Section V

Bewley models: continuous endogenous states

Households

There exists a continuum of ex-ante identical and infinitely lived households, with total mass equal to one.

Each agent's preferences over consumption streams are given by:

$$v_t \equiv \mathbb{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \right], \qquad (1)$$

where $\beta \in (0, 1)$.

The felicity function satisfies the following regularity conditions:

1. $u : R_+ \to R$ is bounded and continuous;
2. $u' > 0$, $u'' < 0$, and $\lim_{c \to 0} u_c(c) = \infty$;
3. u' is convex.

Households own both factors of production, capital and labor, and rent them to the firms on competitive factor markets.

A single homogenous good can be purchased on a competitive market and used for both consumption and investment.

There is no aggregate uncertainty. However, in each period agents face an idiosyncratic shock to their labor endowment, denoted ε_t.

For the sake of simplicity, assume that $\varepsilon_t \in \mathcal{E} = \{\varepsilon_j\}_{j=1}^m$, where $\varepsilon_j > 0$ $\forall j$; furthermore, assume that ε_t evolves according to a stationary first-order discrete Markov chain, independently across agents.

Let π be the corresponding transition matrix, and let $\pi(\varepsilon_i, \varepsilon_j) > 0$ stand for the probability that $\varepsilon_{t+1} = \varepsilon_j$ if $\varepsilon_t = \varepsilon_i$.

Let w_t denote the hourly wage rate: our agents supply inelastically their labor endowment and therefore earn in each period an amount equal to $w_t \varepsilon_t$.

Asset markets are incomplete and the only form of investment available is physical capital accumulation: hence, households cannot fully insure themselves against the risk of unemployment.

For given sequences of factor prices, the dynamic optimization problem of a generic household is as follows:

$$\max_{\{c_s, k_{s+1}\}_{s=t}^{\infty}} \mathbb{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \right] \tag{2}$$

$$\text{s.t.} \quad k_{t+1} \leq (1 - \delta + r_t) k_t + w_t \varepsilon_t - c_t,$$

$$k_{t+1} \geq 0,$$

The following Lagrangian is easily obtained:

$$L_t = \mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left\{ u\left(c_s\right) + \xi_s \left[\left(1 - \delta + r_s\right) k_s + w_s \varepsilon_s - c_s - k_{s+1} \right] + \varphi_s k_{s+1} \right\}.$$

The stationarity, feasibility, and complementary slackness conditions read as:[1]

$$u_c\left(c_t\right) = \xi_t, \quad (4)$$

$$\xi_t - \varphi_t = \mathbb{E}_t \left[\beta \xi_{t+1} \left(1 - \delta + r_{t+1}\right) \right], \quad (5)$$

$$k_{t+1} \leq \left(1 - \delta + r_t\right) k_t + w_t \varepsilon_t - c_t, \quad (6)$$

$$k_{t+1} \geq 0, \quad (7)$$

$$\varphi_t k_{t+1} = 0, \quad (8)$$

$$\xi_t \left[\left(1 - \delta + r_t\right) k_t + w_t \varepsilon_t - c_t - k_{t+1} \right] = 0, \quad (9)$$

$$\varphi_t \geq 0, \quad (10)$$

$$\xi_t \geq 0. \quad (11)$$

Since $\lim_{c \to 0} u_c(c) = \infty$ and $\varepsilon_t > 0$ by assumption, we anticipate that in equilibrium $\xi_t > 0$ $\forall t$.

Hence, the previous conditions boil down to:

$$u_c(c_t) - \varphi_t = \mathbb{E}_t \left[\beta u_c(c_{t+1}) (1 - \delta + r_{t+1}) \right], \tag{12}$$

$$k_{t+1} = (1 - \delta + r_t) k_t + w_t \varepsilon_t - c_t, \tag{13}$$

$$k_{t+1} \geq 0, \tag{14}$$

$$\varphi_t k_{t+1} = 0, \tag{15}$$

$$\varphi_t \geq 0. \tag{16}$$

Remark 1. The Euler equation can be represented as:

$$\begin{cases} u_c(c_t) = \mathbb{E}_t \left[\beta u_c(c_{t+1}) (1 - \delta + r_{t+1}) \right] & \text{if } k_{t+1} > 0 \\ u_c(c_t) \geq \mathbb{E}_t \left[\beta u_c(c_{t+1}) (1 - \delta + r_{t+1}) \right] & \text{if } k_{t+1} = 0 \end{cases}. \tag{17}$$

Firms

The competitive firms are characterized by a constant-returns-to-scale technology; let K_t and L_t stand for the per-capita aggregate capital stock and labor supply, respectively.

Per-capita aggregate output is given by:

$$Y_t = f(K_t, L_t). \tag{18}$$

The first-order conditions for the representative firm read as:

$$w_t = f_L(K_t, L_t), \tag{19}$$

$$r_t = f_K(K_t, L_t). \tag{20}$$

Aggregate states

The vector of individual state variables $s_t \equiv \{k_t, \varepsilon_t\}$ lies in $\mathcal{X} = [0, \infty) \times \mathcal{E}$.

The distribution of individual states across agents is described by an aggregate state, the probability measure λ_t.

λ_t is the unconditional probability distribution of the state vector $\{k_t, \varepsilon_t\}$, defined over the Borel subset of $\mathcal{X}$:

$$\lambda_t(\mathrm{k}, \varepsilon_j) = \Pr(k_t = \mathrm{k}, \varepsilon_t = \varepsilon_j). \tag{21}$$

For the *Law of Large Numbers*, $\lambda_t(\mathrm{s})$ can be interpreted as the mass of agents whose individual state vector is equal to s. Being λ_t a probability measure, the total mass of agents is equal to one.

The policy function for a generic agent satisfies the Euler equation in recursive form:

$$\begin{cases} u_c\left[c\left(s;\lambda\right)\right] = \beta\mathbb{E}\left\{u_c\left[c\left(s';\lambda'\right)\right]\left[1-\delta+r\left(\lambda'\right)\right] \mid s\right\} & \text{if} \quad k'>0 \\ u_c\left[c\left(s;\lambda\right)\right] \geq \beta\mathbb{E}\left\{u_c\left[c\left(s';\lambda'\right)\right]\left[1-\delta+r\left(\lambda'\right)\right] \mid s\right\} & \text{if} \quad k'=0 \end{cases}, \qquad (22)$$

where $s = \{k, \varepsilon\}$, and:

$$k'\left(s;\lambda\right) = \left[1-\delta+r\left(\lambda\right)\right]k + w\varepsilon - c\left(s;\lambda\right). \qquad (23)$$

Remark 2. Note that:

$$\mathbb{E}\left\{u_c\left[c\left(s';\lambda'\right)\right]\left[1-\delta+r\left(\lambda'\right)\right] \mid k,\varepsilon_j\right\} = \sum_{z=1}^{m}\pi\left(\varepsilon_j,\varepsilon_z\right)u_c\left[c\left(k',\varepsilon_z;\lambda'\right)\right]\left[1-\delta+r\left(\lambda'\right)\right]. \qquad (24)$$

The exogenous Markov process driving ε and the optimal policy function $c(s;\lambda)$ *induce* a law of motion for the distribution λ:

$$\lambda'(\mathrm{k},\varepsilon_{\mathrm{j}}) = \sum_{z=1}^{m} \int \mathcal{I}(\mathrm{k},k,\varepsilon_z)\,\pi(\varepsilon_z,\varepsilon_j)\,\lambda(k,\varepsilon_z)\,dk = \int_{\mathcal{X}} \mathcal{I}(\mathrm{k},k,\varepsilon)\,\pi(\varepsilon,\varepsilon_j)\,d\lambda, \tag{25}$$

where:

$$\mathcal{I}(\mathrm{k},k,\varepsilon_j) = \begin{cases} 1 & \text{if } k'(k,\varepsilon_j;\lambda) = \mathrm{k} \\ 0 & \text{if } k'(k,\varepsilon_j;\lambda) \neq \mathrm{k} \end{cases}. \tag{26}$$

Definition of recursive equilibrium

Definition 1. A *recursive equilibrium* is a policy function $c(s;\lambda)$, a couple of sequences $\{w_t, r_t\}$, and a sequence of probability distributions $\{\lambda_t\}$ such that:

1. The policy function $c(s;\lambda)$ solves the individual optimization problem (2).

2. The factor prices $\{w_t, r_t\}$, together with the implied aggregate capital stock $K_t = \int_{\mathcal{X}} k d\lambda_t$ and labor endowment $L_t = \int_{\mathcal{X}} \varepsilon d\lambda_t$, satisfy the first order conditions for the firm, i.e. equations (19) and (20), $\forall t \geq 0$.

3. The market for the final good clears:

$$\int_{\mathcal{X}} \left[c\left(s;\lambda_t\right)+k'\left(s;\lambda_t\right)\right]d\lambda_t=\left(1-\delta\right)K_t+f\left(K_t,L_t\right),\quad \forall t\geq 0.$$

4. The sequence of distributions satisfies the induced law of motion:

$$\lambda_{t+1}\left(\mathrm{k},\varepsilon_{\mathrm{j}}\right)=\int_{\mathcal{X}}\mathcal{I}\left(\mathrm{k},k,\varepsilon\right)\pi\left(\varepsilon,\varepsilon_j\right)d\lambda_t,\quad \forall t\geq 0,\ \forall \mathrm{s}\in X.$$

Given the absence of aggregate uncertainty, in the long run the economy will reach a steady state characterized by a constant aggregate capital stock.

A *stationary equilibrium*, i.e. a steady state, is an equilibrium where factor prices remain constant over time and the probability distribution becomes time invariant.

Remark 3. Note that, in the steady state, the aggregate labor endowment is given by $L = \int_{\mathcal{X}} \varepsilon d\lambda = \pi'_{\infty}\mathcal{E}$, where π_{∞} is the ergodic distribution of the Markov process driving ε.

Definition 2. A *stationary recursive equilibrium* is a policy function $c(s)$, a couple of values $\{w, r\}$, and a probability distribution λ such that:

1. The policy function $c(s)$ solves the individual optimization problem (2).

2. The factor prices $\{w, r\}$, together with $K = \int_{\mathcal{X}} k d\lambda$ and $L = \pi'_{\infty}\mathcal{E}$, satisfy the first order conditions for the firm.

3. The market for the final good clears:

$$\int_{\mathcal{X}} [c(s) + k'(s)] \, d\lambda = (1 - \delta) K + f(K, L).$$

4. The distribution satisfies the induced law of motion:

$$\lambda(\mathrm{k}, \varepsilon_{\mathrm{j}}) = \int_X \mathcal{I}(\mathrm{k}, k, \varepsilon) \, \pi(\varepsilon, \varepsilon_j) \, d\lambda, \quad \forall \mathrm{s} \in X.$$

Solving for the equilibrium

Algorithm 1. *Given the aggregate labor endowment $L = \pi'_{\infty}\mathcal{E}$, and an initial guess for K, say $K_0 > 0$:*

1. *Compute w_j and r_j, where j denotes the current iteration, from (19) and (20) for the current guess K_j.*

2. *Solve the household problem for the individual policy function $c_j(s)$.*

3. *Compute the implied stationary distribution $\lambda_j(s)$.*

4. *Compute the implied aggregate capital stock:*

$$\hat{K}_j = \int_{\mathcal{X}} k d\lambda_j. \tag{27}$$

5. *Given* $\hat{K}$, *compute a new guess for* K:

$$K_{j+1} = \upsilon \hat{K}_j + (1 - \upsilon) K_j \tag{28}$$

where $\upsilon \in (0, 1)$ *is a damping parameter, and iterate until convergence over points* $(1) - (6)$.

Remark 4. From a numerical point of view, a more efficient way to solve the fixed point problem described in the previous Algorithm is to define a function that computes K_{j+1} given K_j and use bisection - or any other robust univariate solution method like Ridder's or Brent's ones - to solve for the fixed point up to the desired precision.

Solving the household's problem

Algorithm 2. *Define a grid for the individual capital stock on* R_+*, say* $\mathbf{k} = \{k_i\}_{i=1}^n$*, where* $k_1 = 0$ *and* $k_n = \bar{k} > 0$*. Furthermore, choose an initial guess for the optimal consumption levels at each grid point, i.e.* m *vectors* $\mathbf{c}_{z,0} = \{c_{z,0,i}\}_{i=1}^n$*, one for each possible realization of the labor endowment shock.*

1. *Given the current guess* $\mathbf{c}_{z,j}$*, where* j *denotes the current iteration, compute the implied future individual capital stock:*

$$\mathbf{k}'_{z,j} = \max\left(\mathbf{y}_z - \mathbf{c}_{z,j}, 0\right), \tag{29}$$

where:

$$\mathbf{y}_z \equiv \left(1 - \delta + r\right)\mathbf{k} + w\varepsilon_z. \tag{30}$$

2. *Given the previously obtained vector* $\mathbf{k}'_{z,j}$, *compute the future optimal consumption levels* $\mathbf{c}'_{q,j}$, *where* $q = 1, 2, ..., m$, *via interpolation (or extrapolation, if needed) on* $\mathbf{k}$ *and* $\mathbf{c}_{q,j}$.

3. *Compute the right hand side of (22) as:*

$$\hat{\mathbf{c}}_{z,j} = \min\left\{u_c^{-1}\left[\beta(1-\delta+r)\sum_{q=1}^{m}\pi(\varepsilon_z,\varepsilon_q)\,u_c\left(\mathbf{c}'_{q,j}\right)\right],\mathbf{y}_z\right\}. \quad (31)$$

4. *Update the guess for* $\mathbf{c}_{z,j}$ *as follows:*

$$\mathbf{c}_{z,j+1} = \upsilon\hat{\mathbf{c}}_{z,j} + (1-\upsilon)\,\mathbf{c}_{z,j}, \quad (32)$$

where $\upsilon \in (0,1)$ *is a damping parameter, and iterate on points* $(1)-(4)$ *until convergence.*

Computing the stationary distribution

The stationary distribution can be computed using a simple "binning" approach.

In other words, the continous distribution λ is approximated with a discrete histogram over a fixed and uniformly distributed grid on $\left[0, \bar{k}\right] \times \mathcal{E}$, say $\{k_i\}_{i=1}^{N} \times \mathcal{E}$, where $k_1 = 0$, $k_N = \bar{k}$, and possibly $N \geq n$.

The histogram can be described as a $(N \times m)$ matrix λ, whose element $\lambda(i, j)$ represents the share of households with capital holdings i and labor endowment j at the beginning of each period.

Hence, the aggregate capital stock can be approximated by:

$$K \approx \sum_{i=1}^{N} \sum_{j=1}^{m} k'_{i,j} \lambda(i,j), \tag{33}$$

where $k'_{i,j}$ denotes the optimal future capital stock at the node $\{k_i, \varepsilon_j\}$, which lies on the previously defined grid, and can be obtained by interpolating the policy function.

Suppose that a strictly positive mass of households, say υ, saves an amount k' such that $k_z \leq k' \leq k_{z+1}$ for some $z \in \{1, 2, ..., N\}$.

The key step in our discrete approximation is to allocate the mass υ to the nodes k_z and k_{z+1} in such a way that the aggregate capital stock remains unaffected.

If ω_z denotes the share of households that end up at node k_z, then the previous requirement boils down to the following constraint:

$$\omega_z k_z + (1 - \omega_z) k_{z+1} = k'.$$

Hence, the mass υ is distributed according to the following rule:[2]

$$\omega_z(k') = \begin{cases} \frac{k'-k_{z-1}}{k_z-k_{z-1}} & \text{if } k_{z-1} \leq k' \leq k_z \\ \frac{k_{z+1}-k'}{k_{z+1}-k_z} & \text{if } k_z < k' \leq k_{z+1} \\ 0 & \text{otherwise} \end{cases}. \tag{34}$$

Remark 5. Note that $\omega_z(k') \geq 0\ \forall z$, $\omega_z(k') > 0$ for at most two values of z, and $\sum_{l=1}^{m} \omega_z\left(\mathrm{k}'_{l,j}\right) = 1\ \forall z, j$.

[2]Note that the two special cases $z = 1$ and $z = N$ have to be taken care separately: if $z = 1$, then $\omega(1, k') = 1 - (k' - k_1)/(k_2 - k_1)$ if $k_1 \leq k' \leq k_2$ and $\omega(1, k') = 0$ otherwise; if $z = N$, then $\omega(N, k') = (k' - k_{N-1})/(k_N - k_{N-1})$ if $k_{N-1} \leq k' \leq k_N$, $\omega(N, k') = 1$ if $k' > k_N$, and $\omega(N, k') = 0$ otherwise.

The law of motion for the wealth distribution described in (25) boils down to the following relationship:

$$\lambda(z,l) = \sum_{j=1}^{m}\sum_{i=1}^{N} \omega_z\left(\mathrm{k}'_{i,j}\right)\pi(\varepsilon_j,\varepsilon_l)\,\lambda(i,j)\,. \tag{35}$$

For the sake of computational convenience, a version of (35) expressed in matrix form is needed.

Let us start from the following obvious remark:

$$\lambda(z,l) = \pi(\varepsilon_1,\varepsilon_l)\begin{bmatrix}\omega_z\left(\mathrm{k}'_{1,1}\right) & \cdots & \omega_z\left(\mathrm{k}'_{N,1}\right)\end{bmatrix}\begin{bmatrix}\lambda(1,1)\\ \vdots \\ \lambda(N,1)\end{bmatrix} + \ldots$$
$$+\,\pi(\varepsilon_m,\varepsilon_l)\begin{bmatrix}\omega_z\left(\mathrm{k}'_{1,m}\right) & \cdots & \omega_z\left(\mathrm{k}'_{N,m}\right)\end{bmatrix}\begin{bmatrix}\lambda(1,m)\\ \vdots \\ \lambda(N,m)\end{bmatrix}. \tag{36}$$

We can rewrite (36) more compactly as:[3]

$$\lambda(z,l) = \left[\begin{array}{c|c|c} \pi(\varepsilon_1,\varepsilon_l)\,\mathbf{g}_{z,1} & \cdots & \pi(\varepsilon_m,\varepsilon_l)\,\mathbf{g}_{z,m} \end{array} \right] \mathrm{vec}(\lambda), \tag{37}$$

where:

$$\mathbf{g}_{z,j} \equiv \left[\begin{array}{ccc} \omega_z\left(\mathrm{k}'_{1,j}\right) & \cdots & \omega_z\left(\mathrm{k}'_{N,j}\right) \end{array} \right].$$

By simply stacking (37) for $z = 1, 2, ..., N$, we get that:

$$\left[\begin{array}{c} \lambda(1,l) \\ \vdots \\ \lambda(N,l) \end{array} \right] = \left[\begin{array}{c|c|c} \pi(\varepsilon_1,\varepsilon_l)\,\mathbf{G}_1 & \cdots & \pi(\varepsilon_m,\varepsilon_l)\,\mathbf{G}_m \end{array} \right] \mathrm{vec}(\lambda)$$

where:

$$\mathbf{G}_j = \left[\begin{array}{c} \mathbf{g}_{1,j} \\ \vdots \\ \mathbf{g}_{N,j} \end{array} \right].$$

Hence, we finally conclude that the law of motion (35) can be written in matrix form as:

$$\text{vec}(\lambda) = \mathbf{P}' \text{vec}(\lambda), \tag{38}$$

where:

$$\mathbf{P} \equiv \begin{bmatrix} \pi(1,1)\mathbf{G}_1 & \pi(1,2)\mathbf{G}_1 & \cdots & \pi(1,m)\mathbf{G}_1 \\ \pi(2,1)\mathbf{G}_2 & \pi(2,2)\mathbf{G}_2 & \cdots & \pi(2,m)\mathbf{G}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \pi(m,1)\mathbf{G}_m & \pi(m,2)\mathbf{G}_m & \cdots & \pi(m,m)\mathbf{G}_m \end{bmatrix} = (\pi \otimes \mathbf{I}_N) \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_m \end{bmatrix}. \tag{39}$$

The $Nm \times Nm$ matrix $\mathbf{P}$ is *right stochastic*, being π so by assumption.

$\text{vec}(\lambda)$ can be interpreted as the ergodic distribution of a discrete Markov chain characterized by the transition matrix $\mathbf{P}$.

Note that $\mathbf{P}$ is quite a sparse matrix: the best way to numerically compute the ergodic distribution $\text{vec}(\lambda)$ in this case is to iterate until convergence on the following recursive scheme, taking the sparsity into account:

$$\text{vec}(\lambda_{k+1}) = \mathbf{P}' \, \text{vec}(\lambda_k) \tag{40}$$

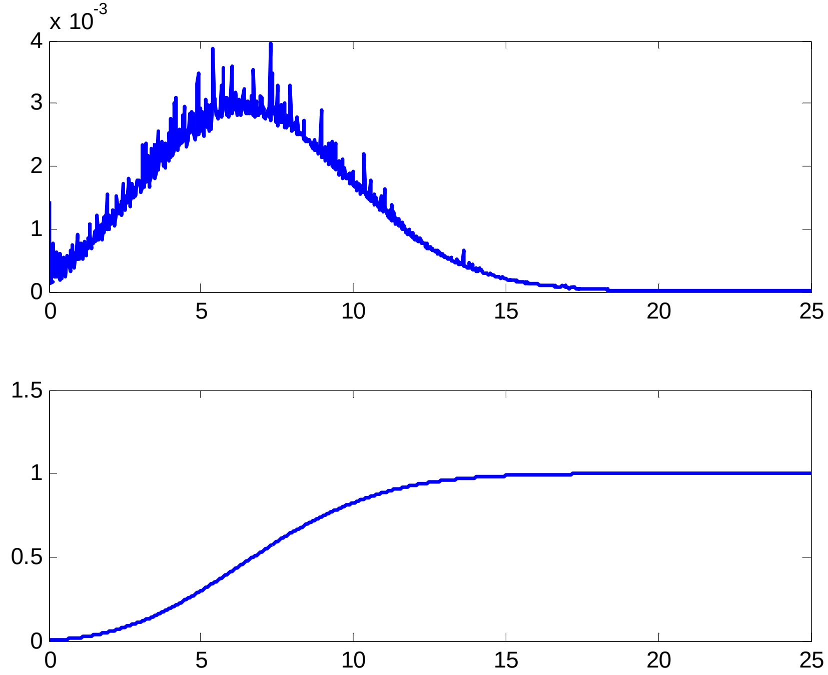
- Following Huggett (1993): CES form for the Bernoulli function, $u(c)=c^{1-\mu}/(1-\mu)$
- Set $\mu=2$, $\beta=0.97$, $w=1$, and $\varphi=1$
- Following Heaton and Lucas (1996): the employment status follows a stationary autoregressive process:

$$\ln s_{t+1} = \rho \ln s_t + \sigma \sqrt{(1-\rho^2)}\, \varepsilon_t$$

where $\varepsilon_t \sim N(0,1)$, $\rho=0.53$, and $\sigma=0.296$

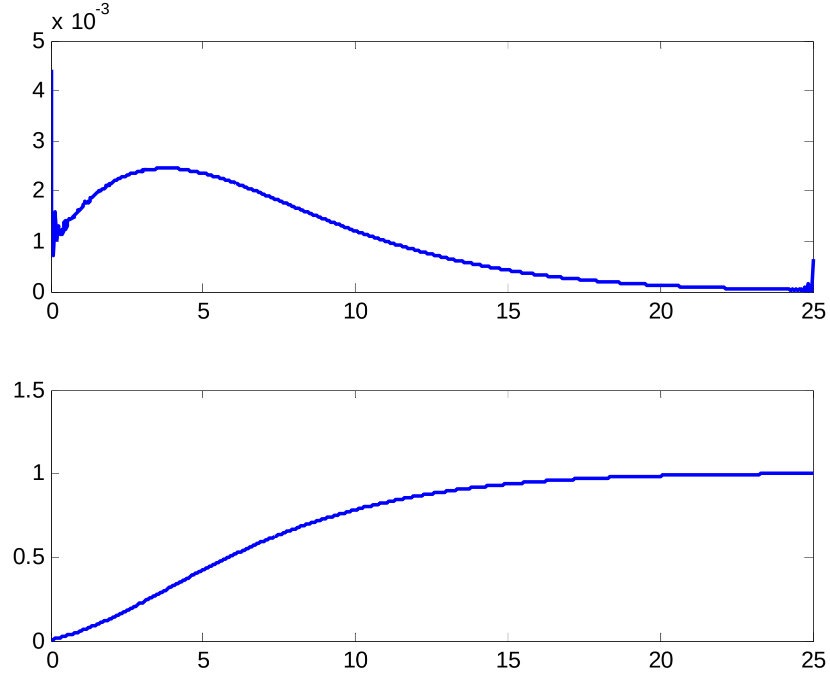
- Using Rouwenhorst's methods, we approximate the continuous-state autoregressive process with a finite-state Markov chain
- Finally, following Aiyagari (1994): $\alpha=0.36$ and $\delta=0.08$

Discretized endo. state + Value Func. Iter.



Nodes for *k* and *k'*: 1000
Realizations of shock: 5
Running time: 27.1 seconds

Continuous endo. state + Fixed Point Iter. on Euler eq.



Nodes for *k*: 1000
Realizations of shock: 5
Running time: 23.9 seconds

Let us now discuss how to compute the transition between two steady states.

Assume that at time 0 the economy is in a steady state characterized by a policy function c_0, a couple of values $\{w_0, r_0\}$, and a distribution λ_0.

Suddenly, and unexpectedly, a change in one of the parameters governing the economy occurs at date 1 and drives the economy to a new steady state.

Even if convergence takes place only in the limit, let us assume that it takes a finite number of periods, say $T = 1000$, to reach the new steady state, characterized by a policy function c_T, a couple of values $\{w_T, r_T\}$, and a distribution λ_T.

The goal of the exercise is to find sequences of policy functions $\{c_t\}_{t=0}^{T}$, prices $\{w_t, r_t\}_{t=0}^{T}$, and distributions $\{\lambda_t\}_{t=0}^{T}$ that satisfy the previously discussed definition of a recursive equilibrium.

Algorithm 3. *The procedure works as follows:*

1. *Compute the initial steady state:* c_0, $\{w_0, r_0\}$, *and* λ_0.

2. *Compute the final steady state:* c_T, $\{w_T, r_T\}$, *and* λ_T.

3. *Once the initial and final steady states are known, guess sequences of factor prices* $\{w_t, r_t\}_{t=1}^{T-1}$, *(i.e. guess sequences of* K_t *and* N_t*), and:*

   (a) *For* $t = T, T-1, ..., 2$, *and for given* c_t, *solve the Euler equation for* c_{t-1}*:*

$$\begin{cases} u_c\left[c_{t-1}\left(s_{t-1}\right)\right] = \beta\mathbb{E}\left\{u_c\left[c_t\left(s_t\right)\right]\left(1-\delta+r_t\right) \mid s_{t-1}\right\} & \text{if } k_t > 0 \\ u_c\left[c_{t-1}\left(s_{t-1}\right)\right] \geq \beta\mathbb{E}\left\{u_c\left[c_t\left(s_t\right)\right]\left(1-\delta+r_t\right) \mid s_{t-1}\right\} & \text{if } k_t = 0 \end{cases}, \tag{41}$$

where:

$$k_t = \left(1-\delta+r_{t-1}\right)k_{t-1} + w_{t-1}\varepsilon_{t-1} - c_{t-1}\left(s_{t-1}\right). \tag{42}$$

(b) Given the previously computed sequence of policy functions, $\{c_t\}_{t=0}^{T}$*, and the initial* λ_0*, compute iteratively the dynamics of the wealth distribution during the convergence process:*

$$\lambda_{t+1}(\mathrm{s}) = \int_X \mathcal{I}_t(\mathrm{k}, k, \varepsilon)\, \pi(\mathrm{z}, \varepsilon)\, d\lambda_t, \quad \forall t \in [0, T-1], \ \forall \mathrm{s} \in X. \tag{43}$$

where:

$$\mathcal{I}_t(\mathrm{k}, k, \varepsilon) = \begin{cases} 1 & \text{if } k'_t = \mathrm{k} \\ 0 & \text{if } k'_t \neq \mathrm{k} \end{cases}. \tag{44}$$

(c) *Given the sequence of distributions, compute* $K_t = \int_X k d\lambda_t$ *and* $L_t = \int_X \varepsilon d\lambda_t$*, and the implied sequences* $\{w_t, r_t\}_{t=1}^{T-1}$.

(d) *If the initial guess and the newly computed sequences do not coincide, update your guess.*

4. *Iterate on (a)-(d) until convergence.*

Solving for the policy functions

Algorithm 4. *Given the grid* $\mathbf{k} = \{k_i\}_{i=1}^{n}$ *and the* m *vectors* $\bar{\mathbf{c}}'_z$*:*

1. *Given the current guess* $\mathbf{c}_{z,j}$*, where* j *denotes the current iteration, compute the implied future individual capital stock:*

$$\mathbf{k}'_{z,j} = \max\left(\mathbf{y}_z - \mathbf{c}_{z,j}, 0\right), \tag{45}$$

where:

$$\mathbf{y}_z \equiv \left(1 - \delta + r\right)\mathbf{k} + w\varepsilon_z. \tag{46}$$

2. *Given the previously obtained vector* $\mathbf{k}'_{z,j}$*, compute the future optimal consumption levels* $\mathbf{c}'_{q,j}$*, where* $q = 1, 2, ..., m$*, via interpolation (or extrapolation, if needed) on* $\mathbf{k}$ *and* $\bar{\mathbf{c}}'_q$*.*

3. Compute the right hand side of (22) as:

$$\hat{\mathbf{c}}_{z,j} = \min \left\{ u_c^{-1} \left[\beta \left(1 - \delta + r\right) \sum_{q=1}^{m} \pi \left(\varepsilon_z, \varepsilon_q\right) u_c \left(\mathbf{c}'_{q,j}\right) \right], \mathbf{y}_z \right\}. \quad (47)$$

4. Update the guess for $\mathbf{c}_{z,j}$ *as follows:*

$$\mathbf{c}_{z,j+1} = \upsilon \hat{\mathbf{c}}_{z,j} + \left(1 - \upsilon\right) \mathbf{c}_{z,j}, \quad (48)$$

where $\upsilon \in (0, 1)$ *is a damping parameter, and iterate on points* $(1) - (4)$ *until convergence.*

Transition after a 1% positive permanent productivity shock