Section VI

Aggregate uncertainty: Krusell and Smith

- The essential feature that makes Bewley models so tractable is the time-invariance of aggregate state variables

- Following Krusell and Smith (1998), let us generalize Aiyagari's framework by assuming the existence of an aggregate productivity shock that follows an exogenous Markov process

- Under complete markets households would fully insure against the risk of idiosyncratic shocks to labor income, and therefore they could be aggregated into a representative household solving:

$$
\begin{aligned}
v(k;K,z) &= \max_{\{c,k'\}} u(c) + \beta E[v(k';K',z')|\{K,z\}] \\
\text{s.t.} \qquad k' &= [1-\delta+r(K,z)]k + w(K,z) - c \\
K' &= \mathcal{K}(K,z)
\end{aligned}
$$

• The current aggregate capital stock K and the current productivity level z where enough to predict the future aggregate state of the economy, via $K(K,z)$ and the Markov process governing z

• Note that, being all agents identical ex-ante by assumption and ex-post via full insurance, the problem can be rewritten as:

$$v(k;K,z) = \max_{\{c,k'\}} u(c) + \beta E[v(k';K',z')|\{K,z\}]$$

$$\text{s.t.} \qquad k' = [1-\delta+r(K,z)]k + w(K,z) - c$$

$$K' = (1-\delta)K + zf(K) - C$$

where $K = k$ and $C = c$

• Each individual is able to forecast the future aggregate per-capita capital stock, and therefore the future factor prices, because she knows that all other individuals are identical and will behave in the same way

- Rule now full insurance out, as in Aiyagari's model

- In this case, individuals may be **heterogenous ex-post**, since their individual capital stock will depend not only on the history of the aggregate productivity shock, but also on the full history of their idiosyncratic labour income shocks

- Hence, the individual, in order to predict the future aggregate capital stock, and therefore future factor prices, needs to know the **actual distribution** of the individual capital stock and employment status

- The state space of the model has to be consistently enriched: the entire distribution $\lambda(k,s)$ becomes an element of the state space

- Hence, the individual needs to figure out a law of motion that describes how the distribution $\lambda(k,s)$ evolves over time as a function of the state space

• More formally, the household would face the following dynamic programming problem:

$$
\begin{aligned}
v(k,s;\lambda,z) &= \max_{\{c,k'\}} u(c) + \beta E[v(k',s';\lambda',z')|\{s,\lambda,z\}] \\
\text{s.t.} \quad k' &= (1-\delta+r)k + ws - c \\
r &= zF_K(K,N) \\
w &= zF_N(K,N) \\
K &= \int k\lambda(k,s)dkds \\
N &= \int s\lambda(k,s)dkds \\
\lambda' &= \mathcal{H}(\lambda,z,z')
\end{aligned}
$$

- According to KS98, a **recursive competitive equilibrium** is a policy function $g(k,s;\lambda,z)$, a pair of pricing functions $r(\lambda,z)$ and $w(\lambda,z)$, and a law of motion $H(\lambda,z,z')$, such that:
  - $g(k,s;\lambda,z)$ solves the household's problem
  - $r(\lambda,z)$ and $w(\lambda,z)$ correspond to the competitive equilibrium factor prices;
  - $H(\lambda,z,z')$ is induced by $g(k,s;\lambda,z)$

- From a numerical point of view, introducing the distribution λ into the state space makes the model literally intractable

- Following KS98, assume that it is not the distribution itself than is part of the state space, but its first h moments $m=\{m_1,m_2,...,m_h\}$:

$$
\begin{aligned}
v(k,s;m,z) &= \max_{\{c,k'\}} u(c) + \beta E[v(k',s';m',z')|\{s,m,z\}] \\
\text{s.t.} \quad k' &= (1-\delta+\tilde{r})k + ws - c \\
r &= zF_K(K,N) \\
w &= zF_N(K,N) \\
K &= \int k\lambda(k,s)dkds \\
N &= \int s\lambda(k,s)dkds \\
m' &= \mathcal{H}(m,z,z')
\end{aligned}
$$

- **Numerical strategy**:

1. Guess a functional form for H and the corresponding initial parameterization
2. Solve the household's problem for the current H_j
3. Use the policy function g_j to simulate the model for a large number of agents and a large number of periods
4. Use the stationary part of the simulate data to estimate the parameters of H and evaluate the goodness of fit
5. Iterate on (2)-(4) until convergence of the parameters of H
6. If the fit is not satisfactory, increase h or change the functional form for H
7. Iterate on (2)-(6) until the fit is satisfactory

- Let us introduce endogenous labour in the framework:

$$v(k,s;\lambda,z) = \max_{\{c,n,k'\}} u(c,n) + \beta E[v(k',s';\lambda',z')|\{s,\lambda,z\}]$$

$$\text{s.t.} \quad k' = (1-\delta+\tilde{r})k + wns - c$$

$$r = zF_K(K,N)$$

$$w = zF_N(K,N)$$

$$K = \int k\lambda(k,s)dkds$$

$$N = \mathcal{N}(\lambda,z)$$

$$\lambda' = \mathcal{H}(\lambda,z,z')$$

Definition 3. A **recursive competitive equilibrium** is a pair of policy functions $g(k, s; \lambda, z)$ and $l(k, s; \lambda, z)$, a pair of pricing functions $r(\lambda, z)$ and $w(\lambda, z)$, and pair of laws of motion $\mathcal{H}(\lambda, z, z')$ and $\mathcal{N}(\lambda, z)$, such that:

1. $g(k, s; \lambda, z)$ and $l(k, s; \lambda, z)$ solve the household's problem.

2. $r(\lambda, z)$ and $w(\lambda, z)$ correspond to the competitive equilibrium factor prices.

3. $\mathcal{H}(\lambda, z, z')$ and $\mathcal{N}(\lambda, z)$ are induced by $g(k, s; \lambda, z)$ and $l(k, s; \lambda, z)$.