, state being s', given that the current state is s, and expressed in terms of current consumption.

In this case, the household faces a sequence of intratemporal budget constraints of the form:

$$c_t(s_t) + \sum_{j=1}^{\infty}\sum_{s_{t+j}|s^t} Q_j(s_{t+j}|s_t)\, a_{t,j}(s_{t+j}|s_t) + p_t(s_t)\, q_{t+1}(s_t) =$$
$$a_{t,0}(s_t) + [p_t(s_t) + d_t(s_t)]\, q_t(s_t)$$

[9] Or, more precisely, that the shares themselves are redundant assets in a general equilibrium framework, since a full set of state-continent securities already does the job.

The household solves the following optimal control problem:[10]

$$\max_{\{c_t(s_t),q_{t+1}(s_t),\{a_{t,j}(s_{t+j}|s_t)\}\}} \quad U = E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] \tag{5.53}$$

$$\text{s.t.} \quad c_t\left(s_t\right)+\sum_{j=1}^{\infty}\sum_{s_{t+j}|s_t} Q_j\left(s_{t+j}|s_t\right)a_{t,j}\left(s_{t+j}|s_t\right)+p_t\left(s_t\right)q_{t+1}\left(s_t\right)=$$

$$a_{t,0}\left(s_t\right)+\left[p_t\left(s_t\right)+d_t\left(s_t\right)\right]q\left(s_t\right)$$

$$\lim_{j\to\infty} q_{t+j}^t\left(s_{t+j}\right)a_{t,j}\left(s_{t+1}\right)\geq 0, \quad \forall s_{t+j}$$

$$a_{0,j}\left(s_0\right) \quad \text{given}$$

The corresponding Lagrangian is:

$$L=\sum_{t=0}^{\infty}\sum_{s_t}\left[\beta^t u\left[c_t\left(s_t\right)\right]\pi_t\left(s_t\right)+\right.$$

$$\eta_t\left(s_t\right)\left\{a_{t,0}\left(s_t\right)+\left[p_t\left(s_t\right)+d_t\left(s_t\right)\right]q\left(s_t\right)+\right.$$

$$\left.\left.-c_t\left(s_t\right)-\sum_{j=1}^{\infty}\sum_{s_{t+j}|s_t}Q_j\left(s_{t+j}|s_t\right)a_{t,j}\left(s_{t+j}|s_t\right)-p_t\left(s_t\right)q_{t+1}\left(s_t\right)\right\}\right]$$

The first order conditions, taking recursivity fully into account, are:

$$\beta^t u'\left[c\left(s_t\right)\right]\pi\left(s_t\right) = \eta\left(s_t\right) \tag{5.54}$$

$$p\left(s_t\right) = \sum_{s_{t+1}|s_t}\frac{\eta\left(s_{t+1}\right)}{\eta\left(s_t\right)}\left[p\left(s_{t+1}\right)+d\left(s_{t+1}\right)\right] \tag{5.55}$$

$$Q_1\left(s_{t+1}|s_t\right) = \frac{\eta\left(s_{t+1}\right)}{\eta\left(s_t\right)}$$

$$Q_2\left(s_{t+2}|s_t\right) = \sum_{s_{t+1}|s_t}\frac{\eta\left(s_{t+2}\right)}{\eta\left(s_t\right)}$$

$$\vdots$$

$$Q_j\left(s_{t+j}|s_t\right) = \sum_{s_{t+j-1}|s_t}\frac{\eta\left(s_{t+j}\right)}{\eta\left(s_t\right)} \tag{5.56}$$

Note that:

$$Q_j\left(s_{t+j}|s_t\right) = \sum_{s_{t+j-1}|s_t}\frac{\beta u'\left[d\left(s_{t+j}\right)\right]}{u'\left[d\left(s_t\right)\right]}\pi\left(s_{t+j}|s_{t+j-1}\right)\cdots\pi\left(s_{t+1}|s_t\right)= \tag{5.57}$$

$$= \sum_{s_{t+j-1}|s_t}\frac{\beta u'\left[d\left(s_{t+j}\right)\right]}{u'\left[d\left(s_t\right)\right]}\prod_{\tau=1}^{j}\pi\left(s_{t+\tau}|s_{t+\tau-1}\right)$$

[10] We impose the following NPG condition: $\lim_{j\to\infty} q_{t+j}^t\left(s_{t+j}\right)a_{t,j}\left(s_{t+j}\right) \geq 0, \quad \forall s_{t+j}$, where $q_{t+j}^t\left(s_{t+j}\right)\equiv q_{t+j}\left(s_{t+j}\right)/q_t\left(s_t\right)$ is the Arrow-Debreu date-t price of claims to consumption good units to be delivered in date $t+j$, contingent on the state s_{t+j}.

Furthermore, note that:

$$Q_2(s_{t+2}|s_t) = \sum_{s_{t+1}|s_t} Q_1(s_{t+2}|s_{t+1}) Q_1(s_{t+1}|s_t) \tag{5.58}$$

$$Q_3(s_{t+3}|s_t) = \sum_{s_{t+2}|s_t} Q_1(s_{t+3}|s_{t+2}) Q_1(s_{t+2}|s_{t+1}) Q_1(s_{t+1}|s_t) = \tag{5.59}$$

$$= \sum_{s_{t+1}|s_t} \left[\sum_{s_{t+2}|s_{t+1}} Q_1(s_{t+3}|s_{t+2}) Q_1(s_{t+2}|s_{t+1}) \right] Q_1(s_{t+1}|s_t) = \tag{5.60}$$

$$= \sum_{s_{t+1}|s_t} Q_2(s_{t+3}|s_{t+1}) Q_1(s_{t+1}|s_t) \tag{5.61}$$

$$\vdots \tag{5.62}$$

Hence, from the first order conditions, imposing the equilibrium conditions:

$$\begin{aligned} c(s_t) &= d(s_t) \\ q(s_t) &= 1 \\ a_{t,j}(s_{t+j}|s_t) &= 0, \quad \forall j \end{aligned}$$

we get:

$$p(s_t) = \sum_{s_{t+1}|s_t} Q_1(s_{t+1}|s_t) [p(s_{t+1}) + d(s_{t+1})] \tag{5.63}$$

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}|s_t} Q_{j-1}(s_{t+j}|s_{t+1}) Q_1(s_{t+1}|s_t) \tag{5.64}$$

$$Q_1(s_{t+1}|s_t) = \frac{\beta u'[d(s_{t+1})]}{u'[d(s_t)]} \pi(s_{t+1}|s_t) \tag{5.65}$$

For the Law of Iterated Expectations:

$$p(s_t) = \sum_{j=1}^{\infty} \sum_{s_{t+j}|s_t} Q_j(s_{t+j}|s_t) d_{t+j}(s_{t+j}) = E_t\left(\sum_{j=1}^{\infty} \tilde{\beta}_{t,t+j} d_{t+j}\right) \tag{5.66}$$

since:

$$\sum_{s_{t+j}|s_t} Q_j(s_{t+j}|s_t) = \sum_{s_{t+j}|s_t} \frac{\beta u'[d(s_{t+j})]}{u'[d(s_t)]} \prod_{\tau=1}^{j} \pi(s_{t+\tau}|s_{t+\tau-1}) = E_t\left(\tilde{\beta}_{t,t+j}\right) \tag{5.67}$$

5.2.2 A "many-trees" extension

Let us extend the Lucas model in a straightforward direction. Assume there are n different kinds of tree, and as before each individual is endowed with one of each kind at date 0. Denote q_{jt} the number of j-kind trees held at the beginning of each period, when all j-kind trees yield d_{jt} dividends, measured in consumption goods. Again, dividends are the only source of income. During any period, each kind of tree can be traded on a competitive market. As before, we denote p_{jt} the price of a j-kind tree measured in consumption goods. Assume that the n-dimensional vector $\{d_{jt}\}_{j=1}^{n}$ follows a multivariate Markov process.

The individual solves the following stochastic optimal control problem[11], taking the sequence of state-contingent prices $\left\{\{p_{jt}\}_{j=1}^{n}\right\}_{t=0}^{\infty}$ and the stochastic process driving $\{d_{jt}\}_{j=1}^{n}$ as given:

$$\max_{\left\{c_t,\{q_{jt+1}\}_j\right\}_{s=t}^{\infty}} \quad U_0 = E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right] \tag{5.68}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} p_{jt}q_{jt+1} = \sum_{j=1}^{n}(p_{jt}+d_{jt})\,q_{jt} - c_t$$

$$\{q_{j0}\}_{j=1}^{n} \text{ given} \tag{5.69}$$

We easily obtain the n Euler equation:

$$\beta E_t\left[u'(c_{t+1})\frac{p_{jt+1}+d_{jt+1}}{p_{jt}}\right] = u'(c_t)\,, \quad \forall j \tag{5.70}$$

As before, we note that, in equilibrium, $c_t = d_t$, where $d_t \equiv \sum_{j=1}^{n} d_{jt}$. Substituting this condition in (5.70) leads to:

$$p_{jt} = E_t\left[\tilde{\beta}_{t,t+1}(p_{jt+1}+d_{jt+1})\right] \tag{5.71}$$

where $\tilde{\beta}_{t,t+1} = \beta^j u'(d_{t+1})/u'(d_t)$.

Iterating on (5.71) and ruling out asset price bubbles, we obtain:

$$p_{jt} = E_t\left(\sum_{s=1}^{\infty}\tilde{\beta}_{t,t+s}d_{jt+s}\right) \tag{5.72}$$

Note that:

- The price of each kind of tree is equal to the corresponding expected discounted future stream of dividends.
- All asset prices share the same stochastic discount factor; this common factor depends on the total flow of dividends d_t only.
- The price of each kind of tree is influenced by the performance of the remaining kinds only via the common discount factor.

We may reconsider the model from a slightly different point of view. We may assume that, as in the basic Lucas model, there exists only one kind of tree for each individual, and that each tree produces a sequence of dividends denoted $\{d_s\}_{s=t}^{\infty}$. However, the corresponding property claims are divided between n equity shares. A j-kind share entitles the owner to receive a sequence of dividends $\{d_{jt}\}_{t=0}^{\infty}$, with $d_t \equiv \sum_{j=1}^{n} d_{jt}$.

[11] Consumption remains the only control variable; there is now a vector of endogenous state variables $\{s_{jt}\}_{j=1}^{n}$ and a vector of exogenous state variable $\{d_{jt}\}_{j=1}^{n}$.

The value of a tree at date t, denoted p_t, is equal to the total value of all shares that represent property claims to its fruits. In other words, from (5.72):

$$p_t = \sum_{j=1}^{n} E_t \left(\sum_{s=1}^{\infty} \tilde{\beta}_{t,t+s} d_{jt+s} \right) = E_t \left(\sum_{s=1}^{\infty} \tilde{\beta}_{t,t+s} d_{t+s} \right) \tag{5.73}$$

The value of a tree depends only on the expected discounted value of its dividends. It is completely independent from the structure of ownership claims. This result is a simplified version of the *Modigliani-Miller theorem.*

5.2.3 Ricardian equivalence

Let us introduce the government in the Lucas model. We assume that the government:

- consumes (*i.e.* throws away, for analytical convenience) a per capita share of current output equal to g_t;
- imposes a lump-sum per capita tax τ_t;
- issues a one-period risk free government bond.

We assume that government consumption follows a nonnegative exogenous stochastic process satisfying each period the constraint $0 \leq g_t < d_t$ with probability one. The lump-sum tax τ_t follows an exogenous stochastic Markov process too. Furthermore, we denote as b_t the amount of government bonds held by the representative individual at date t, and assume that bonds purchased during period t pay with probability one an interest rate $\bar{r}_{t+1}$ at the beginning of period $t+1$. Each period, the government has to satisfy with probability one the following intratemporal budget constraint:

$$g_t = \tau_t + b_{t+1} - (1 + \bar{r}_t)\, b_t \tag{5.74}$$

Since it would be unfeasible for the government to repay existing debt contracting always new debt, we impose a NPG condition, $\lim_{t\to\infty} E_0 \left(R_0^t b_{t+1}\right) \leq 0$, where $R_t^s \equiv \prod_{j=t+1}^{s} (1+\bar{r}_j)^{-1}$ and $R_t^t \equiv 1$.

Iterating on (5.74) and imposing the NPG condition, we obtain the intertemporal government budget constraint:

$$E_0 \left[\sum_{t=0}^{\infty} R_0^t g_t + (1+\bar{r}_0)\, b_0 \right] \leq E_0 \left[\sum_{t=0}^{\infty} R_0^t \tau_t \right] \tag{5.75}$$

The representative individual solves the following stochastic optimal control problem, taking the sequence of state contingent prices $\{p_t, \bar{r}_{t+1}\}_{t=0}^{\infty}$, the initial condition $\{q_0, b_0\}$, and the stochastic process driving d_t, g_t, and τ_t as given:

$$\max_{\{c_t, q_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \quad U_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{5.76}$$
$$\text{s.t.} \quad p_t q_{t+1} + b_{t+1} = (p_t + d_t)\, q_t + (1+\bar{r}_t)\, b_t - c_t - \tau_t$$

The Euler equations are:

$$\beta E_t \left[u' \left(c_{t+1} \right) \frac{p_{t+1} + d_{t+1}}{p_t} \right] = u' \left(c_t \right) \tag{5.77}$$

$$\beta E_t \left[u' \left(c_{t+1} \right) \right] \left(1 + \bar{r}_{t+1} \right) = u' \left(c_t \right) \tag{5.78}$$

In equilibrium, consumption is equal to disposable dividend income, $c_t = d_t - g_t$ (why?).[12] We can substitute this condition in (5.77) and (5.78), solve recursively (5.77) for p_t, and rule out asset price bubbles, obtaining:

$$p_t = E_t \left(\sum_{j=1}^{\infty} \tilde{\beta}_{t,t+j} d_{t+j} \right) \tag{5.79}$$

$$\left(1 + \bar{r}_{t+1} \right)^{-1} = E_t \left(\tilde{\beta}_{t,t+1} \right) \tag{5.80}$$

where $\tilde{\beta}_{t,t+j} = \beta u' \left(d_{t+j} - g_{t+j} \right) / u' \left(d_t - g_t \right)$.

Remark 74 *Equations (5.79) and (5.80) tell us that the equilibrium state-contingent prices, together with the equilibrium allocation consumption, do depend only on the disposable income, and not on how the government finances its consumption.*

In other words, the overall competitive equilibrium is independent from the government financial decisions. This is a straightforward example of the so-called *Ricardian equivalence principle.*

Exercise 75 *Assume that the government issues both one and two-periods risk free bonds. State the relevant NPG conditions. Provide an expression for the government intertemporal budget constraint. Show that the competitive equilibrium is also independent from the term structure (this is a kind of Modigliani-Miller result).*

Exercise 76 *Lump-sum taxes are ruled out. The government finances its consumption by a constant proportional tax on dividends and by issuing a one-period risk free bond. For the sake of simplicity, assume that* $b_t = b > 0$ $\forall t$. *Describe the competitive equilibrium. Does Ricardian equivalence still hold? Comment on your results.*

5.2.4 Unbacked money

We now introduce unbacked money in a simplified endowment economy, similar in spirit to the Lucas model presented in the previous Section, and study its role as a pure store of value. We focus on the deterministic case to obtain a clearer intuition. However, similar results hold in a more general stochastic framework too.

Assume that at date 0 the government puts $M = 1$ units of unbacked and inconvertible currency into circulation; at any later date, then, the money supply is simply equal to unity. Denote π_t the price of money at date t, measured in consumption good units (π_t is simply equal to the reciprocal of the price level) and m_t the amount of money held by the representative individual at date t. At date 0, the government pays the seigniorage revenues back to the representative individual via a lump-sum transfer $z_0 = \pi_0$. At any

[12] Substitute (5.74) into the representative individual's intratemporal budget constraint, and impose the usual equilibrium condition $s_t = 1$ $\forall t$.

later period, transfers are zero, *i.e.* $z_t = 0 \ \forall t \geq 1$. For the sake of simplicity, assume exogenous income is constant across time, *i.e.* that $y_t = y > 0 \quad \forall t$.

The representative individual solves the following optimal control problem, taking the sequence of prices $\{\pi_t\}_{t=0}^{\infty}$ and the sequence of transfers $\{z_t\}_{t=0}^{\infty}$ as given:

$$\max_{\{c_t, m_{t+1}\}_{t=0}^{\infty}} \quad U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{5.81}$$
$$\text{s.t.} \quad \pi_t m_{t+1} = \pi_t m_t + y + z_t - c_t$$

As usual, we easily obtain the Euler equation:

$$\beta u'(c_{t+1})\,\pi_{t+1} = u'(c_t)\,\pi_t \tag{5.82}$$

and the transversality condition:

$$\lim_{j\to\infty} \beta^j u'(c_j)\,\pi_j m_{j+1} = 0 \tag{5.83}$$

In equilibrium, $c_t = y$ and $m_t = 1$. We substitute these equilibrium conditions in (5.82), and solve recursively the result for π_t given π_0:[13]

$$\pi_t = \beta^{-t}\pi_0 \tag{5.84}$$

By substituting (5.84) into (5.83) and imposing the equilibrium conditions, we obtain:

$$\lim_{j\to\infty} \beta^j u'(y)\,\pi_j = u'(y)\,\pi_0 \tag{5.85}$$

Given our assumptions, the TVC can be satisfied only if $\pi_0 = 0$. In other words, we conclude that the only sequence of prices $\{\pi_t\}_{t=0}^{\infty}$ satisfying the first order conditions, the feasibility constraints, and the TVC is $\{\pi_t = 0\}_{t=0}^{\infty}$. The representative individual wants to purchase a positive amount of money only if the price she has to pay for it is zero.

Remark 77 *We just proved a fundamental result: in a representative agent, perfectly competitive economy, unbacked currency has no role as a pure store of value. The only possible equilibrium value of money is zero.*

Exercise 78 *Money is an asset, i.e. a way to transfer purchasing power across periods. We know that individuals love to smooth consumption over time. Explain intuitively why in the previous model individuals do not exploit the possibility of smoothing consumption (please, don't argue that, since non-asset income is constant over time ...)*

Exercise 79 *As before, assume that at date 0 the government puts* $M = 1$ *units of currency into circulation. At any later date, the government distributes as a gift* $\mu > 0$ *units of currency for each unit held by the representative individual. Restate the optimal control problem and derive the Euler equation. Under this assumption, can the value of money be strictly positive in equilibrium?*

[13] Note that $\lim_{t\to\infty} \pi_t = \infty$ if $\pi_0 > 0$, since $\beta \in (0,1)$ by assumption.

Chapter 6

Equilibrium business cycles

In the previous Chapter, we discussed the basic RCK model in a completely deterministic setting. The empirical evidence, however, suggest that the dynamics of all (real and nominal) macroeconomic variables, such as consumption, output, investments, and so on, presents at high frequencies (monthly, quarterly, and even yearly) a clear stochastic component.

In Figure **??** we plot the quarterly US time series for GNP, private consumption (nondurables), private investment (private fixed investment plus consumption durables), changes in inventories, government expenditure (consumption plus investment), and net trade (exports minus imports) over GDP. The variables are expressed in real (1996 prices) per-capita annual terms, and the sample extends from 1947:I to 2000:III.[1] As we can see, output, consumption, investment, and government expenditure[2] are non-stationary, since their first moment is clearly increasing over time. Furthermore, their dynamics seems to be strictly linked, suggesting the existence of cointegration. However, they may also share the same deterministic time trend. Changes in inventories are extremely volatile, and their first moment seems quite constant over time. Finally, the net trade/GDP ratio is extremely volatile too, but is characterized by a decreasing time trend.

Figure 6.2 plots the quarterly employment rate and the average worked hours (expressed as the ratio between the average weekly worked hours in non-agricultural establishments and the personal discretionary weekly time endowment, *i.e.* nine hours a day for seven days in a week) for the US, over the 1948:I-2000:III period for employment and the 1964:I-2000:III period for hours worked. The employment rate is characterized by a more or less constant first moment until the end of the 60s, and by a positive time trend thereafter, caused mainly by the increasing participation of women in the labor force. Worked hours present instead a clear decreasing time trend until the beginning of the 90s. The most striking properties of these data is that employment is far more volatile than worked hours.

The overall impression we get from a simple visual inspection of Figure **??** is that the stochastic nature of these variables can be hardly denied.[3] Once we agreed upon the view that macroeconomic variables are expressions of a underlaying multivariate stochastic process, we are interested in characterizing their main stochastic properties, known as *business cycle properties*. To isolate the cyclical components from the (deterministic

[1] The quarterly National Income and Production Accounts (NIPA) data are freely available from the Bureau of Economic Analysis.

[2] Note the positive shocks to government expenditure at the beginning of the 50s, at the end of the 60s, and from the middle of the 80s to the beginning of the 90s. Any idea of the causes?

[3] For an alternative view, see the book edit by Jess Benhabib (1992).

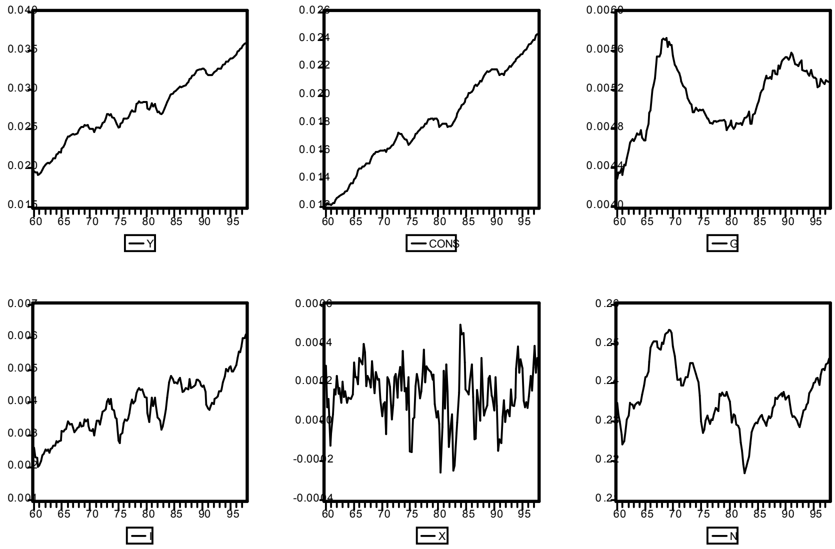

Figure 6.1: Main US aggregate variables.

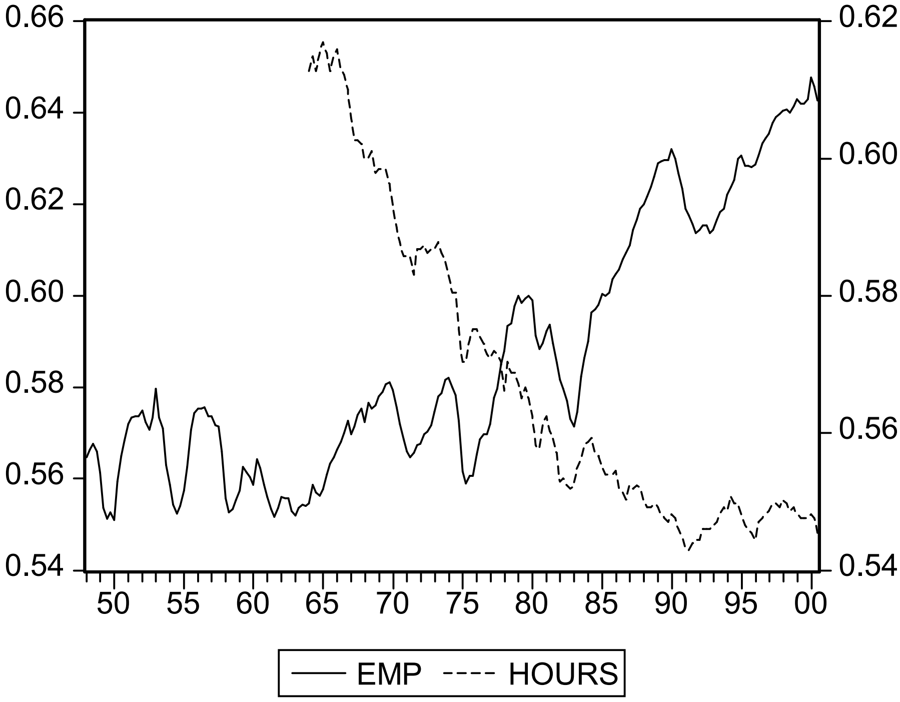

Figure 6.2: Employment rate and hours worked in the US.

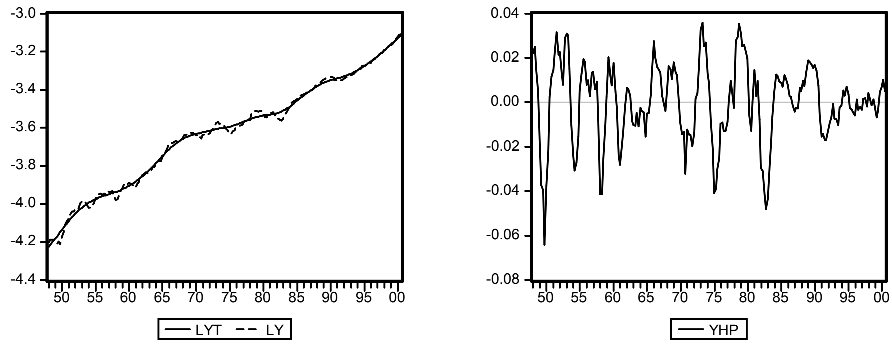

Figure 6.3: HP trend and cycle in US GDP.

or stochastic) trend, and actually make the series stationary, we need some de-trending methodology. We follow the standard procedure and apply the so-called Hodrick-Prescott filter (HP). Figure 6.3 plots the log of quarterly GDP, its HP trend, and the corresponding cycle component, expressed in percentage deviations from the trend.

Table 6.1 summarizes the relevant stochastic properties of HP filtered quarterly per-capita output, consumption, investment, employment, *Solow residual*,[4] and net trade (exports minus imports), across eleven OECD countries.[5] We focus on a set of standard statistics, as the relative volatility, defined as the ratio between the standard deviation of each variables and the standard deviation of output, the autocorrelation coefficient, the national comovement with output, defined as the contemporaneous correlation coefficient with output, and the international correlation, defined as cross-country correlation of each variable. We provide the statistics' cross-country averages and the corresponding standard deviations (the international correlations are averaged across country pairs).

The main stylized facts about business cycles at the quarterly frequency are the following:

1. consumption, employment, net trade, and the Solow residual are all less volatile than output;
2. investment is three times more volatile than output;
3. all variables are highly autocorrelated; in particular, employment and output;
4. consumption, investment, employment, and the Solow residual are *procyclical* (positively correlated with output);

[4] The Solow residual is an empirical proxy for TFP, defined as $\hat{a}_t = \hat{y}_t - s_N \hat{n}_t - s_K \hat{k}_t$, where $s_N \equiv wn/y$ and $s_K \equiv rk/y$ are respectively the labor and capital shares in income, and a hat identifies variables in logarithms. Three strong assumptions are hidden in the previous definition: (i) the aggregate production function is CRS; (ii) factor markets are competitive, so that factors are paid their marginal productivity; (iii) factor shares are constant over time. Unfortunately, no time series for the quarterly capital stock are available. If the focus is on the stochastic properties of the Solow residual, we can bypass the problem by assuming a low variability of capital at the quarterly frequency, and approximate the former (quite crudely) by linearly detrending the quantity $\hat{a}_t = \hat{y}_t - s_N \hat{n}_t$.

[5] The data set regards eleven OECD countries (Australia, Austria, Canada, France, Germany, Italy, Japan, The Netherlands, Switzerland, UK and USA). The sample period is 1970:I-1997IV (for Australia, Germany, the Netherlands, and Switzerland the sample period is shorter for some variables). Variables are GDP at constant prices, private consumption, private fixed investment, civilian employment, standard Solow residuals, and net exports over output. The table has been quoted from Maffezzoli (2000).

	Rel. Vol.		Auto.		Nat. Cor.		Int. Cor.	
	Avg.	*Std.*	*Avg.*	*Std.*	*Avg.*	*Std.*	*Avg.*	*Std.*
Y	–	–	0.79	0.04	–	–	0.42	0.20
C	0.93	0.19	0.70	0.18	0.74	0.12	0.23	0.22
I	3.01	0.55	0.76	0.17	0.78	0.10	0.31	0.21
N	0.70	0.14	0.88	0.06	0.64	0.15	0.35	0.21
A	0.78	0.09	0.69	0.13	0.90	0.05	0.36	0.16
NX	0.56	0.40	0.68	0.17	−0.35	0.12	–	–

Table 6.1: Observed stochastic properties

5. net trade is *countercyclical*;[6]

6. all variables are positively correlated across countries.

The modern equilibrium business cycle theory interprets this stochastic behavior as the reaction of the economy to a limited number of unpredictable shocks that hit specific parts of the system. These shocks are transmitted to the macroeconomic aggregates via an internal propagation mechanism, specific to the particular model at hand. Shocks may generally be: (*i*) *real shocks*, as shocks to TFP or government consumption; (*ii*) *nominal shocks*, as shocks to international prices or money supply; (*iii*) *preference shocks*, as sudden variations in the rate of time preference.

The standard literature on equilibrium business cycles focused on the effects of real shocks in competitive economies, and is consequently known as *Real Business Cycle* (RBC) theory. Recently, the increasing interest in other sources of randomness and in alternative transmission mechanism has driven researchers to incorporate many non-Walrasian features in the basic framework, like sticky prices or monopolistic competition.

However, the easiest way, both from an analytical and computational point of view, to introduce randomness in a general equilibrium framework is to generalize the RCK model by assuming that the aggregate production function is subject to a sequence of persistent productivity shocks. This explains why a stochastic version of the RCK model, developed initially by Brock and Mirman (1972), lies at the hart of the Real Business Cycle theory that originated from the seminal work of Kydland and Prescott (1982).

6.1 The Brock-Mirman model

Consider the RCK model with exogenous growth discussed in Section 3.2, p. 53. Assume that TFP, denoted $a_t \in R_{++}$, follows a stationary Markov process. In particular, assume that the natural logarithm of a_t follows an AR(1) process of the form $\ln(a_{t+1}) = \rho \ln(a_t) + \varepsilon_t$, where $0 < \rho < 1$ is the persistence parameter and $\varepsilon_t \sim N(0, \sigma^2)$ the *iid* innovation. Assume furthermore that the Bernoulli utility function and the aggregate production function belong respectively to the isoelastic and "Cobb-Douglas" families.

Under these assumptions, the planner solves the following stochastic optimal control

[6] Imports are more correlated with output that exports; as a result, net trade is negatively correlated with output.

problem:

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \quad U_0 = E_0\left(\sum_{t=0}^{\infty}\tilde{\beta}^t \frac{c_t^{1-\mu}}{1-\mu}\right) \tag{6.1}$$

$$\text{s.t.} \quad \gamma k_{t+1} = a_t k_t^{\alpha} + (1-\delta)\,k_t - c_t$$

$$\ln(a_{t+1}) = \rho\ln(a_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0,\sigma^2) \tag{6.2}$$

$$\{k_0, a_0\} \quad \text{given} \tag{6.3}$$

where $\tilde{\beta} = \beta\gamma^{1-\mu} \in (0,1)$, $\alpha \in (0,1)$, and $\gamma > 1$.

As usual, we build a Lagrangian and derive it with respect to c_t, k_{t+1}, and λ_t, to obtain the following first order conditions:

$$c_t^{-\mu} = \lambda_t \tag{6.4}$$

$$\varphi\lambda_t = E_t\left[\lambda_{t+1}\left(a_{t+1}\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right)\right] \tag{6.5}$$

$$\gamma k_{t+1} = a_t k_t^{\alpha} + (1-\delta)\,k_t - c_t \tag{6.6}$$

where $\varphi \equiv \gamma/\tilde{\beta}$. Conditions (6.4)-(6.6), together with the stochastic TVC, are necessary and sufficient for problem (6.1).

6.1.1 Solution methods

Equations (6.4)-(6.6) form a system of deterministic/stochastic difference equations, whose solution is an infinite sequence of probability measures converging in the long-run to a so-called invariant distribution. This invariant distribution can be interpreted as the variables' joint unconditional distribution. Given the recursive structure of our problem, the solution may equivalently be seen as a collection of time-invariant policy functions.

From both points of view, a purely qualitative analysis is impossible or overwhelmingly difficult. We may however study the numerical properties if a closed form solution in terms of policy functions were actually available. Unfortunately, being the system so highly non-linear, no exact close form solution does generally exist. To explicitly solve it, we need some kind of approximation.

We solve our model using two alternative approaches. First, we apply the well-known standard King, Plosser, and Rebelo (1988, KPR) procedure, which log-linearly approximates the Euler equations around the steady-state under a certainty equivalence assumption. Second, we apply the projection method described in Sections and .

There are two main reasons for doing this. First of all, the old-fashioned log-linear approach (a standard instrument in the quantitative macroeconomist toolbox) remains useful when the number of state variables in the system is high. However, the KPR procedure, being based on a certainty equivalence assumption, rules precautionary saving out; by comparing the KPR results with the alternative ones, obtained using a procedure that takes the system's non-linearity fully into account, we can evaluate the importance of precautionary saving at the business cycle frequency.

The King, Plosser, and Rebelo procedure

The KPR procedure works in three steps. We start by assuming certainty equivalence. In other words, we assume that the planner acts under uncertainty as if future random variables will turn out equal to their conditional mean. Any behavior induced by uncertainty alone, as precautionary savings, is ruled out. A direct consequence is that the unconditional mean of the invariant distribution equals the deterministic steady-state, obtained by dropping the conditional expectation operator. The steady-state becomes the ideal locus where to approximate our system. Then, we transform the system in logarithms and linearly approximate it around the steady-state, using a first-order Taylor expansion. Finally, we solve the resulting linear system of expectational difference equations with the standard Blanchard and Khan (1980) algorithm. The whole procedure is outlined in the Appendix.

The projection method

As in Section 1.5, p. 26, we exploit the model's recursive structure to approximately solve the following functional equation for the policy function $c(k, a)$:

$$E\left[c\left(k', \mathrm{e}^{\ln(a')}\right)^{-\mu}\left[\alpha \mathrm{e}^{\ln(a')}\left(k'\right)^{\alpha-1}+1-\delta\right] \mid k, \mathrm{e}^{\ln(a)}\right]=\varphi c\left(k, \mathrm{e}^{\ln(a)}\right)^{-\mu} \tag{6.7}$$

where:

$$k' = \frac{\mathrm{e}^{\ln(a)}k^{\alpha}+(1-\delta)k-c\left(k, \mathrm{e}^{\ln(a)}\right)}{\gamma}$$
$$\ln(a') = \rho\ln(a)+\varepsilon$$
$$\varepsilon \sim N(0,\sigma^2)$$

As in our previous numerical exercises, we solve (6.7) using the *collocation* method (see the Appendix for details)

6.1.2 Calibration

To actually solve the system and perform any numerical exercise, we need to specify a sensible parameterization. This task is quite demanding, for some parameters are difficult, or practically impossible, to estimate. The approach we follow here, known as calibration, is extensively discussed in Cooley (1997). He states that "calibration is a strategy for finding numerical values for the parameters of artificial economic worlds...[it] uses economic theory extensively as the basis for restricting a general framework and mapping that framework into the measured data." The parameters are chosen "so that the behavior of the model economy matches features of the measured data in as many dimensions as there are unknown parameters." In our case, we will match long-run features of the model with the corresponding long-run features of US data, since our main interest are the model's short-run cyclical properties.[7]

Now, the parameters to pin down are the following: μ, the intertemporal elasticity of substitution, $\tilde{\beta}$, the intertemporal discount factor, α, the "Cobb-Douglas" coefficient, δ, the depreciation rate, γ, the long-run growth rate, ρ, the TFP persistence parameter, and σ, the standard deviation of productivity shocks.

A large sub-set of them can be easily estimated. In particular, we can fit a linear trend to the logarithm of quarterly US GDP, and estimate a long-run quarterly growth factor equal to $\gamma = 1.0046$. The TFP persistence parameter and the standard deviation of

[7] The interested readers may refer also to Favero (2000, ch. 8, joint with M. Maffezzoli).

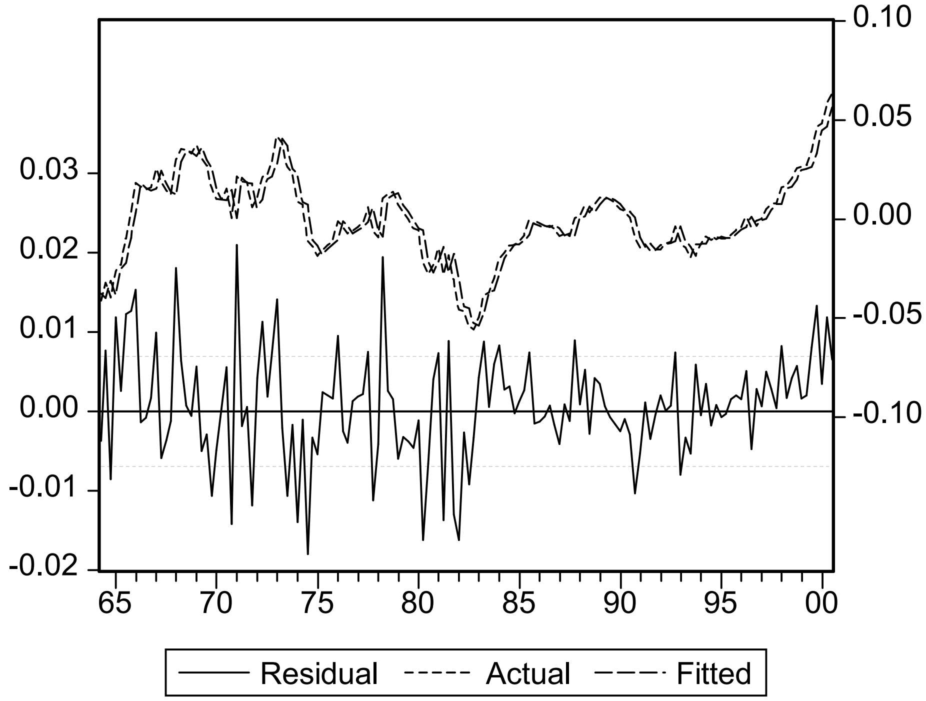

Figure 6.4: The Solow residual for the US.

productivity shocks may be estimated by fitting an AR(1) model on the standard Solow residual in logs; we obtain $\rho = 0.965$ and $\sigma = 0.007$. Figure 6.4 plots the logarithm of the Solow residual (Actual), the fitted time series (Fitted), and the implied innovations to TFP (Residual).Finally, most authors agree on an elasticity of intertemporal substitution between 0 and 2; we chose a standard value of 0.5, *i.e.* $\mu = 2$.

The remaining parameters, namely $\tilde{\beta}$, α, and δ, are left for our calibration exercise. As already anticipated, we choose these parameters to reproduce some long-run features of actual US data.

The Cobb-Douglas technology implies that the factor shares in income are always constant; in particular, the capital share equals $s_K = \alpha$, while that labor share, given constant returns to scale, equals $s_N = 1 - \alpha$. In order to replicate the observed long-run factors share in total income reported by Cooley and Prescott (1995) we set $\alpha = 0.4$.

As already noted, if certainty equivalence holds, the unconditional mean of the invariant distribution matches the deterministic steady-state. Evaluate equations (6.33) and (6.34) at the steady-state, and solve them for the steady state capital-output ratio, $r_{ky} = s_K/(\varphi - 1 + \delta)$, and the consumption share in income, $s_c = 1 - (\gamma - 1 + \delta)r_{ky}$. Empirical estimates of the long-run capital-output ratio and the consumption share are readily available. Cooley and Prescott (1995) obtain a long-run quarterly capital-output ratio equal to 13.28; we calculate a consumption share equal to 0.81. Manipulating the previous equations, we can express $\tilde{\beta}$ and δ as functions of these observable long-run properties, $\delta = 1 - \gamma + (1 - s_c)/r_{ky}$ and $\tilde{\beta} = \gamma/\left(\alpha/r_{ky} + 1 - \delta\right)$. The implied values are

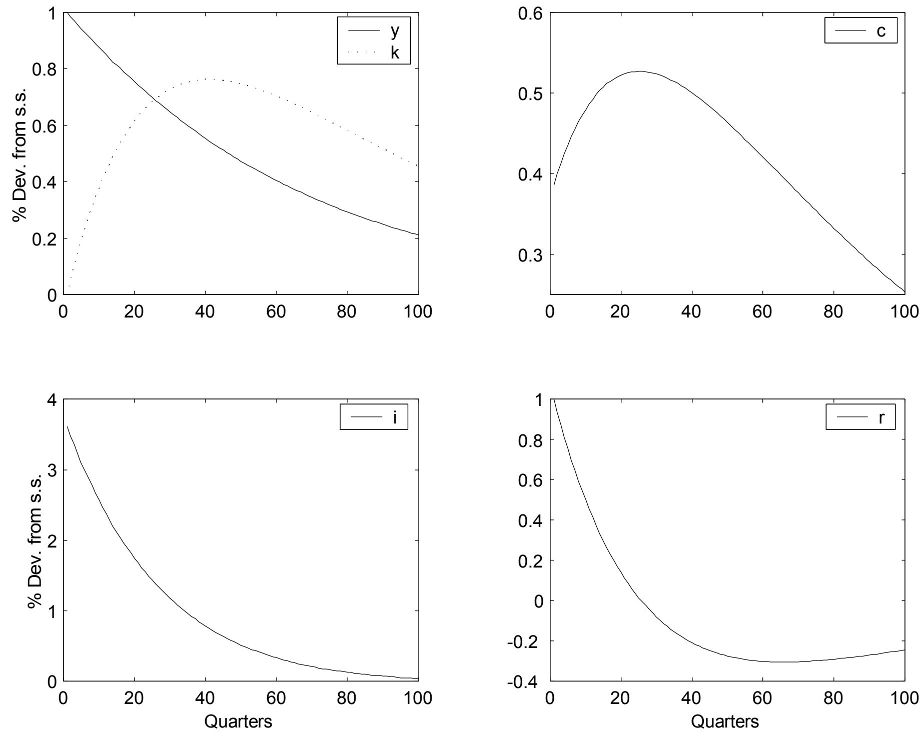

Figure 6.5: Impulse response (benchmark).

$\tilde{\beta} = 0.985$ and $\delta = 0.01$. To summarize, then, our benchmark parameterization will be:

$$\mu = 2, \quad \gamma = 1.0046, \quad \rho = 0.965, \quad \sigma = 0.007$$
$$\alpha = 0.40, \quad \tilde{\beta} = 0.985, \quad \delta = 0.01$$

6.1.3 Numerical experiments

Impulse response functions

We may be interested in studying the effects of an unexpected shock to one of the (endogenous or exogenous) state variables, for instance a 1% sudden increase in TFP. In other words, we may consider the impulse response functions of our system, *i.e.* the adjustment paths for all variables of interest that describe the system's transitional dynamics.

In Figure 6.5 we plot the impulse response functions, expressed in percentage deviations from the steady-state[8], for the log-linearly approximated system. Admire consumption smoothing at work! On impact, the system's reacts in the following way:

1. the rise in TFP produces a parallel increase in current output, being the current capital stock fixed;

2. the persistent but transitory increase in output translates into a less than proportional increase in permanent income;

[8] In other words, for a generic variable x_t we plot $\tilde{x}_t \equiv (x_t - x)/x$, where x is the corresponding steady-state value.

3. consumption increase, but only to a limited extent, since the representative individual reacts to the increase in her permanent income by smoothing consumption over time;

4. the limited reaction of consumption drives investment up, whose increase is more than proportional to the rise in output;

5. the interest rate (the marginal productivity of capital minus depreciation) increases;

6. the slope of the consumption path increases, driven by the rise in the interest rate.

During the transition back to the steady-state, we observe that:

1. being the productivity shock highly persistent, output converges slowly to its steady-state level;

2. the accumulation of capital and the decrease in TFP drive jointly down the interest rate, back to its steady-state level and even further down;

3. at some point (quarter nineteen, more or less), the interest rate becomes lower than in steady-state, and the slope of the consumption path turns negative;

4. net investment becomes negative, and the capital stock starts to be eaten up;

5. the interest rate increases again and converges back to its steady state level from below;

6. consumption, output, investment, and capital converge slowly to their steady-state levels.

This mechanism depends heavily on the high persistence of the productivity shocks. If the persistence is low, the increase in permanent income is only marginal, and the interest rate remains high for a limited number of quarters only. Figure 6.6 shows the reaction of our model to a positive shock to TFP when the persistence parameter is $\rho = 0.10$. As we can see, the reaction of consumption is extremely limited, while both output and investment increase sharply on impact, but then converge quickly to the steady-state. The interest rate returns to its steady-state value in only three quarters.

The previous results are illuminating: the internal transmission mechanism of the Brock-Mirman model is extremely weak. A highly transitory shock to productivity generates only highly transitory deviations from the steady-state for all variables except physical capital. The deviation from steady-state experienced by capital is still persistent, but its scale is definitely smaller.

Consider now the opposite extreme, and assume that the technology shocks are permanent, *i.e.* that the stochastic process driving TFP is a random walk, with $\rho = 1$. The impulse response functions are plotted in Figure 6.7. A permanent and positive shock to TFP increases the productivity of a given resource endowment, and translates in a one-to-one increase in the representative individual's permanent income. In a partial equilibrium framework, this would imply a one-to-one increase in consumption. However, in general equilibrium, the interest rate is not constant across time, but it is endogenously determined. Actually, a permanent increase in TFP translates directly in a permanent increase of the interest rate. The consumption path, then, becomes steeper, and consumption increases less than output on impact. Investment increases with consumption;

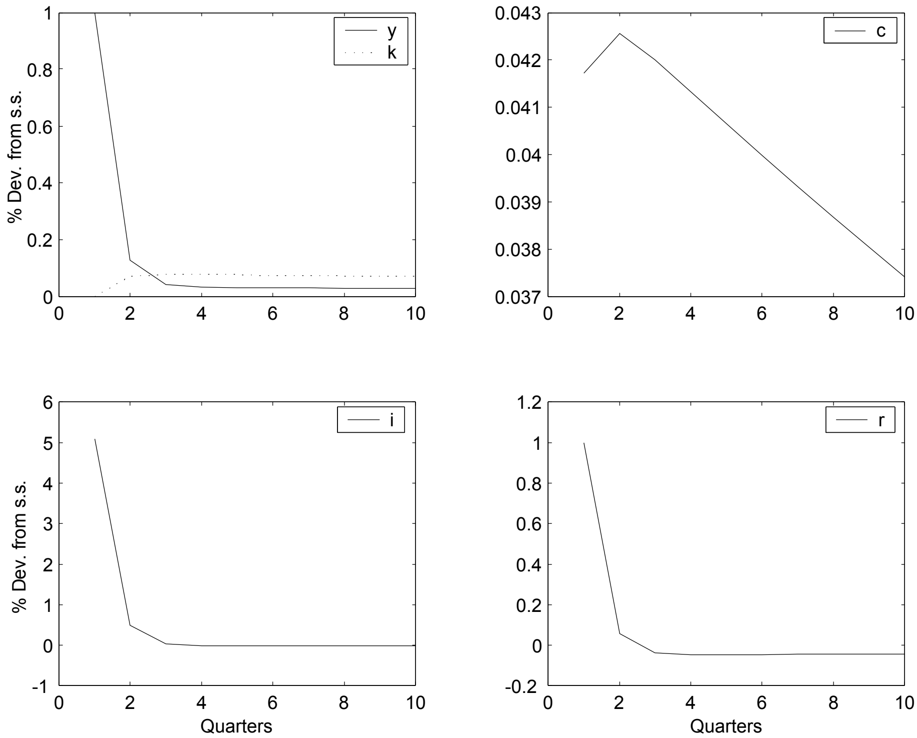

Figure 6.6: Impulse response ($\rho = 0.1$)

while the consequent increase in the capital stock drives slowly the interest rate down to its new steady-state value. The slope of the consumption path decreases (without becoming negative), and all variables, namely output, consumption, investment, and capital slowly converge to a new steady-state, since the unit root in the stochastic process driving TFP makes the system non-stationary. In other words, a persistent shock to productivity simply "reallocates" the whole system to a new steady-state.

Stochastic properties

We are mainly interested in the small sample stochastic properties of our model at business cycles frequencies. To estimate these properties, we perform a so-called Montecarlo experiment. We draw from a random number generator a finite sequence of *iid* Gaussian innovations (the simulation horizon is typically $T = 100$) and iterate the policy functions to obtain the simulated deviations from the steady-state for all endogenous and exogenous variables. To isolate the dynamics at business cycles frequencies, we filter the simulated series applying the HP filter, with a smoothing parameter equal to 1600. Then, we calculate the statistics of interests, namely the relative volatility, the autocorrelation, and the correlation with output. We repeat this procedure for N times (in our experiments, $N = 100$) and summarize the empirical distribution of our statistics of interest by calculating their mean, standard deviation, and so on, across the N replications. The results for our benchmark parameterization are summarized in Table 6.2.

The stochastic Brock-Mirman model is able to crudely reproduce some of the stylized facts characterizing US data. In particular, the simulated series for consumption and investment present a volatility that is on average respectively lower and higher than output. Furthermore, all simulated series present a positive autocorrelation coefficient.

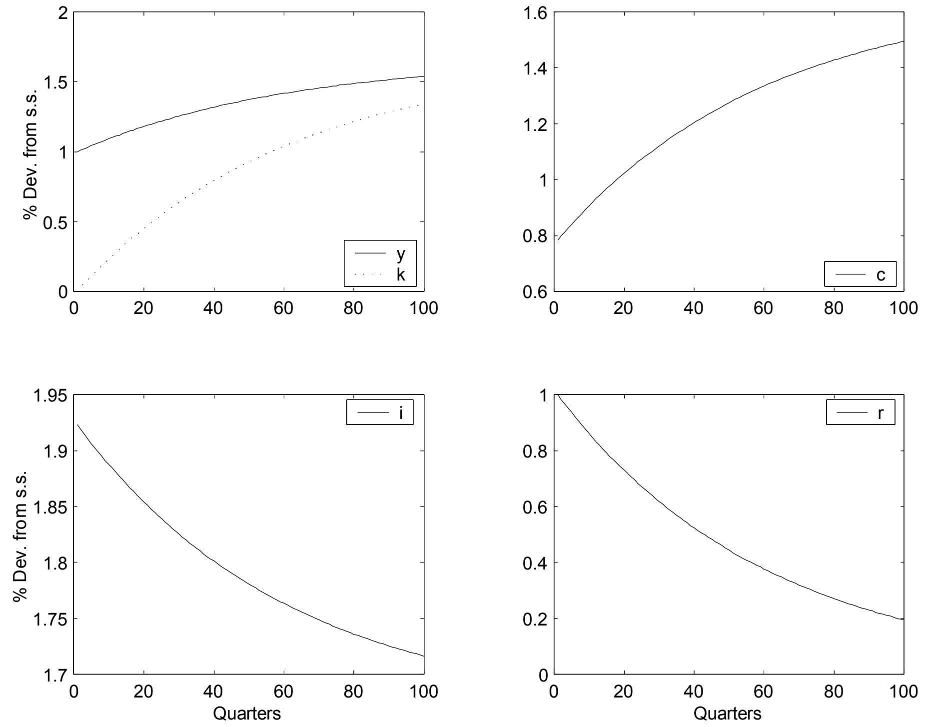

Figure 6.7: Impulse response ($\rho = 1$).

	US				Model (KPR)			
	Std.%	Volat.	Auto.	Cor. Y	*Std.%*	Volat.	Auto	Cor. Y
Y	*1.59*	–	0.84	–	*0.89*	–	0.68	–
C	–	0.63	0.81	0.86	–	0.39	0.70	0.98
I	–	3.27	0.80	0.92	–	3.63	0.67	1.00

Table 6.2: Stochastic properties (benchmark)

	US				Model (KPR)			
	Std.	Volat.	Auto.	Cor. Y	*Std.*	Volat.	Auto	Cor. Y
Y	*1.59*	–	0.84	–	*0.67*	–	0.01	–
C	–	0.63	0.81	0.86	–	0.06	0.69	0.46
I	–	3.27	0.80	0.92	–	5.16	0.01	1.00

Table 6.3: Stochastic properties ($\rho = 0.1$)

Finally, all variables present a highly positive correlation coefficient with output.

The actual values of all these coefficients are fairly different from the observed counterparts, but without any formal metric we can not decide whether this differences are significative or not. According to a pure aesthetic criteria, we may conclude that, given the differences between the simulated and observed statistics, our model is a far to simple representation of reality. In particular, we note that:

1. the standard deviation of output is 70 basis points lower than the observed one;
2. the relative volatility of consumption is 24 points lower, while that of investment is 36 points higher;
3. all variables are not autocorrelated enough.

However, the capacity of such a simple model (actually, the simplest one we could think of) to catch many qualitative features of the data is striking.

As usual, there is an important *caveat*: the properties of the simulated series descent directly from the properties of the exogenous stochastic process governing the Solow residual. Table 6.3 summarizes the stochastic properties of the model when the productivity shocks are less persistent, *i.e.* when the persistence parameter ρ is equal to 0.10. As we can see, under this parameterization, the model is completely unable to reproduce many relevant observed properties. In particular, consumption becomes almost constant over time, since the contribution of the productivity shocks to the permanent income is now negligible. For the same reason, investment becomes even more volatile than before. The autocorrelation coefficients of output and investment are almost zero.

We conclude that the Brock-Mirman model behaves well only if a highly persistent Solow residual is assumed. This critique is slightly unfair, since the persistence parameter ρ was actually estimated from observed data. However, it is undeniable that the internal propagation mechanism of the simple Brock-Mirman model is weak, since it simply transmits the stochastic properties of the Solow residual to all other variables. Furthermore, it is also undeniable that the Solow residual is, citing Abramovitz (1956), "nothing more than a measure of our ignorance"; it is certainly be related to technology, but surely also to capacity utilization, labor hoarding, and so on.

The Brock-Mirman model reproduces the observed business cycles properties of the data by assuming the existence of an exogenous process that presents itself these business cycle properties. It may be, to some extend, successful in reproducing the empirical stylized facts, but still leaves the origin of business cycles in a competitive economy unexplained.

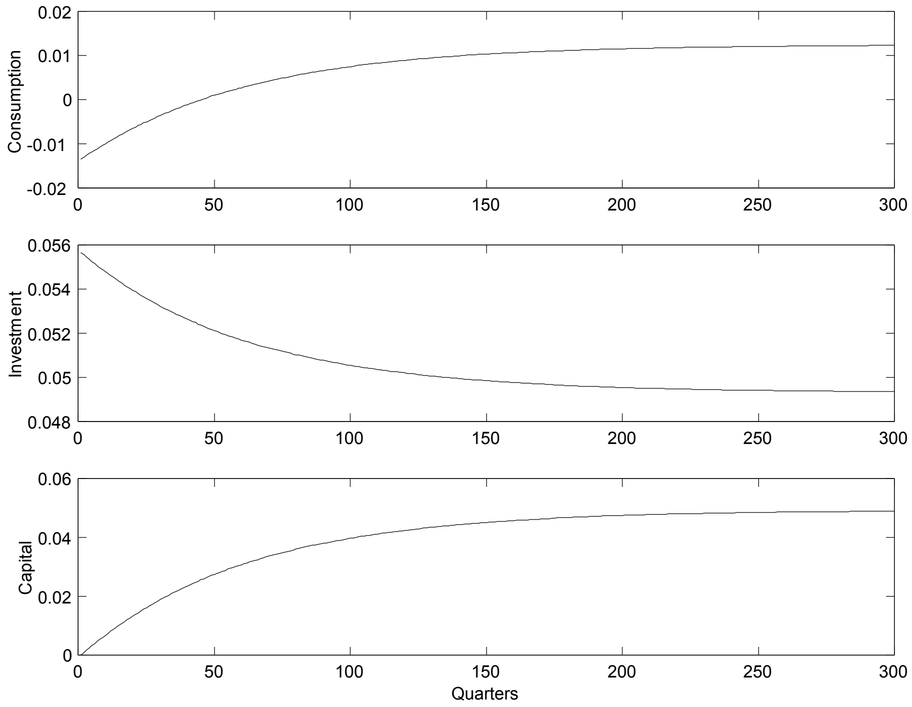

Figure 6.8: Precautionary saving.

6.1.4 The role of precautionary savings

All results presented in the previous Section were obtained under a certainty equivalence assumption, using the KPR solution procedure. We are now interested in the possible role of precautionary saving at the business cycles frequency; as already known, the variability of productivity shocks should translate into a higher level of saving/investment, just for a precautionary motif. To study the quantitative dimensions of this phenomenon, we solve our model using the collocation method, and simulate it over a long time horizon (300 quarters). The initial conditions correspond to the deterministic steady-state, *i.e.* to the unconditional mean of the model under certainty equivalence; the steady-state value of a_t is simply one, while $\bar{k} = [s_K / (\varphi - 1 + \delta)]^{\frac{1}{\alpha - 1}}$. Note furthermore that $\bar{y} = \bar{k}(\varphi - 1 + \delta) / s_K$ and $\bar{c} = s_c \bar{y}$.

In Figure 6.8 we plot the percentage deviation of consumption, investment, and capital from their *deterministic* steady-state values. As expected, consumption is initially lower than its deterministic steady-state level, while investment is higher; in other words, saving is higher than under certainty equivalence, just for a precautionary motif. During the transition, the system converges to a steady-state characterized by higher values for all variables, since an increase in savings generates obviously an increase in the steady-state capital level. We should note, however, that under our benchmark parameterization the precautionary motif seems *quantitatively* irrelevant: the steady-state capital level is only 0.05 percentage points higher than under certainty equivalence.

We may wonder now if the precautionary motif, while having irrelevant effects in the long-run, influences the short-run dynamics of the system, and in particular its stochastic properties. Furthermore, we may be interested in evaluating the overall accuracy of the log-linear approximation implied by the KPR solution procedure. Table 6.4 summaries the statistics produced by our two solution methods, KPR and collocation, under the

	Collocation				KPR			
	Std.%	Volat.	Auto.	Cor. Y	*Std.%*	Volat.	Auto	Cor. Y
Y	*0.89*	-	0.68	-	*0.89*	-	0.68	-
C	-	0.39	0.69	0.98	-	0.39	0.70	0.98
I	-	3.57	0.67	1.00	-	3.63	0.67	1.00

Table 6.4: Stochastic properties (benchmark)

benchmark parameterization. As we can see, differences are marginal. We can conclude that: (i) the precautionary motif seems to be irrelevant at the business cycle frequency too, at least for our benchmark parameterizations; (ii) the KPR method is quite accurate, at least if the system's dynamics remains near the steady-state.

6.2 The standard RBC model

We introduce now a fully-fledged labor market in the previously described Brock-Mirman framework. The labor supply curve can be made endogenous by confronting the individuals with a *labor/leisure choice problem.* This choice has two dimensions: each individual can choose between working or not working at all, *i.e.* being employed or not, and, once employed, between different work loads, *i.e.* total amounts of hours of work. The first dimension is called *extensive margin*, the second *intensive margin*. We already discussed Figure 6.2, where both the intensive and the extensive margins were separately plotted. Figure 6.9 shows instead the quarterly series for the aggregate time share devoted to labor, *i.e.* the product of employment and hours worked divided by the time endowment. This approximated aggregate measure of labor employed in production takes both margins jointly into account. Note that the decreasing trend in hours worked seems to be counterbalanced by the increasing trend in employment: the time share devoted to labor does not present a clear positive or negative trend over the whole sample period.

Focusing on the intensive margin alone is the easiest way to introduce endogenous labor in our framework. We assume that all individuals are actually employed, and own a fixed time endowment, normalized to unity. The representative individual has to choose the time share devoted to labor, denoted n_t; the remaining time is devoted instead to leisure, $l_t = 1 - n_t$; leisure, being a good, is valued in the utility function.

The competitive real wage, equal to the marginal productivity of labor, grows at the same exogenous long-run rate as the other variables in the system. An increase in the real wage is expected to cause an increase in the supply of labor, but this cannot be the case in steady-state, because labor is a time share, and cannot grow indefinitely. Further assumptions on the preferences are necessary to guarantee that leisure is constant in steady-state, even if the real wage grows at a constant rate. In other words, we have to impose that the wealth, income, and substitution effects of any permanent increase in the real wage rate cancel themselves perfectly out.

As shown in King, Plosser and Rebelo (1987), only Bernoulli utility functions of the form:

$$u(c,l) = \begin{cases} \frac{c^{1-\mu}}{1-\mu} v(l), & \text{if } \mu \neq 1 \\ \ln(c) + v(l), & \text{if } \mu = 1 \end{cases} \tag{6.8}$$

are compatible with a constant steady-state labor share. To guarantee also strict concavity, we assume that:

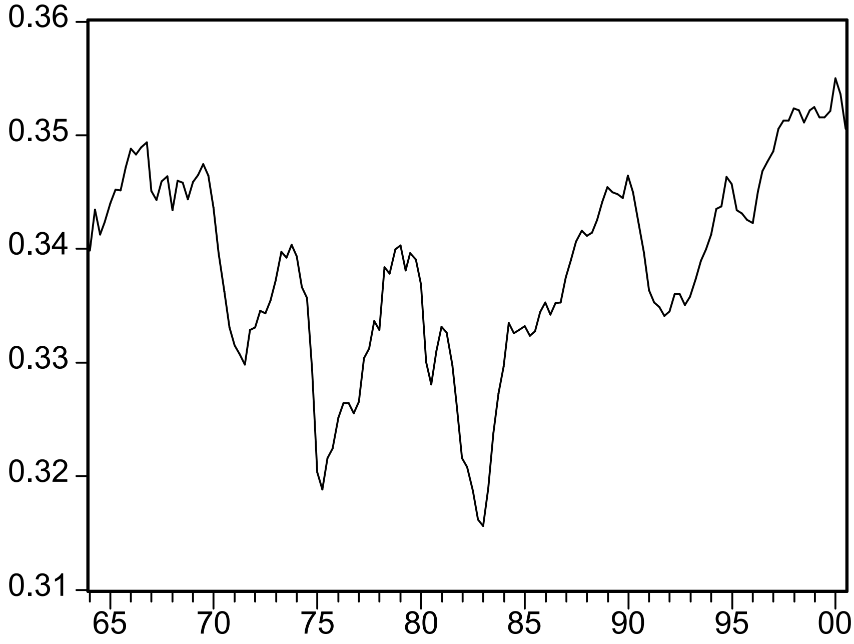

Figure 6.9: Time share devoted to labor in the US.

1. v is increasing and strictly concave if $\mu \leq 1$;
2. v is decreasing and convex if $\mu > 1$;
3. $-\mu(v''/v') > (1-\mu)(v'/v)$ holds.

A utility function satisfying these conditions is $u(c,l) = [c^{1-\mu}/(1-\mu)]\, l^{\tau(1-\mu)}$, where $\tau > 0$ when $\mu > 1$ and $0 < \tau < 1/(1-\mu)$ when $\mu < 1$.

Assuming this particular functional form, the planner solves the usual stochastic optimal control problem, taking the initial condition $\{k_0, a_0\}$ and the stochastic process for a_t as given (note that n_t is a further control variable):

$$\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} \quad U_0 = E_0\left[\sum_{t=0}^{\infty}\tilde{\beta}^t \frac{c_t^{1-\mu}\,(1-n_t)^{\tau(1-\mu)}}{1-\mu}\right] \tag{6.9}$$
$$\text{s.t.} \quad \gamma k_{t+1} = a_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t - c_t$$

The first order conditions with regard to n_t, c_t, k_{t+1}, and λ_t are:

$$\tau c_t^{1-\mu}(1-n_t)^{\tau(1-\mu)-1} = \lambda_t\,(1-\alpha)\,a_t k_t^{\alpha} n_t^{-\alpha} \tag{6.10}$$
$$c_t^{-\mu}(1-n_t)^{\tau(1-\mu)} = \lambda_t \tag{6.11}$$
$$\varphi\lambda_t = E_t\left[\alpha\lambda_{t+1}a_{t+1}k_{t+1}^{\alpha-1}n_t^{1-\alpha} + \lambda_{t+1}(1-\delta)\right] \tag{6.12}$$
$$\gamma k_{t+1} = (1-\delta)k_t + a_t k_t^{\alpha} n_t^{1-\alpha} - c_t \tag{6.13}$$

where $\varphi \equiv \gamma/\tilde{\beta}$.

Substitute (6.11) into (6.10) :

$$\tau c_t^{1-\mu}(1-n_t)^{\tau(1-\mu)-1} = c_t^{-\mu}(1-n_t)^{\tau(1-\mu)}\,(1-\alpha)\,a_t k_t^{\alpha} n_t^{-\alpha} \tag{6.14}$$

Equation (6.14) has a clear economic interpretation. Along an optimal path, the marginal utility of leisure (left hand side) has to equal the marginal productivity of labor times the marginal utility of consumption (right hand side), since a marginal decrease in leisure translates into a marginal increase in labor, which in turn translates into a marginal rise in output (and possibly consumption), equal to the marginal productivity of labor.

We can rewrite (6.14) as:

$$\tau c_t = l_t w_t \tag{6.15}$$

where w_t is the competitive wage rate. Note that any permanent increase in the wage rate implies *ceteris paribus* a permanent increase in the representative individual's permanent income, and would cause a one-to-one increase in consumption, leaving leisure unaltered: as expected, the wealth, income, and substitution effects cancel out.

The Euler equation (6.12) can be rewritten as:

$$E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\mu}\left(\frac{l_{t+1}}{l_t}\right)^{\tau(1-\mu)}(r_{t+1}+1-\delta)\right]=\varphi \tag{6.16}$$

As we can see, the interest rate influences jointly the *slope* of both the consumption and leisure paths, because the marginal utility of consumption influences and is itself influenced by the marginal utility of leisure.

The calibration procedure describe in the previous Section remains valid. There is however a further preference parameter to calibrate, τ. Simplifying (6.14) and evaluating it at the steady-state, we obtain $\tau = [(1-\alpha)l]\,/\,(ns_c)$. We already know that the steady-state US investment share is equal to 0.19. The steady-state consumption share is then $s_c = 0.81$. The empirical evidence reported by Ghez and Becker (1975) suggests that on average one third of the available discretionary time (net of sleep and personal care) is devoted to market activities in the US. Our results confirm their conclusions: we obtain a long-run time share devoted to labor equal to $n = 0.34$. We calibrate the parameter τ to reproduce these long-run properties, obtaining $\tau = 1.44$.

The model is solved with the KPR procedure, and the analytical details are summarized in the Appendix.

6.2.1 Numerical experiments

The system's impulse response to a 1% positive productivity shock under the benchmark parameterization is summarized in Figure 6.10. We observe that, on impact:

1. the persistent but transitory shock to TFP translates into a non-negligible increases in the representative individual's permanent income;

2. consumption increases again less than output because the representative individual is able to smooth consumption over time;

3. investment increases proportionally more than output, again to implement consumption smoothing;

4. the interest rate increases, and so the slope of both the consumption and leisure paths (the slope of the labor path decreases by construction);

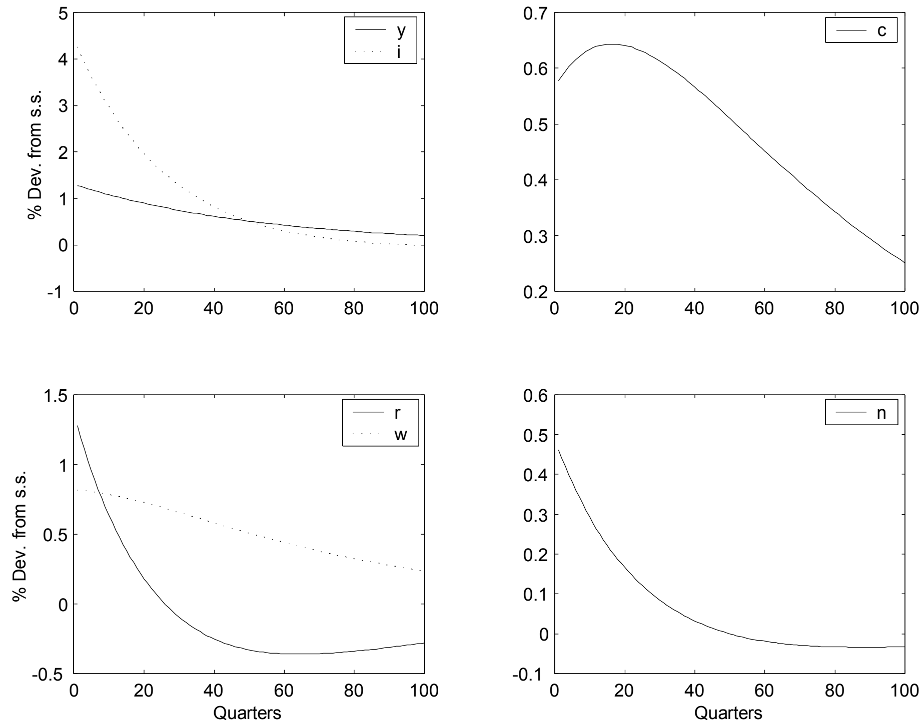

Figure 6.10: Impulse response (benchmark).

5. the shock to TFP translates into a persistent but transitory increase in the wage rate too;

6. the positive shock to permanent income tends to rise leisure, while the increase in the leisure path's slope tends to decrease it; furthermore, the labor supply is expected to positively deviate from its steady-state value as long as the wage rate does so;

7. these opposite forces make the labor share increase, but less than TFP;

8. the joint increase in TFP and labor makes the output level react more than proportionally to the productivity shock.

In other words, the increase in the wage rate pushes the representative individual to substitute current leisure for future leisure, since time devoted to labor is more productive now than in the future. The amplitude of this effect depends directly on the *elasticity of intertemporal substitution* between leisure at different points in time.

During the transition back to the steady-state, we observe that:

1. capital accumulation drives the interest rate back to its steady-state value and even below;

2. the slope of the consumption path turns quickly negative, and consumption converges slowly back to its steady-state value (note that the strict link between the consumption path's slope and the interest rate disappears);

3. the time share devoted to labor decreases over time, oversteps its steady-state value in about forty quarters, and then converges back to it from below;

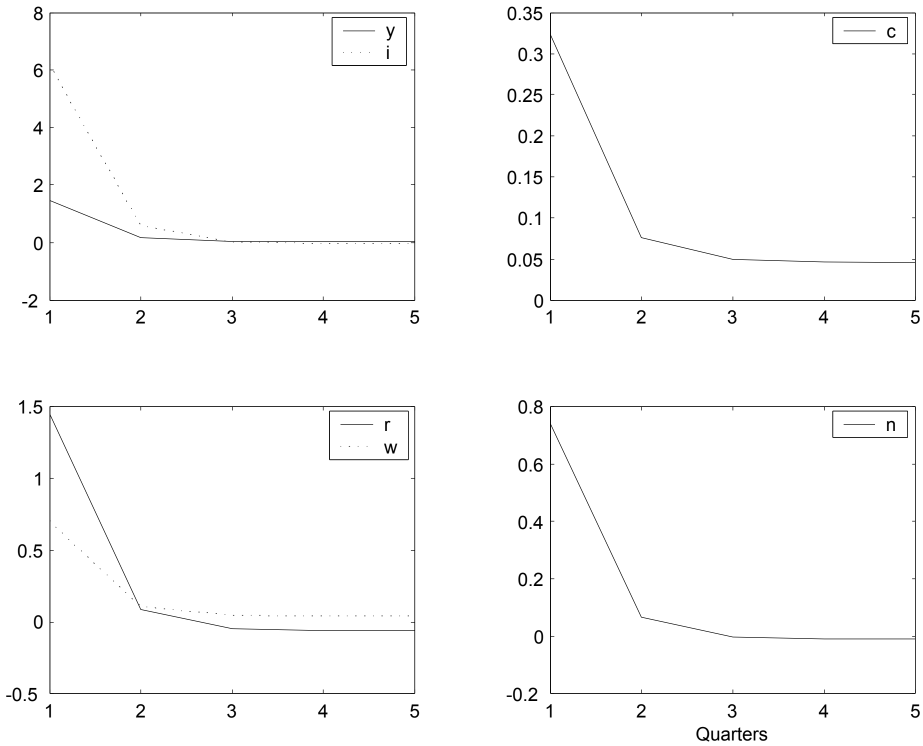

Figure 6.11: Impulse response ($\rho = 0.1$).

4. the slope of the labor path increases slowly, and turns positive after about ninety quarters.

Exercise 80 *Compare Figures 6.5 and 6.10; why does consumption increase more in the RBC model than in the Brock-Mirman one? Provide just the intuition.*

The introduction of variable labor supply enhances to some extent the model's internal propagation mechanism, since the reactions on impact of output, consumption, and investment to a productivity shock are amplified, with respect to the Brock-Mirman model. However, the subsequent adjustment is again mainly driven by the shock's persistence. If the persistence is low, as in Figure 6.11, all the variables increase sharply on impact, but then converge quickly to the steady-state; note that all variables except consumption reach their steady-state values in only three quarters.

Consider now the effects of a permanent increase in total factor productivity, summarize in Figure 6.12. Note that the impulse response of investment is not a straight line, even if ti seems so compared to the impulse responses of consumption and output; investment increases sharply on impact and then slightly over time. The permanent productivity shock translates into a permanently higher wage rate. We know that, given our assumptions on the Bernoulli function, the substitution, income, and wealth effects of a permanent increase in the wage rate cancel themselves completely out. If so, why does labor increase on impact?

Well, the productivity shock drives up the interest rate, increasing the slope of the leisure path. As soon as the capital stock increases, the interest rate converges to its new steady-state level, and the slope of the leisure path goes back to zero. The representative individual offers more labor than usual as long as the interest rate is higher than its new steady-state value. Note that the steady-state labor share does not change.

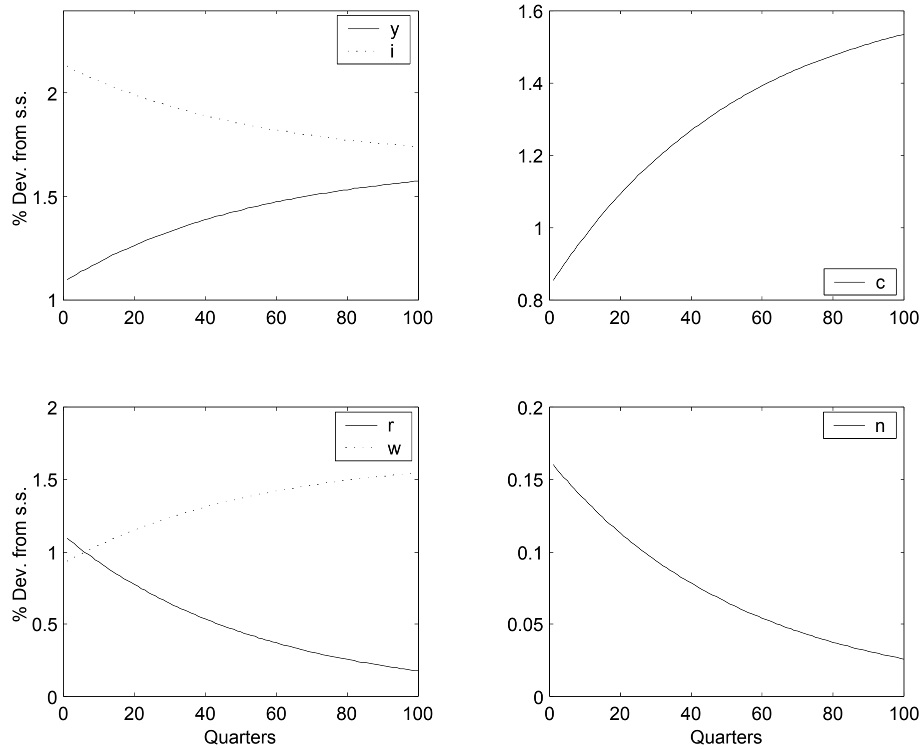

Figure 6.12: Impulse response ($\rho = 1$).

	US				Model (KPR)			
	Std.	Volat.	Auto.	Cor. Y	*Std.*	Volat.	Auto	Cor. Y
Y	*1.59*	-	0.84	-	*1.11*	-	0.67	-
C	-	0.63	0.81	0.86	-	0.45	0.68	0.99
I	-	3.27	0.80	0.92	-	3.35	0.67	1.00
n	-	0.77	0.87	0.89	-	0.36	0.67	0.99
P	-	0.48	0.64	0.65	-	0.64	0.67	1.00

Table 6.5: Stochastic properties (benchmark)

Table 6.5 reports the usual statistics, summarizing the model's stochastic properties. We report the statistics not only for the time share devoted to labor, but also for the average productivity of labor, *i.e.* the ratio between output and labor.[9] We note that:

1. output is more volatile than in the Brock-Mirman model, since endogenous labor supply enhances the model's internal transmission mechanism;
2. consumption is again more volatile, since leisure influences directly the marginal utility of consumption;
3. investment is slightly less volatile (in models where income is spent only in consumption and investment, their dynamics are inversely related);
4. the volatility of labor is less than half of the observed one;
5. the average productivity of labor is perfectly correlated with output in the model, but not in the data.

6.2.2 Conclusions

As the Brock-Mirman model before, the simple standard RBC model is, quite surprisingly, able to qualitatively reproduce the stylized facts described in the introduction. However, there are still many problems left. In particular, it clearly fails in reproducing the stochastic properties of the labor input; furthermore, volatility of output is too low compared to the observed one; finally, the transmission mechanism amplifies the shocks to technology, but cannot explain the high autocorrelations in the data, since the model's transitional dynamics is generated mainly by the persistence of productivity shocks, and only to a very limited extent by capital accumulation.

Two more subtle problems hide themselves behind the theoretical structure of the model. First of all, the dynamics of labor is exclusively driven by the intertemporal substitution of leisure, and this implies that the elasticity of the individual labor supply to transitory changes in the wage rate has to be very high. The available empirical evidence suggests however that the elasticity of labor supply is actually very low, almost equal to zero for middle-aged males. In other words, the main amplification mechanism in the standard RBC model is based on a single assumption clearly rejected by the data.

A second problem concerns the characteristic of TFP implied by the standard Solow residual. We already know that the measured Solow residual, derived under the assumptions outlined in note 4, p. 106, is very persistent and volatile. Figure 6.13 plots its growth rate, which, as expected, seems very volatile too. We note immediately that the quarterly variations are substantial, ranging form -1.8% to 2% with a standard deviation equal to 0.7 points. Furthermore, we observe that the growth rate is often negative, *i.e.* technological regress seems a common phenomenon. This is not what we observe in reality: technological regress is extremely rare. Our overall impression is that the standard Solow residual is a clearly inadequate measure of TFP.

[9] The Cobb-Douglas technology implies that the average productivity of labor has the same dynamics of the wage rate.

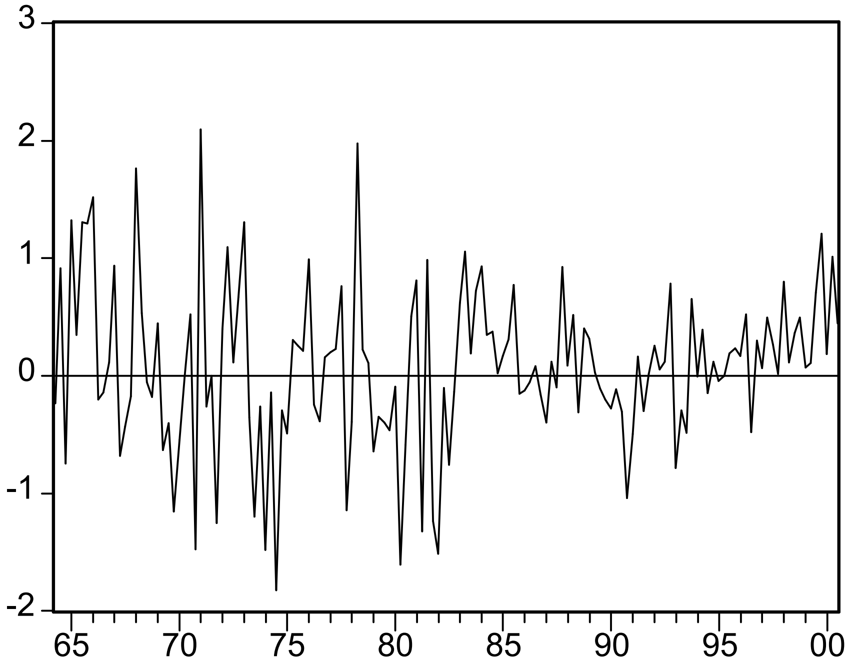

Figure 6.13: Growth rate of the Solow residual.

6.3 Extensions

6.3.1 The Hansen-Rogerson-Wright model

Our previous results suggest that modelling the choice along the intensive margin is not enough to reproduce the actual dynamics of labor in the US. We describe now a well-known theoretical framework, developed originally by Rogerson (1988), extended by Rogerson and Wright (1988), and then introduced in the RBC literature by Hansen (1985), which uses the so-called *employment lotteries* to model the choice along the extensive margin in general equilibrium.[10]

We assume that the intensive margin is not operating at all: each household member has to choose between working a fixed number of hours (normalized to one) and not working at all. The choice set is not convex, *i.e.* intermediate possibilities are not admitted, even if the individuals would prefer them. However, it may be convexified by introducing employment lotteries, that can be considered as *efficient contracts* on which individuals agree in equilibrium. By entering a lottery, an individual can choose to work a fraction n of her days and to remain unemployed the other $1 - n$ days; the allocation of individuals to work or leisure is completely random, and the lottery's outcome are independent over time. Note that, from the individual point of view, the variable of choice is the *ex-ante* probability of being employed, and not the *ex-post* actual number of hours worked.

Being all individuals identical, they will choose the same the same ex-ante probability of employment, *i.e.* they will agree on the same efficient contract. Under these assumptions, the current *aggregate* employment rate equals the chosen *ex-ante* probability of being employed at the individual level. Prior to the lottery draw, the expected

[10] The Rogerson-Wright setup is extensively described in King and Rebelo (2000, pp. 46-49).

intratemporal utility assumes then the following form (we abstract from time indexes):

$$n\left[\tilde{c}_e v(0)\right]^{1-\mu} + (1-n)\left[\tilde{c}_u v(1)\right]^{1-\mu} \tag{6.17}$$

where $\tilde{c}_e$ represent the consumption level of employed individuals (a tilde identifies individual variables), $\tilde{c}_u$ the consumption level of idle ones, and v the utility of leisure.

If asset markets are complete, individuals can perfectly insure themselves against the idiosyncratic risk of being unemployed.[11] As already known, under perfect risk sharing the marginal utilities of consumption are equal across employed and unemployed individuals; this implies that:

$$\tilde{c}_u = \tilde{c}_e \left(\frac{v_1}{v_0}\right)^{\frac{1-\mu}{\mu}} \tag{6.18}$$

where $v_0 \equiv v(0)$ and $v_1 \equiv v(1)$ are two constants.

By substituting (6.18) into (6.17), we obtain:

$$n\tilde{c}_e^{1-\mu} + (1-n)\,\tilde{c}_e^{1-\mu}\left(\frac{v_1}{v_0}\right)^{\frac{1-\mu}{\mu}} \tag{6.19}$$

The average consumption level, defined as $c = n\tilde{c}_e + (1-n)\,\tilde{c}_u$, can be interpreted as the consumption level chosen by a representative individual. Using the previous definition, we can rewrite the Bernoulli utility function (6.19) as:

$$c^{1-\mu}\left[n + (1-n)\left(\frac{v_1}{v_0}\right)^{\frac{1-\mu}{\mu}}\right]^{\mu} \tag{6.20}$$

Under these results, the stand-in representative individual's intertemporal utility function becomes:

$$U_t = E_t\left\{\sum_{s=t}^{\infty}\tilde{\beta}^{s-t}\frac{\left[c_s\varphi(n_s)\right]^{1-\mu}}{1-\mu}\right\} \tag{6.21}$$

where:

$$\varphi(n_t) \equiv \left[n_t + (1-n_t)\left(\frac{v_1}{v_0}\right)^{\frac{1-\mu}{\mu}}\right]^{\frac{\mu}{1-\mu}} \tag{6.22}$$

We still assume that the labor market is perfectly competitive, *i.e.* that both the individuals and the firms are small enough to take the market prices as given. In particular, we assume that, at the *individual* level, the choice of the ex-ante probability of being employed cannot influence the wage rate. Under perfect risk sharing, all individual factor incomes are pooled, so that our stand-in representative individual receives an aggregate income equal to $w_t n_t + r_t k_t$, where w_t is the wage rate, r_t the interest rate, and k_t the aggregate capital stock (note that unemployed individuals receive no labor income). Given constant returns to scale and competitive markets, aggregate factor income exhausts aggregate output, *i.e.* $y_t = w_t n_t + r_t k_t$.

All the previous results imply that the Hansen-Rogerson-Wright (HRW) model is, from a computational point of view, a simple extension of the standard RBC model. The

[11] Alternatively, we can assume the existence of a competitive insurance market characterized by a zero-profit condition for the firms offering insurance; in this case, the relative price of consumption under employment and unemployment will be $n/(1-n)$.

representative individual solves the following optimal control problem, taking as usual the initial conditions $\{k_0, a_0\}$ and the stochastic process for a_t as given:

$$\max_{\{c_t, n_t, k_{t+1}\}_{s=t}^{\infty}} \quad U_0 = E_0\left[\sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\left[c_t \varphi\left(n_t\right)\right]^{1-\mu}}{1-\mu}\right] \tag{6.23}$$

$$\text{s.t.} \quad \gamma k_{t+1} = a_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t - c_t$$

The first order conditions with regard to c_t and n_t, after some manipulations, become:

$$c_t^{-\mu} \varphi\left(n_t\right)^{1-\mu} = \lambda_t \tag{6.24}$$

$$c_t \varphi'\left(n_t\right) = -\varphi\left(n_t\right) w_t \tag{6.25}$$

where w_t is the competitive wage again.

Note that, in steady-state, condition (6.25) collapses to:

$$\frac{\varphi'(n)\, n}{\varphi(n)} = -\frac{s_N}{s_c} \tag{6.26}$$

We can use (6.26) to calibrate the preference parameter v_1/v_0:

$$\frac{v_1}{v_0} = \left(\frac{\frac{\mu}{1-\mu} + \frac{s_N}{s_c}}{\frac{\mu}{1-\mu} - \frac{s_N}{s_c}\frac{1-n}{n}}\right)^{\frac{\mu}{1-\mu}} \tag{6.27}$$

Furthermore note that:

$$\frac{\varphi''(n)\, n}{\varphi'(n)} = \frac{1-2\mu}{\mu}\frac{s_N}{s_c} \tag{6.28}$$

Using (6.26) and (6.28), we can log-linearize the first order conditions around the steady-state, obtaining:

$$-\mu \hat{c}_t - (1-\mu)\frac{s_N}{s_c}\hat{n}_t = \hat{\lambda}_t \tag{6.29}$$

$$\hat{c}_t + \frac{1-\mu}{\mu}\frac{s_N}{s_c}\hat{n}_t = \hat{w}_t \tag{6.30}$$

As noted in King and Rebelo (2000, p. 49), combining (6.30) and (6.29) obtains:

$$\hat{w}_t = -\frac{\hat{\lambda}_t}{\mu} \tag{6.31}$$

Equation (6.31) implies the first, important property of the HRW setup: the stand-in representative individual has an *infinite* λ-constant elasticity of labor supply, *i.e.* the elasticity of her labor supply is infinite for any given shadow value of installed physical capital. In other words, she supplies any amount of work for the wage rate implicitly defined in (6.31). The total number of hours worked, from a partial equilibrium point of view, will be determined by the labor demand schedule only. Of course, since the Lagrange multiplier λ_t is endogenous, the determinants of labor dynamics in general equilibrium are slightly more complex, for the role played by capital accumulation. Note that the high elasticity of labor supply displayed by the representative individual is not related to the elasticity of labor supply at the *individual* level: the individual elasticity of substitution

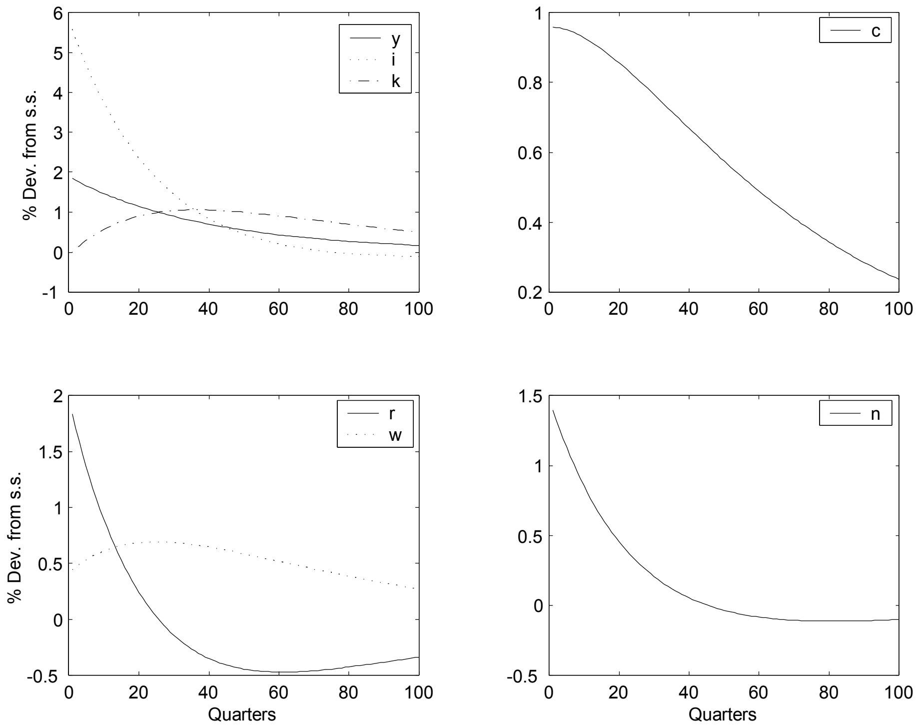

Figure 6.14: Impulse response (benchmark).

may be very small, or even zero, but the aggregate labor supply curve is, *ceteris paribus* again, completely flat.[12]

In the HRW model, as in the standard RBC one, the individuals face a trade-off between leisure and consumption; in the former, however, leisure can be consumed only when the individuals are unemployed. Choosing to bear some uncertainty by entering the employment lottery is a way to substitute consumption for leisure. In this sense, the possibly positive unemployment rate arising in equilibrium is Pareto-efficient. The optimally of unemployment is the second important property of the HRW setup.

We can solve the model with the usual KPR procedure, and simulate it under our benchmark parameterization. Figure 6.14 plots the impulse response functions to a positive 1% shock to productivity.

As we can see, the transmission mechanism built in the HRW model is able to considerably amplify the shock to productivity: the output level and the interest rate increase on impact by almost 2 percentage points, while labor rises by nearly 1.5 percentage points. Note furthermore the humped shape of the wage rate: equation (6.31) implies that the wage rate follows strictly the dynamics of the shadow value of installed capital.

Table 6.6 summarizes the usual statistics. Surprisingly, the HRW model is able to perfectly reproduce the volatility of labor. As already noted, it achieves this result without assuming a high elasticity of labor supply at the individual level. Furthermore, the HRW model reproduces the standard deviation of output quite well, and generates a higher volatility of consumption, as a by-product of the higher volatility of leisure. The overall fit of the model seems remarkably good, at least better than the fit of the standard RBC model.

[12] As shown in Rogerson (1988, pp. 14-15), this property does not depend on the strict homogeneity of individuals over time, but holds also under a limited degree of heterogeneity.

	USA				Model (KPR)			
	Std.	Volat.	Auto.	Cor. Y	*Std.*	Volat.	Auto	Cor. Y
Y	*1.59*	-	0.84	-	*1.63*	-	0.68	-
C	-	0.63	0.81	0.86	-	0.52	0.68	1.00
I	-	3.27	0.80	0.92	-	3.05	0.68	1.00
n	-	0.77	0.87	0.89	-	0.77	0.67	1.00
P	-	0.48	0.64	0.65	-	0.25	0.71	0.96

Table 6.6: Stochastic properties (benchmark)

6.3.2 Capacity utilization

To be added ...

6.4 Appendix

6.4.1 The Brock-Mirman model

The KPR procedure

Start by considering a deterministic version of conditions (6.4)-(6.6):

$$c_t^{\xi_{cc}} = \lambda_t \tag{6.32}$$

$$s_K \lambda_{t+1} a_{t+1} k_{t+1}^{-s_N} + \lambda_{t+1} (1-\delta) = \varphi \lambda_t \tag{6.33}$$

$$\gamma k_{t+1} = (1-\delta) k_t + a_t k_t^{s_K} - c_t \tag{6.34}$$

where $\xi_{cc} \equiv -\mu$, $s_K \equiv \alpha$, $s_N \equiv 1-\alpha$.

Once the time index has been dropped, equations (6.32)-(6.34) can be easily solved for the deterministic steady-state. We will now linearly approximate them, expressing the result in percentage deviations from the steady-state.

Consider condition (6.32), and simply substitute $\exp(\ln(x_t))$ to x_t, where the latter is a generic variable, and consider $\ln(x_t)$ as a variable on its own. In other words, write it as (c_t and λ_t are always strictly positive) $\exp(\xi_{cc}\tilde{c}_t) = \exp(\tilde{\lambda}_t)$, where $\tilde{x} \equiv \ln(x_t)$. The first-order Taylor expansion around the steady-state is:[13]

$$\xi_{cc} e^{\xi_{cc}\tilde{c}} (\tilde{c}_t - \tilde{c}) = e^{\tilde{\lambda}} \left(\tilde{\lambda}_t - \tilde{\lambda}\right) \tag{6.35}$$

Since in steady-state $\exp(\xi_{cc}\tilde{c}) = \exp(\tilde{\lambda})$, and since $\tilde{x}_t - \tilde{x} = \ln(x_t/x)$, equation (6.35) can be simplified as $\xi_{cc}\hat{c}_t = \hat{\lambda}_t$, where $\hat{x}_t \equiv \ln(x_t/x)$. The last expression is a log-linearized version of condition (6.32), expressed in percentage deviation from the steady-state, since $\hat{x}_t \approx (x_t - x)/x$.

Consider now condition (6.33), and applying the procedure outlined in the previous paragraph rewrite it as:

$$s_K e^{\tilde{\lambda}_{t+1}} e^{\tilde{a}_{t+1}} e^{-s_N \tilde{k}_{t+1}} + e^{\tilde{\lambda}_{t+1}} (1-\delta) = \varphi e^{\tilde{\lambda}_t} \tag{6.36}$$

[13] The first-order Taylor expansion of a non-linear function $f(x)$ around a point x_0 is given by $f(x) \approx \nabla f(x_0)(x - x_0)$.

The first-order Taylor approximation of (6.36) around the steady-state is:

$$s_K e^{\tilde{\lambda}} e^{-s_N \tilde{k}} (\tilde{\lambda}_{t+1} - \tilde{\lambda}) + s_K e^{\tilde{\lambda}} e^{-s_N \tilde{k}} \tilde{a}_{t+1} + \tag{6.37}$$
$$-s_N s_K e^{\tilde{\lambda}} e^{-s_N \tilde{k}} (\tilde{k}_{t+1} - \tilde{k}) + (1-\delta) e^{\tilde{\lambda}} (\tilde{\lambda}_{t+1} - \tilde{\lambda}) = \varphi e^{\tilde{\lambda}} (\tilde{\lambda}_t - \tilde{\lambda})$$

Equation (6.37) can be rewritten as:

$$\left(\frac{s_K}{r_{ky}} + 1 - \delta\right) \lambda \hat{\lambda}_{t+1} + \frac{s_K}{r_{ky}} \lambda \hat{a}_{t+1} - \frac{s_N s_K}{r_{ky}} \lambda \hat{k}_{t+1} = \varphi \lambda \hat{\lambda}_t \tag{6.38}$$

since $\exp(\tilde{x}) = x$ and $s_K \exp\left(\tilde{\lambda}\right) \exp\left(-s_N \tilde{k}\right) = (s_K \lambda)/r_{ky}$, where $r_{ky} \equiv k/y$. Taking into account that $(s_K/r_{ky} + 1 - \delta)\lambda = \varphi\lambda$, divide everything by $(s_K\lambda)/r_{ky}$ and rewrite (11) as:

$$-s_N \hat{k}_{t+1} + \varphi\vartheta\hat{\lambda}_{t+1} - \varphi\vartheta\hat{\lambda}_t = -\hat{a}_{t+1} \tag{6.39}$$

where $\vartheta \equiv r_{ky}/s_K$. For the sake of future notational convenience, in equation (6.39) all endogenous state and costate variables were grouped on the left-hand side, while the unique exogenous state variable was isolated on the right-hand side.

Condition (6.34) can be log-linearly approximated as (check!):

$$-\gamma r_{ky} \hat{k}_{t+1} + \left[(1-\delta) r_{ky} + s_K\right] \hat{k}_t = s_c \hat{c}_t - \hat{a}_t \tag{6.40}$$

where $s_c \equiv c/y$. Finally, note that the stochastic process driving TFP can be rewritten as $\hat{a}_{t+1} = \rho \hat{a}_t + \varepsilon_t$. Now, define a vector of control variables $\hat{u}_t \equiv [\hat{c}_t]$, a vector of state and costate variables $\hat{s}_t \equiv [\hat{k}_t | \hat{\lambda}_t]'$, a vector of exogenous state variables $\hat{e}_t \equiv [\hat{a}_t]$, and a vector of costate variables $\hat{l}_t \equiv [\hat{\lambda}_t]$. We can write the linearized version of (6.32) as:

$$M_{uu} \hat{u}_t = M_{us} \hat{s}_t + M_{ue} \hat{e}_t \tag{6.41}$$

where $M_{uu} \equiv \xi_{cc}$, $M_{us} \equiv [0|1]$, and $M_{ue} \equiv 0$.

Conditions (6.39) and (6.40) can instead be jointly written as:

$$(M_{ss}^0 + M_{ss}^1 L)\hat{s}_{t+1} = (M_{su}^0 + M_{su}^1 L)\hat{u}_{t+1} + (M_{se}^0 + M_{se}^1 L)\hat{e}_{t+1} \tag{6.42}$$

where:

$$M_{ss}^0 = \begin{bmatrix} -s_N & \varphi\vartheta \\ -\gamma r_{ky} & 0 \end{bmatrix}, \quad M_{ss}^1 = \begin{bmatrix} 0 & -\varphi\vartheta \\ (1-\delta) r_{ky} + s_K & 0 \end{bmatrix} \tag{6.43}$$

Solving (6.41) for $\hat{u}_t$, substituting the result into (6.42), and solving the result for $\hat{s}_{t+1}$ gets:

$$\hat{s}_{t+1} = W\hat{s}_t + R\hat{e}_{t+1} + Q\hat{e}_t \tag{6.44}$$

where W, R and Q are adequately defined matrices.

Under our certainty equivalence assumption, randomness can be reintroduced by simply taking expectations of (6.45). Being $E_t(\hat{e}_{t+1}) = \rho\hat{e}_t$, we obtain:

$$E_t(\hat{s}_{t+1}) = W\hat{s}_t + (R\rho + Q)\,\hat{e}_t = W\hat{s} + A\hat{e}_t \tag{6.45}$$

Equation (6.46) is a linear system of expectational difference equations. We can solved it using the standard Blanchard-Khan algorithm (details are omitted). The algorithm provides three matrices, denoted M_v, U_v, and L_v, that characterize the following linear policy functions:

$$\hat{v}_{t+1} = M_v \hat{v}_t + \varepsilon_t, \quad \hat{u}_t = U_v \hat{v}_t, \quad \hat{l}_t = L_v \hat{v}_t \tag{6.46}$$

where $\hat{v}_t \equiv [\hat{k}_t | \hat{a}_t]'$ and $\epsilon_t \equiv [0|\epsilon_t]'$.

The system implied by (6.47) completely characterizes the approximated solution to our stochastic optimal control problem.

Finally, note that there are two other variables of interest we would like to recover, *i.e.* output and investment. We can log-linearize them as (again, check!):

$$y_t = \hat{a}_t + s_K \hat{k}_t, \quad \hat{\imath}_t = \frac{1}{s_i}\hat{a}_t + \frac{s_K}{s_i}\hat{k}_t - \frac{s_c}{s_i}\hat{c}_t \tag{6.47}$$

and write them more compactly as $\hat{f}_t = FV_u \hat{u}_t + FV_v \hat{v}_t + FV_l \hat{l}_t$, where $\hat{f}_t \equiv [\hat{y}_t | \hat{\imath}_t]'$ and:

$$FV_u = \begin{bmatrix} 0 \\ -\frac{s_c}{s_i} \end{bmatrix}, \quad FV_v = \begin{bmatrix} s_K & 1 \\ \frac{s_K}{s_i} & \frac{1}{s_i} \end{bmatrix}, \quad FV_l = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.48}$$

Substituting (6.47), we get $\hat{f}_t = F_v \hat{v}_t$, where $F_v \equiv FV_u U_v + FV_v + FV_l L_v$. For the sake of notational simplicity, define a vector $\hat{h}_t \equiv [\hat{u}_t | \hat{f}_t]'$ such that $\hat{h}_t = H_v \hat{v}_t$, where $H_v \equiv [U_v | F_v]$.

To simulate the model's transitional dynamics, we start by assuming that the system at date 0 is in steady-state, *i.e.* that $\hat{v}_0 = 0$. Iterating on $\hat{v}_t$ and assuming $\epsilon_j = 0$ for $j \geq 1$, we obtain $\hat{v}_t = M_v^t \epsilon_0$. The matrix $d\hat{v}_t / d\epsilon_0' = M_v^t$ summarizes the effect on $\hat{v}_t$ of an unexpected shock at date 0 when $\varepsilon_j = 0$ for $j \geq 1$. The derivative $\partial \hat{v}_{it} / \partial \epsilon_j 0 = [M_v^t]_{ij}$, considered as a function of t, is called the impulse response function for the state variable i when a shock hits the state variable j at date 0. Given the policy functions (6.46), it is easy to recover the impulse response functions of all control variables and all variables of interest, once the impulse response functions for the state variables are available.

Operationally, we start by choosing the state variable to shock. Then, we define an initial vector of innovations, for instance $\varepsilon_0 = [0|1]'$, if we are assuming a 1% increase in TFP. Given ϵ_0 and $\hat{v}_0 = 0$, we obtain $\hat{v}_1$, $\hat{u}_1$ and $\hat{f}_1$ from (6.46). Assuming now $\varepsilon_t = 0 \ \ \forall t \geq 1$, we iterate the procedure for finite number T of periods. Finally, we plot the simulate series, obtaining the impulse response functions.

The projection method

The policy function is approximated over a rectangle $D \equiv [\underline{k}, \overline{k}] \times [\underline{a}, \overline{a}] \in R^2$ with a linear combination of multidimensional basis functions taken from a *2*-fold tensor product of Chebyshev polynomials. In other words, we approximate $c(k, a)$ with:

$$\widehat{c}(k, a; \boldsymbol{\theta}) = \sum_{i=0}^{d} \sum_{j=0}^{d} \theta_{ij} \psi_{ij}(k, a) \tag{6.49}$$

where:

$$\psi_{ij}(k, a) \equiv T_i\left(2\frac{k - \underline{k}}{\overline{k} - \underline{k}} - 1\right) T_j\left(2\frac{a - \underline{a}}{\overline{a} - \underline{a}} - 1\right) \tag{6.50}$$

Given that $\ln(a') = \rho \ln(a) + \sigma\sqrt{2}z$, where $z \sim N(0, 1)$, the Euler equation (6.7) becomes:

$$\int_{-\infty}^{\infty} \hat{c}\left(k', \mathrm{e}^{\rho \ln(a)+\sigma z}; \boldsymbol{\theta}\right)^{-\mu} \left[\alpha \mathrm{e}^{\rho \ln(a)+\sigma z} \left(k'\right)^{\alpha-1} + 1 - \delta\right] \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = \varphi \hat{c}\left(k, \mathrm{e}^{\ln(a)}; \boldsymbol{\theta}\right)^{-\mu} \tag{6.51}$$

where:

$$k' = \frac{\mathrm{e}^{\ln(a)} k^{\alpha} + (1-\delta) k - \hat{c}\left(k, \mathrm{e}^{\ln(a)}; \boldsymbol{\theta}\right)}{\gamma} \tag{6.52}$$

The integral in (6.51) can be numerically approximated using the Gauss-Hermite quadrature formula:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{c}\left(k', \mathrm{e}^{\rho \ln(a)+\sigma z}; \boldsymbol{\theta}\right)^{-\mu} \left[\alpha \mathrm{e}^{\rho \ln(a)+\sigma z} \left(k'\right)^{\alpha-1} + 1 - \delta\right] e^{-\frac{z^2}{2}} dz = \\ &\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \hat{c}\left(k', \mathrm{e}^{\rho \ln(a)+\sigma z}; \boldsymbol{\theta}\right)^{-\mu} \left[\alpha \mathrm{e}^{\rho \ln(a)+\sigma z} \left(k'\right)^{\alpha-1} + 1 - \delta\right] e^{-z^2} dz \approx \\ &\frac{1}{\sqrt{\pi}} \sum_{j=1}^{n} a_j \hat{c}\left(k', \mathrm{e}^{\rho \ln(a)+\sigma z_j}; \boldsymbol{\theta}\right)^{-\mu} \left[\alpha \mathrm{e}^{\rho \ln(a)+\sigma z_j} \left(k'\right)^{\alpha-1} + 1 - \delta\right] \end{aligned} \tag{6.53}$$

where the z_j's and the a_j's are respectively the Gauss-Hermite quadrature nodes and weights.

6.4.2 The standard RBC model

The first order conditions can be easily log-linearized:

$$(\xi_{ll}\omega - s_K)\hat{n}_t - \xi_{lc}\hat{c}_t = -s_K\hat{k}_t - \hat{\lambda}_t - \hat{a}_t \tag{6.54}$$

$$-\xi_{cl}\omega\hat{n}_t + \xi_{cc}\hat{c}_t = \hat{\lambda}_t \tag{6.55}$$

$$-s_L\hat{k}_{t+1} + \varphi\vartheta\hat{\lambda}_{t+1} - \varphi\vartheta\hat{\lambda}_t = -s_N\hat{n}_{t+1} - \hat{a}_{t+1} \tag{6.56}$$

$$-\gamma r_{ky}\hat{k}_{t+1} + [(1-\delta)r_{ky} + s_K]\hat{k}_t = -s_N\hat{n}_t + s_c\hat{c}_t - \hat{a}_t \tag{6.57}$$

where $\xi_{cc} \equiv -\mu$, $\xi_{cl} \equiv \tau(1-\mu)$, $\xi_{lc} \equiv 1-\mu$, $\xi_{ll} \equiv \xi_{cl} - 1$, $\vartheta \equiv r_{ky}/s_K$, and $\omega = n/(1-n)$.

As before, defining $\hat{u}_t \equiv [\hat{n}_t, \hat{c}_t]'$, $\hat{s}_t \equiv [\hat{k}_t, \hat{\lambda}_t]'$, and $\hat{e}_t \equiv [\hat{a}_t]$, we can rewrite (6.54) and (6.55) as:

$$M_{uu}\hat{u}_t = M_{us}\hat{s}_t + M_{ue}\hat{e}_t \tag{6.58}$$

where:

$$M_{uu} = \begin{bmatrix} \xi_{ll}\omega - s_K & -\xi_{lc} \\ -\xi_{cl}\omega & \xi_{cc} \end{bmatrix}, \quad M_{us} = \begin{bmatrix} -s_K & -1 \\ 0 & 1 \end{bmatrix}, \quad M_{ue} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{6.59}$$

Conditions (6.56)-(6.57) can instead be rewritten as:

$$(M_{ss}^0 + M_{ss}^1 L)\hat{s}_{t+1} = (M_{su}^0 + M_{su}^1 L)\hat{u}_{t+1} + (M_{se}^0 + M_{se}^1 L)\hat{e}_{t+1} \tag{6.60}$$

where:

$$M_{ss}^{0} = \begin{bmatrix} -s_N & \varphi\vartheta \\ -\gamma r_{ky} & 0 \end{bmatrix}, \ M_{ss}^{1} = \begin{bmatrix} 0 & -\varphi\vartheta \\ (1-\delta)\, r_{ky} + s_K & 0 \end{bmatrix} \tag{6.61}$$

$$M_{su}^{0} = \begin{bmatrix} -s_N & 0 \\ 0 & 0 \end{bmatrix}, \ M_{su}^{1} = \begin{bmatrix} 0 & 0 \\ -s_N & s_c \end{bmatrix} \tag{6.62}$$

$$M_{se}^{0} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ M_{se}^{1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{6.63}$$

There are two other variables of interest that we would like to recover, namely output and investment. From the following expressions:

$$y_t = a_t k_t^{s_K} n_t^{s_N}, \ \ i_t = a_t k_t^{s_K} n_t^{s_N} - c_t, \ \ p_t = a_t k_t^{s_K} n_t^{s_N - 1} \tag{6.64}$$

we can compute the corresponding log-linear approximations:

$$y_t = \hat{a}_t + s_K \hat{k}_t + s_N \hat{n}_t \tag{6.65}$$

$$\hat{\imath}_t = \frac{1}{s_I}\hat{a}_t + \frac{s_K}{s_I}\hat{k}_t + \frac{s_N}{s_I}\hat{n}_t - \frac{s_C}{s_I}\hat{c}_t \tag{6.66}$$

$$p_t = \hat{a}_t + s_K \hat{k}_t + (s_N - 1)\, \hat{n}_t \tag{6.67}$$

We can write the previous system in a more compact form as:

$$\hat{f}_t = FV_u \hat{u}_t + FV_v \hat{v}_t + FV_l \hat{l}_t \tag{6.68}$$

where $\hat{f}_t \equiv [\hat{y}_t | \hat{\imath}_t]'$, $\hat{v}_t \equiv [\hat{k}_t | \hat{a}_t]'$, and $\hat{l}_t \equiv [\hat{\lambda}_t]$, and:

$$FV_u = \begin{bmatrix} s_N & 0 \\ \frac{s_N}{s_i} & -\frac{s_c}{s_i} \\ s_N - 1 & 0 \end{bmatrix}, \ FV_v = \begin{bmatrix} s_K & 1 \\ \frac{s_K}{s_i} & \frac{1}{s_i} \\ s_K & 1 \end{bmatrix}, \ FV_l = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.69}$$

6.4.3 The Hansen-Rogerson-Wright model

To be added ...

Chapter 7

Incomplete markets models

7.1 Markov chains

A time-invariant, discrete-state Markov chain is characterized by:

- An n-dimensional state space $S = \{\mathrm{s}_1, \mathrm{s}_2, ..., \mathrm{s}_n\}$.
- A $n \times n$ non-negative *transition matrix* Π, such that $\sum_{j=1}^{n} \Pi_{ij} = 1$ for $i = 1, 2, ..., n$. The matrix $\mathbf{\Pi}$ is being a *right stochastic matrix*, and records the transition probabilities from state i into state j:

$$\Pi_{ij} = \text{Prob}\left(x_{t+1} = \mathrm{s}_j | x_t = \mathrm{s}_i\right). \tag{7.1}$$

- A $n \times 1$ non-negative vector π_0, such that $\sum_{i=1}^{n} \pi_{i0} = 1$, representing the initial, unconditional probability distribution on x_0:

$$\pi_{i0} = \text{Prob}\left(x_0 = \mathrm{s}_i\right). \tag{7.2}$$

Note that:

$$\begin{aligned}
&\text{Prob}\left(x_{t+2} = \mathrm{s}_j | x_t = \mathrm{s}_i\right) \\
&\qquad = \sum_{m=1}^{n} \text{Prob}\left(x_{t+2} = \mathrm{s}_j | x_{t+1} = \mathrm{s}_m\right) \text{Prob}\left(x_{t+1} = \mathrm{s}_m | x_t = \mathrm{s}_i\right) \\
&\qquad = \sum_{m=1}^{n} \Pi_{im}\Pi_{mj} = \Pi_{ij}^{(2)},
\end{aligned}$$

where $\Pi_{ij}^{(2)}$ is the (i, j) element of $\mathbf{\Pi}^2$. Hence, in general:

$$\text{Prob}\left(x_{t+k} = \mathrm{s}_j | x_t = \mathrm{s}_i\right) = \Pi_{ij}^{(k)}. \tag{7.3}$$

Remark 81 *This implies that the unconditional distribution of x_t is given by:*

$$\pi_t = \left(\mathbf{\Pi}'\right)^t \pi_0, \tag{7.4}$$

where π_{it} =Prob$(x_{it} = \mathrm{s}_i)$.

Note furthermore that:

$$\pi_{t+1} = \mathbf{\Pi}'\pi_t. \tag{7.5}$$

Definition 82 *An* ***unconditional distribution*** *is called* ***stationary (ergodic)*** *if it remains constant over time, and satisfies:*

$$\pi = \mathbf{\Pi}'\pi. \tag{7.6}$$

Note that equation (7.6) can be rewritten as:

$$(\mathbf{1} - \mathbf{\Pi}')\,\pi = \mathbf{0}. \tag{7.7}$$

In other words, π is just an eigenvector associated with a unit eigenvalue of Π', pinned down by the previously discussed normalization, $\sum_{j=1}^{n} \pi_j = 1$.

Remark 83 *The matrix* $\mathbf{\Pi}$ *is right stochastic, i.e. has nonnegative elements and rows that sum up to one. This implies that* $\mathbf{\Pi}'$ *has at least one (possibly more) unit eigenvalue, and that there is at least one (again, possibly more) eigenvector satisfying equation (7.7).*

Definition 84 *If there is one and only one vector* π_∞ *that satisfies equation (7.7), and:*

$$\lim_{t\to\infty} \pi_t = \pi_\infty. \tag{7.8}$$

for all possible initial distributions π_0*, then the Markov chain is* ***asymptotically stationary with a unique invariant (ergodic) distribution****.*

Theorem 85 *Let* $\mathbf{\Pi}$ *be a right stochastic matrix such that* $\Pi_{ij} > 0$ *for all* (i, j)*: the associated Markov chain is asymptotically stationary and has a unique stationary distribution.*

From an operational point of view, there are three ways to calculate the invariant distribution π_∞ given the stochastic matrix:

1. Iterate until convergence on:
$$\pi_{k+1} = \mathbf{\Pi}'\pi_k. \tag{7.9}$$

2. Calculate the eigenvalues and eigenvectors of $\mathbf{\Pi}'$ and take the normalized eigenvector associated to $\lambda = 1$:
$$\pi_\infty = \frac{v_1}{\sum_{i=1}^{n} v_{1i}}. \tag{7.10}$$

3. Define:
$$\hat{\mathbf{A}} \equiv \left(\mathbf{A}'\mathbf{A}\right)^{-1}\mathbf{A}', \tag{7.11}$$
where:
$$\underset{(n+1)\times n}{\mathbf{A}} \equiv \begin{bmatrix} \mathbf{I}_n - \mathbf{\Pi}' \\ \mathbf{1}_n' \end{bmatrix}, \tag{7.12}$$
and $\mathbf{1}_n$ is a $n \times 1$ vector of ones. It turns out that π_∞ is equal to the $n + 1$ column of $\hat{\mathbf{A}}$.

4. Note that $\mathbf{1}_{n\times n}\pi = \mathbf{1}_n$, where $\mathbf{1}_{n\times n}$ is a $n \times n$ matrix of ones, since π sums to one. Hence: $\mathbf{1}_n = \pi - \mathbf{\Pi}'\pi + \mathbf{1}_{n\times n}\pi = (\mathbf{I}_n - \mathbf{\Pi}' + \mathbf{1}_{n\times n})\,\pi$.This implies that:
$$\pi_\infty = \left(\mathbf{I}_n - \mathbf{\Pi}' + \mathbf{1}_{n\times n}\right)^{-1}\mathbf{1}_n. \tag{7.13}$$

7.2 Bewley models

Bewley models are characterized by a large number of ex-ante identical and ex-post heterogeneous households that trade a set of non-state-contingent securities. For the sake of simplicity, the basic framework will be characterized by the absence of aggregate uncertainty and aggregate dynamics. In other words, the aggregate variables are assumed to be deterministic and constant over time, as in the steady state of a deterministic representative agent economy. Uncertainty, however, plays an essential role at the individual level: idiosyncratic shocks to labour income introduce an incentive for self insurance. The availability of a single, non-contingent asset, will help households to buffer consumption against adverse shocks. This inability of fully insure against bad shocks will generate precautionary savings.

7.2.1 The basic framework

At the individual level, the employment status evolves according to a m-state discrete Markov chain characterized by the state space $S = \{\mathrm{s}_0, \mathrm{s}_1, ..., \mathrm{s}_m\}$ and the transition matrix Π. In the simples case, $S = \{0, 1\}$ so that the individual is employed when $s = 1$ and unemployed when $s = 0$. Hence, in each period the labour income simply corresponds to $w_t s_t$, where w_t is the equilibrium wage rate and s_t the individual employment status.

Households are allowed to invest in a single asset, and a_t denotes individual asset holdings at the beginning of period t. For computational reasons, let us discretize the state space and constrain asset holdings on this finite-dimensional grid:

$$\mathcal{A} = \{-\phi < \mathrm{a}_1 < \mathrm{a}_2 < ... < \mathrm{a}_n\}. \tag{7.14}$$

The parameter $\phi > 0$ represents a borrowing constraint, that may possibly be more stringent than the natural borrowing limit (recall that in general a borrowing constraint is necessary in order to prevent the household form running Ponzi schemes).

Being aggregate dynamics shut down by assumption, factor prices will remain constant over time, i.e. $w_t = w$ and $r_t = r$ for all $t \geq 0$. Hence, given the aggregate factor prices $\{w, r\}$ and the initial conditions $\{a_0, s_0\}$ the household solves the following problem:

$$\begin{aligned} \max_{\{a_{t+1}\}_{t=0}^{\infty}} \quad & U = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right], \\ \text{s.t.} \quad & a_{t+1} = (1+r) a_t + w s_t - c_t, \\ & a_{t+1} \in \mathcal{A}, \end{aligned} \tag{7.15}$$

where $\beta \in (0, 1)$ is the intertemporal discount factor, $u(\cdot)$ is a C^2, strictly increasing and strictly concave Bernoulli function such that $\lim_{c \to 0} du(c)/dc = +\infty$. We also impose that $\beta(1+r) < 1$.

The Bellman equation for the previous recursive problem is (note that the constraint has been substituted into the objective function):

$$v(\mathrm{a}_i, \mathrm{s}_j) = \max_{a' \in \mathcal{A}} \left\{ u[(1+r)\mathrm{a}_i + w\mathrm{s}_j - a'] + \beta \sum_{z=1}^{m} \Pi_{jz} v(a', \mathrm{s}_z) \right\}. \tag{7.16}$$

A solution to this problem can be represented as a *value function* $v(a, s)$ and the associated *policy function* $a' = g(a, s)$. Being the objective function concave and the constraint set convex, there is one and only one optimal future asset stock for each current state vector, i.e. the policy function is a *deterministic* single-value function of the current state vector. This implies that we can define a single-valued *indicator function* such that:

$$\mathcal{I}(a_i, a_h, s_j) = \begin{cases} 1 & \text{if } g(\mathrm{a}_h, \mathrm{s}_j) = \mathrm{a}_i \\ 0 & \text{if } g(\mathrm{a}_h, \mathrm{s}_j) \neq \mathrm{a}_i \end{cases}. \tag{7.17}$$

Numerical strategy

Consider the Bellman equation described in (7.16). To identify the unique value function that solves this equation we can take advantage of the constructive proof of Banach's theorem and the fact that the operator implied by the right hand side of (7.16) turns out to be a contraction, and iterate until convergence on the following recursive scheme, given an initial guess for v_0:

$$v_{k+1}(\mathrm{a}_i, \mathrm{s}_j) = \max_{a' \in \mathcal{A}} \left\{ u\left[(1+r)\,\mathrm{a}_i + w\mathrm{s}_j - a'\right] + \beta \sum_{z=1}^{m} \Pi_{jz} v_k(a', \mathrm{s}_z) \right\}. \tag{7.18}$$

The previous recursive scheme can be represented in matrix notation. Define a set of $n \times 1$ vectors $\mathbf{v}_j$ and $n \times n$ matrices $\mathbf{R}_j$, with $j = 1, 2, ..., m$, such that:

$$\mathbf{v}_j(i) = v(\mathrm{a}_i, \mathrm{s}_j), \tag{7.19}$$

$$\mathbf{R}_j(i, h) = u\left[(1+r)\,\mathrm{a}_i + w\mathrm{s}_j - \mathrm{a}_h\right], \tag{7.20}$$

for all $i = 1, 2, ...n$ and $h = 1, 2, ..., n$. Furthermore, define:

$$\underset{(mn)\times 1}{\mathbf{v}} \equiv \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}, \quad \underset{(mn)\times n}{\mathbf{R}} \equiv \begin{bmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_m \end{bmatrix}. \tag{7.21}$$

The recursive scheme (7.18) can be represented in matrix notation as:[1]

$$\mathbf{v}_{k+1} = \max\left[\mathbf{R} + \beta\left(\mathbf{\Pi} \otimes \mathbf{1}_n\right)\mathbf{v}_k'\right]. \tag{7.22}$$

Note that the policy function, and the indicator function $\mathcal{I}(a_i, a_h, s_j)$, can be represented by a set of $n \times n$ matrices $\mathbf{G}_j$, with $j = 1, 2, ..., m$, such that:

$$G_j(i, h) = \begin{cases} 1 & \text{if } g(\mathrm{a}_i, \mathrm{s}_j) = \mathrm{a}_h \\ 0 & \text{if } g(\mathrm{a}_i, \mathrm{s}_j) \neq \mathrm{a}_h \end{cases}. \tag{7.23}$$

Wealth distribution

Denote as $\boldsymbol{\lambda}_t$ the unconditional probability distribution of the state vector $\{a_t, s_t\}$, and represent it as a $n \times m$ matrix, with non-negative elements that sum to one, such that:

$$\lambda_t(\mathrm{a}_i, \mathrm{s}_j) = \mathrm{Prob}(a_t = \mathrm{a}_i, s_t = \mathrm{s}_j). \tag{7.24}$$

[1]The max operator applied to a $n \times m$ matrix M returns a $n \times 1$ vector whose ith element is the maximum of the ith row of M.

The exogenous Markov chain for s and the optimal policy function $g(a,s)$ *induce* a law of motion for the distribution λ_t:

$$\begin{aligned}
&\underbrace{\text{Prob}\left(a_{t+1}=\mathrm{a}_i, s_{t+1}=\mathrm{s}_j\right)}_{\text{Unconditional } t+1} \qquad (7.25)\\
&= \sum_{z=1}^{m}\sum_{h=1}^{n} \underbrace{\text{Prob}\left(a_{t+1}=\mathrm{a}_i|a_t=\mathrm{a}_h, s_t=\mathrm{s}_z\right)}_{\text{Policy function}} \times \\
&\underbrace{\text{Prob}\left(s_{t+1}=\mathrm{s}_j|s_t=\mathrm{s}_z\right)}_{\text{Transition probability}} \times \\
&\underbrace{\text{Prob}\left(a_t=\mathrm{a}_h, s_t=\mathrm{s}_z\right)}_{\text{Unconditional } t}.
\end{aligned}$$

Note that:

$$\text{Prob}\left(a_{t+1}=\mathrm{a}_i|a_t=\mathrm{a}_h, s_t=\mathrm{s}_z\right) = \begin{cases} 1 & \text{if } g(\mathrm{a}_h,\mathrm{s}_z)=\mathrm{a}_i \\ 0 & \text{if } g(\mathrm{a}_h,\mathrm{s}_z)\neq \mathrm{a}_i \end{cases}, \qquad (7.26)$$

$$\text{Prob}\left(s_{t+1}=\mathrm{s}_j|s_t=\mathrm{s}_z\right) = \Pi_{zj}. \qquad (7.27)$$

The law of motion (7.25) can be compactly rewritten as:

$$\lambda_{t+1}(\mathrm{a}_i,\mathrm{s}_j) = \sum_{z=1}^{m}\sum_{h=1}^{n} G_z(h,i)\,\Pi_{zj}\lambda_t(\mathrm{a}_h,\mathrm{s}_z) = \sum_{z=1}^{m}\sum_{\{a:\mathrm{a}_i=g(a,\mathrm{s}_z)\}} \Pi_{zj}\lambda_t(a,\mathrm{s}_z). \qquad (7.28)$$

Definition 86 *A* ***stationary distribution*** *is a time-invariant distribution λ that solves:*

$$\lambda(\mathrm{a}_i,\mathrm{s}_j) = \sum_{z=1}^{m}\sum_{h=1}^{n} G_z(h,i)\,\Pi_{zj}\lambda(\mathrm{a}_h,\mathrm{s}_z), \qquad (7.29)$$

for $i=1,2,...,n$ and $j=1,2,...,n$.

Note that the relationship (7.29) can be written in matrix notation as:[2]

$$\text{vec}(\boldsymbol{\lambda}) = \mathbf{Q}'\,\text{vec}(\boldsymbol{\lambda}), \qquad (7.30)$$

where:

$$\mathbf{Q} \equiv \begin{bmatrix} \Pi_{11}\mathbf{G}_1 & \Pi_{21}\mathbf{G}_2 & \cdots & \Pi_{m1}\mathbf{G}_m \\ \Pi_{12}\mathbf{G}_1 & \Pi_{22}\mathbf{G}_2 & \cdots & \Pi_{m2}\mathbf{G}_m \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{1m}\mathbf{G}_1 & \Pi_{2m}\mathbf{G}_2 & \cdots & \Pi_{mm}\mathbf{G}_m \end{bmatrix} =$$

$$(\boldsymbol{\Pi}\otimes\mathbf{I}_n)\begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_m \end{bmatrix}. \qquad (7.31)$$

[2] The vec operator transforms the $n\times m$ matrix $\boldsymbol{\lambda}$ into a $mn\times 1$ vector by simply stacking its columns below one another.

The $mn \times mn$ matrix $\mathbf{Q}$ is right stochastic, being $\mathbf{\Pi}$ so by assumption: by comparing (7.6) and (7.30), we immediately realize that $\text{vec}(\boldsymbol{\lambda})$ can be interpreted as the ergodic distribution of a discrete Markov chain characterized by the transition matrix $\mathbf{Q}$, and constructed combining the dynamics of both the exogenous stochastic shock and the endogenous state variable. Note that $\mathbf{Q}$ is quite a sparse matrix: the best way to numerically compute the ergodic distribution $\text{vec}(\boldsymbol{\lambda})$ in this case is to iterate until convergence on the following recursive scheme, taking the sparsity into account:

$$\text{vec}(\boldsymbol{\lambda}_{k+1}) = \mathbf{Q}' \text{vec}(\boldsymbol{\lambda}_k) \tag{7.32}$$

For the *Law of Large Numbers*, the stationary distribution $\boldsymbol{\lambda}$ will reproduce, in the limit, the fraction of time that individual households spend in each state $\{a_i, s_j\}$.

Remark 87 *From the aggregate point of view, if the economy is populated by a continuum of ex-ante identical households, the stationary distribution will reproduce the fraction of the total population in state $\{a_i, s_j\}$ along the stationary equilibrium. In other words, $\boldsymbol{\lambda}$ can be interpreted as the* ***steady-state distribution of financial wealth****.*

7.2.2 Applications

Pure credit (Huggett, 1993)

Huggett (1993) studies the simplest version of the framework described in the previous Sections. Assume that households have access to a centralized loan market in which they can borrow or lend at a constant risk-free interest rate r. No other assets are available in the economy.

Definition 88 *Given ϕ, a* ***stationary equilibrium*** *is an interest rate r, a policy function $a' = g(a,s)$, and a distribution $\lambda(a,s)$, such that:*

1. *$g(a,s)$ solves the household's problem;*
2. *$\lambda(a,s)$ is the stationary distribution induced by Π and $g(a,s)$, i.e. $\lambda(a,s)$ satisfies (7.29) given Π and $g(a,s)$.*
3. *The loan market clears:*

$$\sum_{h=1}^{n} \sum_{z=1}^{m} \lambda(\mathrm{a}_h, \mathrm{s}_z) \underbrace{g(\mathrm{a}_h, \mathrm{s}_z)}_{a'} = 0 \tag{7.33}$$

To compute the equilibrium, we can use the following numerical algorithm:

Algorithm 89

1. *Choose an initial guess for r, say $r_j > 0$ where $j = 0$.*
2. *Given r_j, solve the household problem for $g_j(a,s)$ and $\lambda_j(a,s)$.*
3. *Check whether the loan market clears by computing the excess demand (or supply) of loans:*

$$\sum_{h=1}^{n} \sum_{z=1}^{m} \lambda_j(\mathrm{a}_h, \mathrm{s}_z)\, g_j(\mathrm{a}_h, \mathrm{s}_z) = E_j \tag{7.34}$$

4. *If* $E_j > 0$, *then set* $r_{j+1} < r_j$. *If, instead,* $E_j < 0$, *then set* $r_{j+1} > r_j$.[3]

5. *Iterate until convergence over points* $(2) - (5)$.

Productive capital (Aiyagari, 1994)

Following Aiyagari (1994), we will now study a more developed version of the previous model. Assume that households are allowed to invest in a single, homogenous, capital good, and denote k_t the household's capital holdings. No other assets exist, in particular households are not allowed to borrow or lend on a loan market. Note that in this case the borrowing constraint is redundant since $k \geq 0$ by assumption.

The individual capital stock evolves according to the following accumulation equation:

$$k_{t+1} = (1 - \delta + \tilde{r})\, k_t + w s_t - c_t \tag{7.35}$$

where $\tilde{r} \equiv r + \delta$ is the rental rate and w the competitive wage rate.

Denote $\lambda(k, s)$ the stationary distribution of capital across households; the aggregate per-capita steady-state capital stock and the aggregate employment rate are respectively equal to:

$$K = K' = \sum_{h=1}^{n} \sum_{z=1}^{m} \lambda(\mathrm{k}_h, \mathrm{s}_z) \underbrace{g(\mathrm{k}_h, \mathrm{s}_z)}_{k'} \tag{7.36}$$

$$N = \sum_{h=1}^{n} \sum_{z=1}^{m} \lambda(\mathrm{k}_h, \mathrm{s}_z)\, \mathrm{s}_z = \pi'_\infty S \tag{7.37}$$

where π_∞ is the invariant distribution associated with $\mathbf{\Pi}$ and $S = [\mathrm{s}_0, \mathrm{s}_1, ..., \mathrm{s}_m]'$ the corresponding state space.

A representative competitive firm combines capital and labor to produce the single consumption/investment good via the following aggregate "Cobb-Douglas" production function:

$$Y \equiv F(K, N) = K^\alpha N^{1-\alpha} \tag{7.38}$$

where $\alpha \in (0, 1)$. The first order conditions for the problem of the firm imply that:

$$w = (1 - \alpha)\left(\frac{K}{N}\right)^\alpha \tag{7.39}$$

$$\tilde{r} = \alpha \left(\frac{K}{N}\right)^{\alpha - 1} \tag{7.40}$$

Definition 90 *A* ***stationary equilibrium*** *is a policy function* $k' = g(k, s)$, *a distribution* $\lambda(k, s)$, *and a triple of positive real numbers* $\{K, \tilde{r}, w\}$, *such that:*

1. $g(k, s)$ *solves the household's problem;*

2. $\lambda(k, s)$ *is the stationary distribution induced by* Π *and* $g(k, s)$*;*

3. *The factor prices satisfy conditions (7.39) and (7.40);*

[3] From a practical point of view, the most robust approach (but not the quinckest) is to use

4. *The aggregate capital stock K is implied by the households individual decisions:*

$$K = \sum_{h=1}^{n} \sum_{z=1}^{m} \lambda\left(\mathrm{k}_h, \mathrm{s}_z\right) g\left(\mathrm{k}_h, \mathrm{s}_z\right) \tag{7.41}$$

To compute the equilibrium, we can use the following numerical algorithm:

Algorithm 91

1. *Choose an initial guess for K, say $K_j > 0$ where $j = 0$.*
2. *Compute w_j and $\tilde{r}_j$ from (7.39)-(7.40).*
3. *Given w_j and $\tilde{r}_j$, solve the household problem for $g_j(k, s)$ and $\lambda_j(k, s)$.*
4. *Compute the aggregate capital stock:*

$$\hat{K}_j = \sum_{h=1}^{n} \sum_{z=1}^{m} \lambda_j\left(\mathrm{k}_h, \mathrm{s}_z\right) g_j\left(\mathrm{k}_h, \mathrm{s}_z\right) \tag{7.42}$$

5. *Given a fixed "relaxation" parameter $\kappa \in (0, 1)$, compute a new estimate of K from:*

$$K_{j+1} = \kappa K_j + (1 - \kappa) \hat{K}_j \tag{7.43}$$

6. *Iterate until convergence over points $(2) - (5)$.*

7.2.3 Calibration and simulations

Following Huggett (1993), assume a CES form for the Bernoulli function, $u(c) = c^{1-\mu}/(1-\mu)$, and set $\mu = 2$. Furthermore, set $\beta = 0.97$, $w = 1$, and $\phi = 1$.

Following Heaton and Lucas (1996), assume that labor income follows a stationary autoregressive process:

$$\ln s_{t+1} = \rho \ln s_t + \sigma \sqrt{(1-\rho^2)} \varepsilon_t \tag{7.44}$$

where $\varepsilon_t \sim N(0, 1)$, $\rho = 0.53$, and $\sigma = 0.296$. Using Tauchen's method, we can approximate the previously described continous-state autoregressive process with a finite-state Markov chain.

Finally, following Aiyagari (1994), se set $\alpha = 0.36$ and $\delta = 0.08$.

Pure credit

Figures 7.1-7.4.

Productive capital

Figures 7.5-7.9

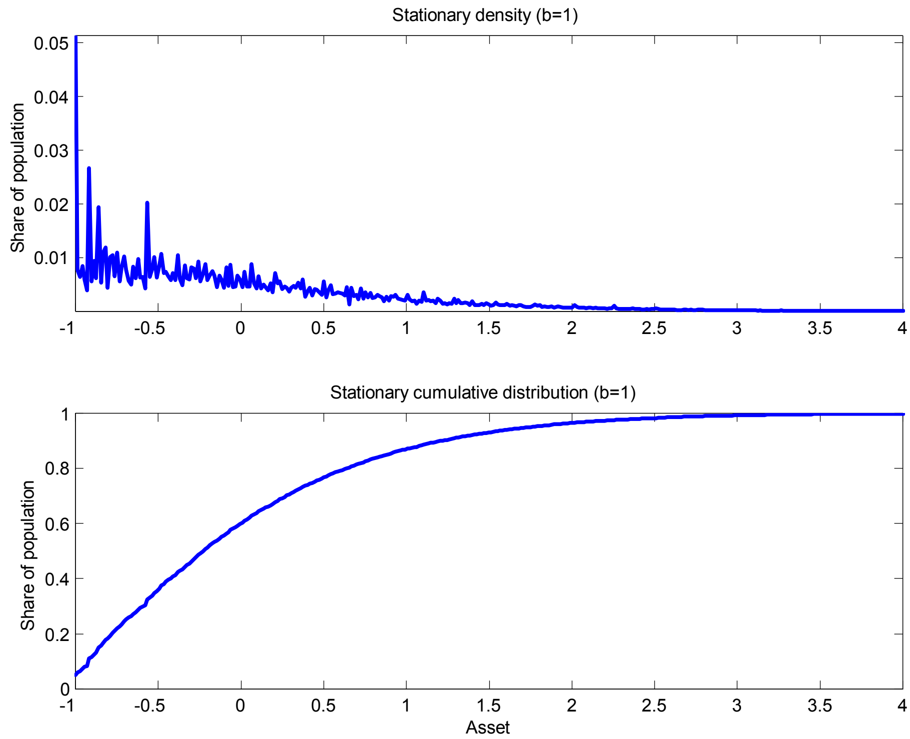

Figure 7.1:

7.3 Aggregate uncertainty

The essential feature that makes Bewley models so tractable is the time-invariance of aggregate state variables. Of course, this assumption is questionable form an empirical and theoretical point of view. Following Krusell and Smith (1998), let us generalize Aiyagari's framework by assuming the existence of an aggregate productivity shock that follows an exogenous Markov process.

Note that under complete markets households would fully insure against the risk of idiosyncratic shocks to labour income, and therefore they could be aggregated into a representative household in charge of solving the following dynamic programming problem:

$$\begin{aligned} v\left(k;K,z\right) &= \max_{\{c,k'\}} u\left(c\right)+\beta E\left[v\left(k';K',z'\right)\mid\{K,z\}\right] \\ \text{s.t.} \quad & k'=\left[1-\delta+\tilde{r}\left(K,z\right)\right]k+w\left(K,z\right)-c \\ & K'=\mathcal{K}\left(K,z\right) \end{aligned} \tag{7.45}$$

In this case, the knowledge of the current aggregate capital stock K and the current productivity level z where enough to predict the future aggregate state of the economy, via the law of motion $\mathcal{K}\left(K,z\right)$ and the properties of the exogenous Markov process governing z. Note that, being all agents identical ex-ante by assumption and ex-post thanks to full

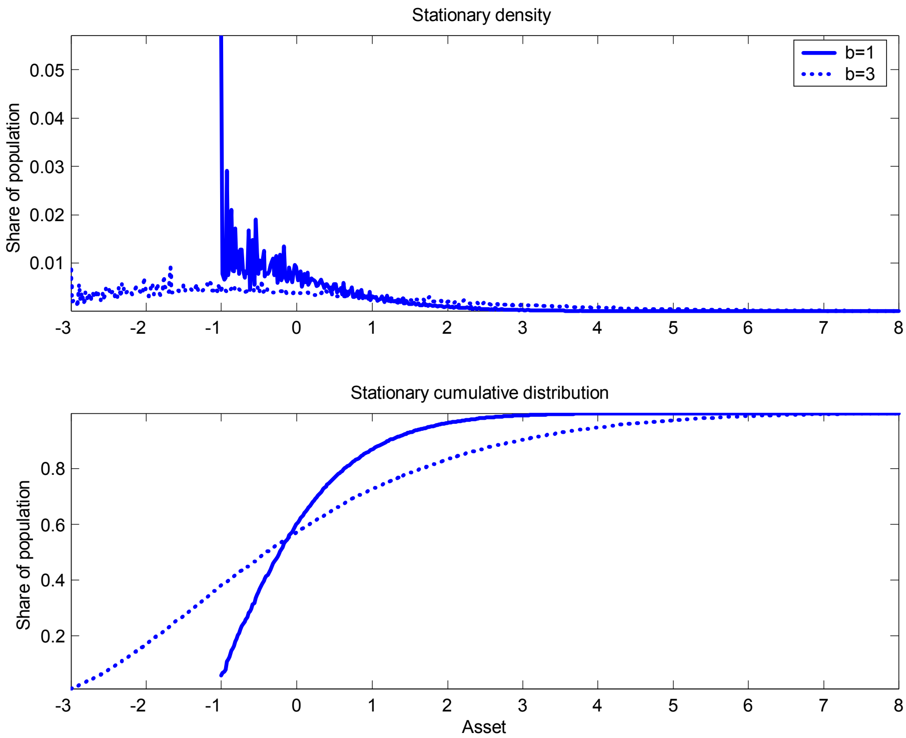

Figure 7.2:

insurance, problem (7.45) can be rewritten as:

$$\begin{aligned} v(k;K,z) &= \max_{\{c,k'\}} u(c) + \beta E\left[v(k';K',z') \mid \{K,z\}\right] \\ \text{s.t.} \quad & k' = \left[1-\delta+\tilde{r}(K,z)\right]k + w(K,z) - c \\ & K' = (1-\delta)K + zf(K) - C \end{aligned} \tag{7.46}$$

where $K = k$ and $C = c$. In other words, each single individual is able to perfectly forecast the future aggregate per-capita capital stock, and therefore the future factor prices, because she knows that all other individuals are identical and will behave in the same way. Hence, even if she still has no control over the aggregate capital stock, she knows that it will be equal to her own future individual capital stock.

Rule now full insurance out, as in Aiyagari's model. In this case, individuals may be heterogenous ex-post, since their individual capital stock will depend not only on the history of the aggregate productivity shock, but also on the full history of their idiosyncratic labour income shocks. Hence, our individual household, in order to exactly predict the future aggregate capital stock, and therefore future factor prices, needs to know the actual distribution of the individual capital stocks and employment status. The state space of the model has to be consistently enriched: the entire distribution $\lambda(k,s)$ becomes an element of the state space.

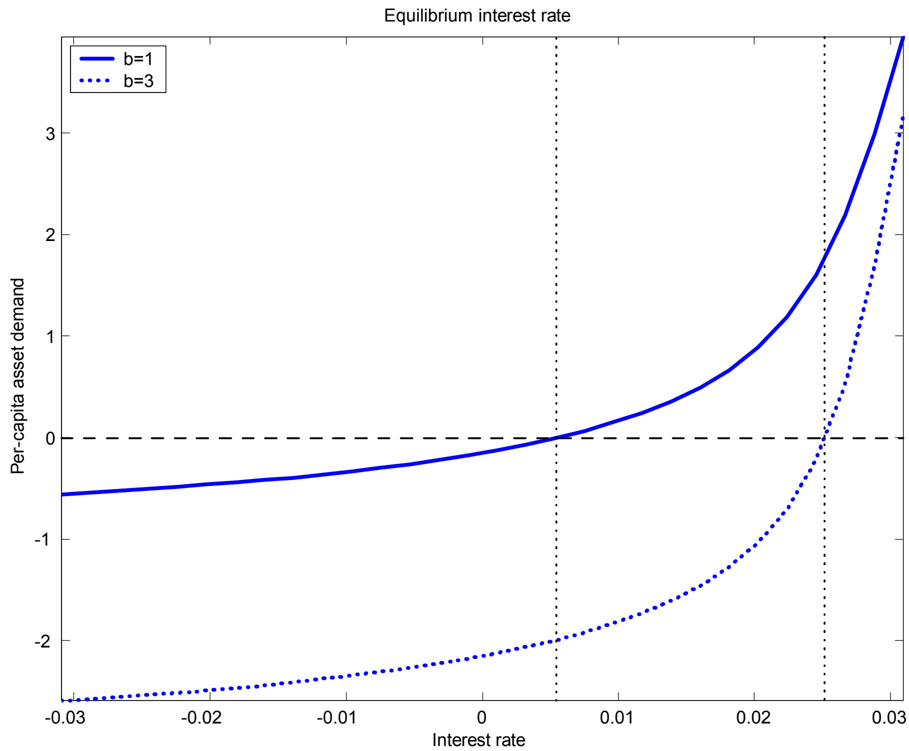

Figure 7.3:

More formally, the household would face the following dynamic programming problem:

$$
\begin{aligned}
v(k,s;\lambda,z) &= \max_{\{c,k'\}} u(c) + \beta E\left[v(k',s';\lambda',z') \,|\, \{s,\lambda,z\}\right] \qquad (7.47)\\
\text{s.t.} \quad & k' = (1-\delta+\tilde{r})k + ws - c\\
& \tilde{r} = zF_K(K,N)\\
& w = zF_K(K,N)\\
& K = \int k\lambda(k,s)\,dkds\\
& N = \int s\lambda(k,s)\,dkds\\
& \lambda' = \mathcal{H}(\lambda,z,z')
\end{aligned}
$$

where $\mathcal{H}$ represents the perceived law of motion that maps the current aggregate state space into the future distribution.

Definition 92 *A* ***recursive competitive equilibrium*** *is a policy function* $g(k,s;\lambda,z)$*, a pair of pricing functions* $\tilde{r}(\lambda,z)$ *and* $w(\lambda,z)$*, and a law of motion* $\mathcal{H}(\lambda,z,z')$*, such that:*

1. $g(k,s;\lambda,z)$ *solves the household's problem (7.47);*

2. $\tilde{r}(\lambda,z)$ *and* $w(\lambda,z)$ *correspond to the competitive equilibrium factor prices;*

3. $\mathcal{H}(\lambda,z,z')$ *is induced by* $g(k,s;\lambda,z)$*.*

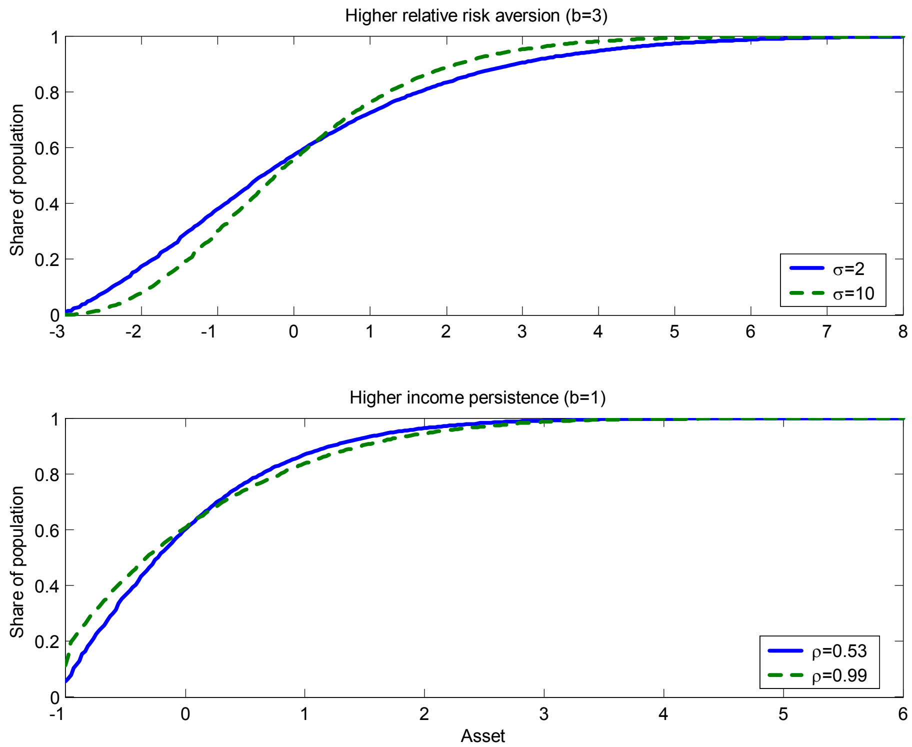

Figure 7.4:

From a numerical point of view, introducing the distribution λ into the state space makes the model literally intractable. One way to go, following Krusell and Smith (1998), is to assume that it is not the distribution itself than is part of the state space, but its first h moments $m = \{m_1, m_2, ..., m_h\}$. Hence, we approximate problem (7.47) with:

$$\begin{aligned} v(k,s;m,z) &= \max_{\{c,k'\}} u(c) + \beta E\left[v(k',s';m',z') \mid \{s,m,z\}\right] \qquad (7.48) \\ \text{s.t.} \quad & k' = (1-\delta+\tilde{r})\,k + ws - c \\ & \tilde{r} = zF_K(K,N) \\ & w = zF_K(K,N) \\ & K = \int k\lambda(k,s)\,dkds \\ & N = \int s\lambda(k,s)\,dkds \\ & m' = \mathcal{H}(m,z,z') \end{aligned}$$

Numerical strategy:

Algorithm 93

1. *Guess a functional form for $\mathcal{H}$ and the corresponding initial parameterization.*
2. *Solve the household's problem for the current $\mathcal{H}_j$.*
3. *Use the policy function g_j to simulate the model for a large number of agents and a large number of periods.*

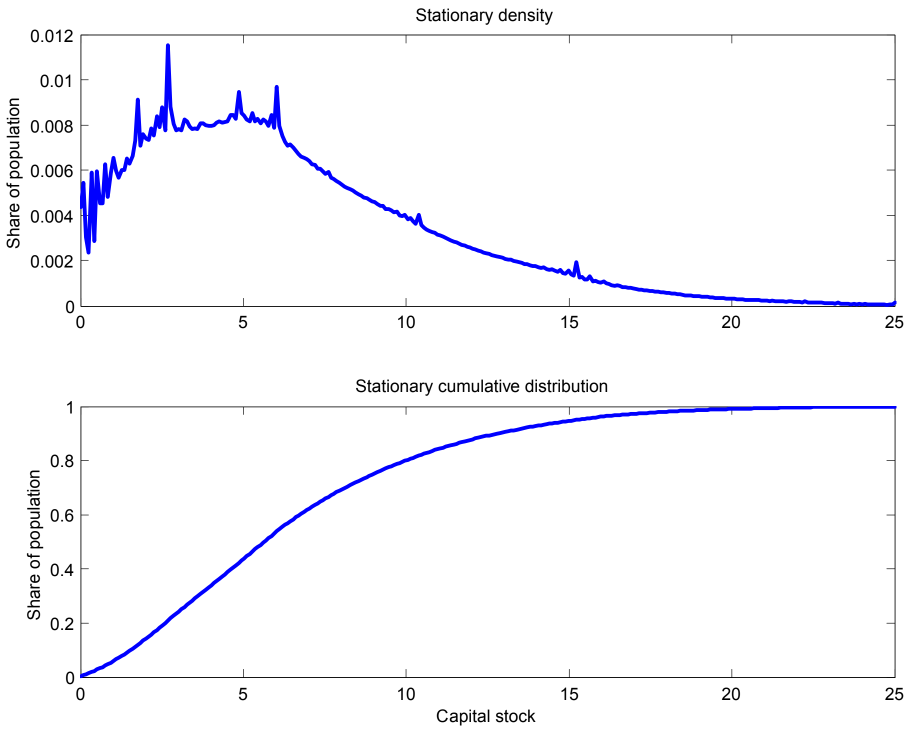

Figure 7.5:

4. *Use the stationary part of the simulate data to estimate the parameters of $\mathcal{H}$ and evaluate the goodness of fit.*

5. *Iterate on* $(2)-(4)$ *until convergence of the parameters of $\mathcal{H}$.*

6. *If the fit is not satisfactory, increase h or change the functional form for $\mathcal{H}$.*

7. *Iterate on* $(2)-(6)$ *until the fit is satisfactory.*

Let us introduce endogenous labour in the framework:

$$
\begin{aligned}
v(k,s;\lambda,z) &= \max_{\{c,n,k'\}} u(c,n)+\beta E\left[v(k',s';\lambda',z')\,|\,\{s,\lambda,z\}\right] \qquad (7.49)\\
\text{s.t.} \quad & k'=(1-\delta+\tilde{r})\,k+wns-c\\
& \tilde{r}=zF_K(K,N)\\
& w=zF_K(K,N)\\
& K=\int k\lambda(k,s)\,dkds\\
& N=\mathcal{N}(\lambda,z)\\
& \lambda'=\mathcal{H}(\lambda,z,z')
\end{aligned}
$$

Definition 94 *A* ***recursive competitive equilibrium*** *is a pair of policy functions $g(k,s;\lambda,z)$ and $l(k,s;\lambda,z)$, a pair of pricing functions $\tilde{r}(\lambda,z)$ and $w(\lambda,z)$, and pair of laws of motion $\mathcal{H}(\lambda,z,z')$ and $\mathcal{N}(\lambda,z)$, such that:*

1. *$g(k,s;\lambda,z)$ and $l(k,s;\lambda,z)$ solve the household's problem (7.49);*

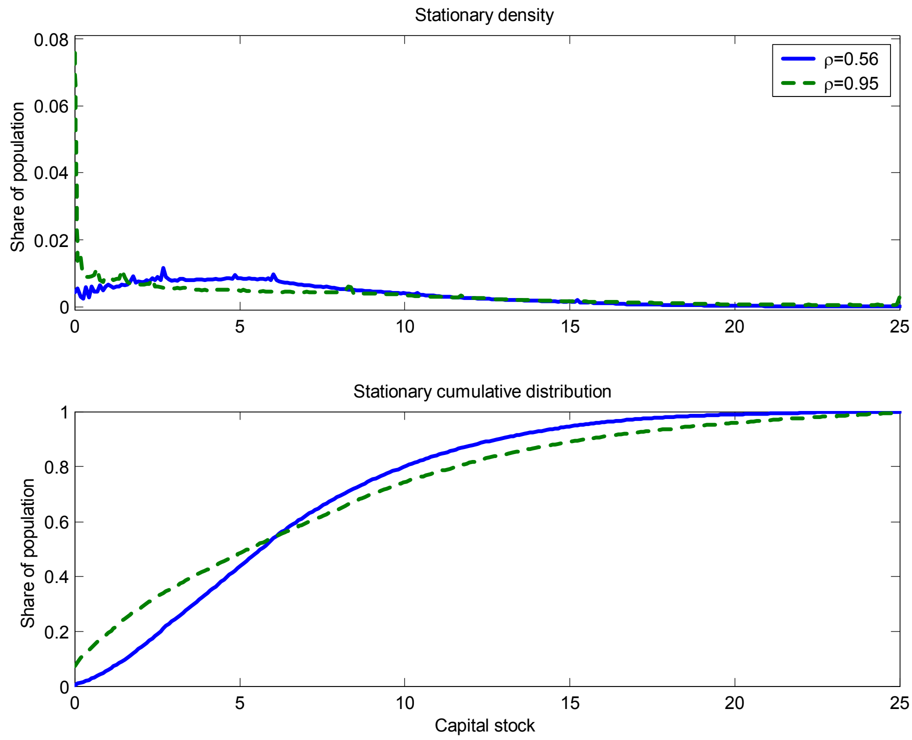

Figure 7.6:

2. *$\tilde{r}(\lambda, z)$ and $w(\lambda, z)$ correspond to the competitive equilibrium factor prices;*

3. *$\mathcal{H}(\lambda, z, z')$ and $\mathcal{N}(\lambda, z)$ are induced by $g(k, s; \lambda, z)$ and $l(k, s; \lambda, z)$.*

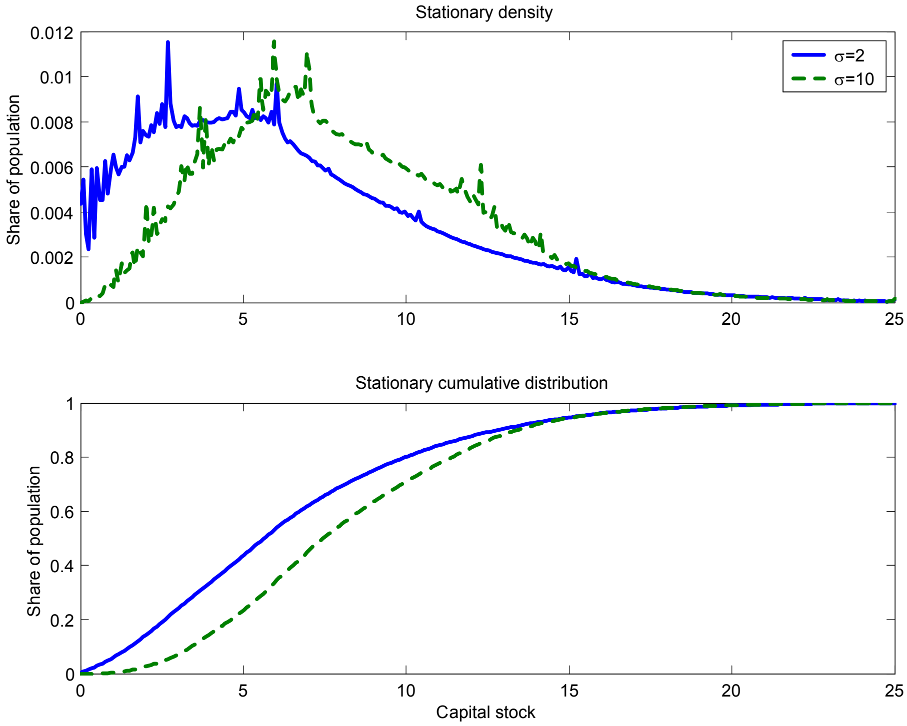

Figure 7.7:

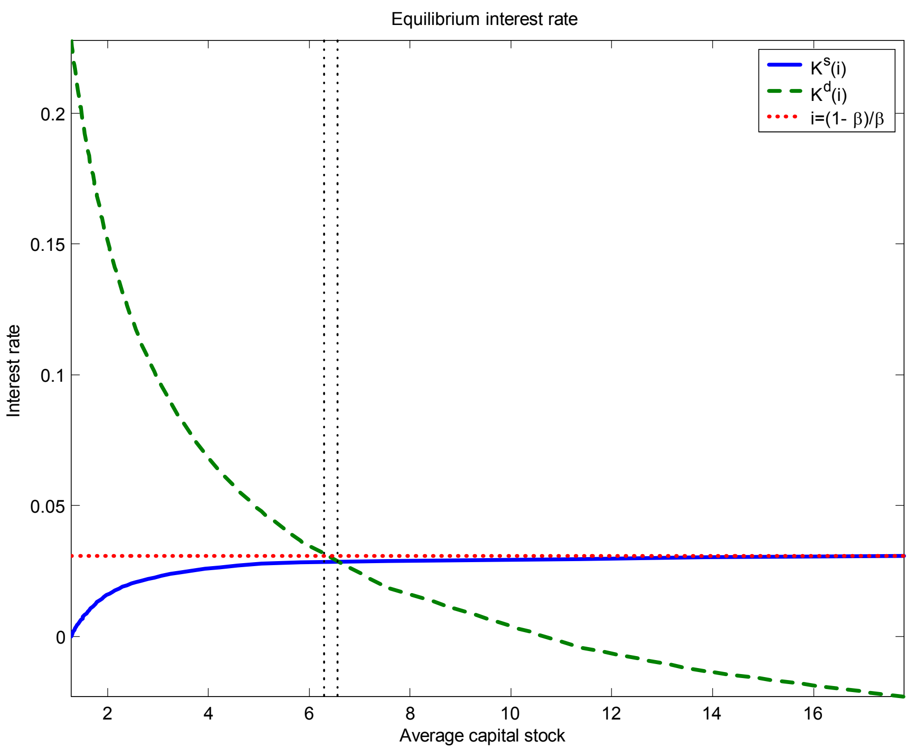

Figure 7.8:

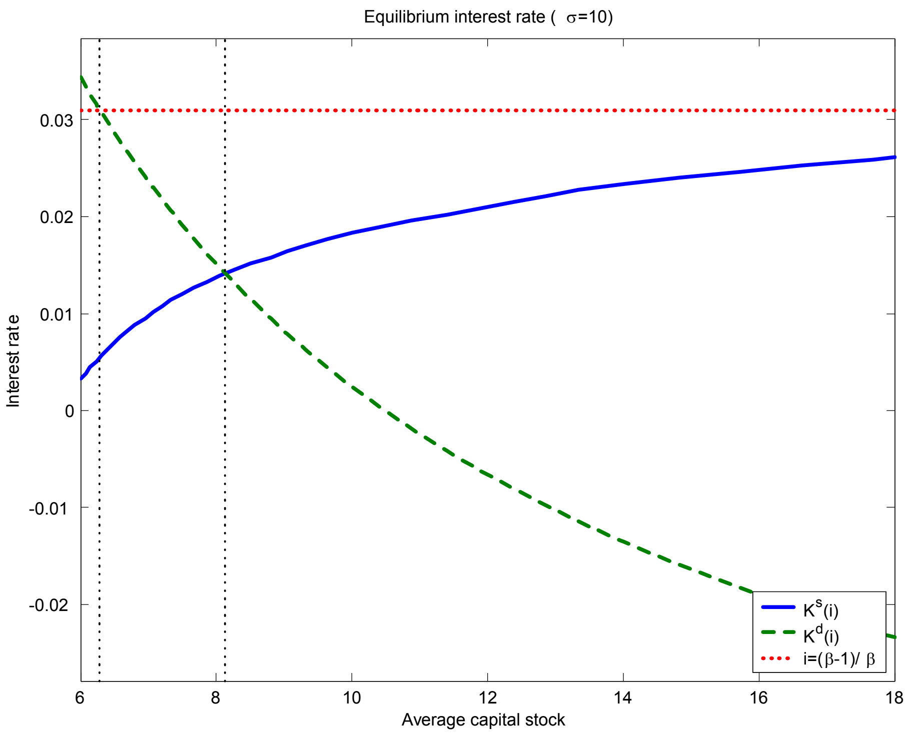
Equilibrium interest rate (σ=10)
0.03
0.02
0.01
0
-0.01
-0.02
Interest rate
6
8
10
12
14
16
18
Average capital stock
K^s(i)
K^d(i)
i=(β-1)/ β

Figure 7.9: