Lecture 2:

Aggregation

Macroeconomics 4

March 2015

Introduction

- **Positive representative household**:
  - An economy admits a *positive representative household* when the preference (demand) side of the economy can be represented as if there were a single household making the aggregate consumption and saving decision subject to an aggregate budget constraint.
  - This description is purely *positive*: it just states that the behavior can be represented as if it were generated by a single household, but does not imply that we can use this representation for normative purposes.
- **Normative representative household**:
  - An economy admits a *normative representative household* when it admits a positive representative households and we are allowed to use the latter's utility function for welfare comparisons.

A trivial example

- Consider an economy with a unit measure of *infinitely lived households* and no aggregate nor idiosyncratic uncertainty.
- Suppose that all households are and **identical**; i.e. they share the same:
  - discount factor β,
  - instantaneous utility function $u(c_t)$,
  - sequence of effective labor endowments $\{e_t\}_{t=0}^{\infty}$.
- This economy trivially admits a *Representative Household* (RH), whose preferences:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t),$$

can be used for **positive and normative analysis**.

Potential difficulties

- Let us consider a simple exchange economy with a finite number N of commodities.
- The equilibrium can be characterized in terms of excess demand correspondences.
- Let the equilibrium be represented by the *aggregate excess demand function* $\mathbf{x}(\mathrm{p})$ where the vector of prices is p.
- The demand side of this economy admits a RH if $\mathbf{x}(\mathrm{p})$ can be obtained as a solution to the maximization problem of a single household.
- The following *Debreu-Mantel-Sonnenschein (DMS) Theorem* shows that **this is not possible in general**.

“Anything Goes Theorem”

Theorem

Let $\varepsilon > 0$ *and* $N \in \mathbb{N}$*. Consider a set of prices:*

$$\mathbf{P}_\varepsilon = \left\{ p \in \mathbb{R}_+^N : \frac{p_j}{p_z} \geq \varepsilon \;\; \forall j, z \right\},$$

*and **any** continuous function* $\mathbf{x} : \mathbf{P}_\varepsilon \to \mathbb{R}_+^N$ *that satisfies Walras's Law and is homogenous of degree* 0*.*
Then there exists an exchange economy with N *commodities and* $H < \infty$ *households where the aggregate excess demand is given by* $\mathbf{x}\,(\mathrm{p})$ *over the set* $\mathbf{P}_\varepsilon$*.*

Implications of the DMS Theorem

- Individual excess demands satisfy the **weak axiom of revealed preferences** and have **symmetric and negative semi-definite** Slutsky matrices.
- The "Anything Goes Theorem" shows that these properties do not necessarily hold for the agg. excess demand function $\mathbf{x}(\mathrm{p})$ that results from aggregating the optimizing behavior of households.
- **Hence, without imposing further structure it is generally impossible to derive $\mathbf{x}(\mathrm{p})$ from the max. behavior of a single household.**
- However, the result is an outcome of strong income effects: restrictions on preferences and on the dist. of income across households can rule out arbitrary agg. excess demand functions.

Existence of a Positive RH

- Consider a finite set of H households who differ in their preferences (over N commodities) and wealth.
- Consider a particular good, and let $x_i(\mathrm{p}, w_i)$ denote the demand function of consumer i for this good, given prices p and wealth w_i.
- Let $\mathrm{w} = \{w_1, w_2, ..., w_H\}$ be the vector of wealth levels for all H households.
- Aggregate demand in this economy can be written as:

$$x(\mathrm{p}, \mathrm{w}) = \sum_{i=1}^{H} x_i(\mathrm{p}, w_i).$$

- The key question is, when are we allowed to write:

$$x(\mathrm{p}, \mathrm{w}) = x\left(\mathrm{p}, \sum_{i=1}^{H} w_i\right)?$$

Existence of a Positive RH

- For the wealth distribution not to matter, we need agg. demand to not change for any redistribution of wealth that keeps aggregate wealth constant, so that $\sum_{i=1}^{H} dw_i = 0$.

- Hence, for all possible redistributions:

$$\sum_{i=1}^{H} \frac{\partial x_i\left(\mathrm{p}, w_i\right)}{\partial w_i} dw_i = 0.$$

- This can be true only if $\frac{\partial x_i(\mathrm{p},w_i)}{\partial w_i} = \frac{\partial x(\mathrm{p},w)}{\partial w}$ $\forall i$, so that:

$$\frac{\partial x\left(\mathrm{p}, w\right)}{\partial w} \sum_{i=1}^{H} dw_i = 0.$$

- The key condition is that households share the same **marginal propensity to consume** (MPC) out of wealth.

Gorman's aggregation theorem

Theorem

Consider an economy with $N < \infty$ commodities and H consumers. Suppose that the preferences of each household i can be represented by an indirect utility function of the form:

$$v_i(\mathrm{p}, w_i) = a_i(\mathrm{p}) + b(\mathrm{p})\, w_i.$$

Suppose furthermore that each household i has a positive demand for each commodity. Then, these preferences can be aggregated and represented by those of a RH with indirect utility:

$$v(\mathrm{p}, W) = A(\mathrm{p}) + b(\mathrm{p})\, W,$$

where $A(\mathrm{p}) \equiv \sum_{i=1}^{H} a_i(\mathrm{p})$ and $W \equiv \sum_{i=1}^{H} w_i$ is aggregate wealth.

Gorman aggregation

- Gorman preferences imply that all households have, for each commodity, linear Engel curves that share the same slope.

- In particular, assuming that $a_i(\mathrm{p})$ and $b(\mathrm{p})$ are differentiable, *Roy's identity* implies, for a given commodity:

$$x_i(p, w_i) = -b(\mathrm{p})^{-1}\left(\frac{\partial a_i}{\partial p} + \frac{\partial b}{\partial p} w_i\right).$$

- Note that if preferences are not of the Gorman form, then by definition the Engel curves of some households have different slopes, and there exists a specific scheme of income redistribution that would affect aggregate demand.

Corollary

Gorman pref. are necessary for the economy to admit a Positive RH.

CES preferences

- Suppose that each household has wealth w_i and preferences defined over N commodities given by a standard CES utility function:

$$u_i(x_i) = \left(\sum_{j=1}^{N} x_{i,j}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

  where $\sigma \in (0, \infty)$; the *elasticity of substitution* between any two commodities is equal to σ.

- The indirect utility function for a generic household is homogenous of degree 0 in p and w_i, and satisfies the Gorman form:

$$v_i(\mathrm{p}, w_i) = b(\mathrm{p})\, w_i,$$

  where:

$$b(\mathrm{p}) = \frac{1}{\left(\sum_{j=1}^{N} p_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}}}.$$

CES preferences

- Therefore, this economy admits a RH with an indirect utility function given by:

$$v\left(\mathrm{p}, W\right) = \frac{W}{\left(\sum_{j=1}^{N} p_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}.$$

- The utility function is obviously given by:

$$u\left(x\right) = \left(\sum_{j=1}^{N} x_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

Existence of a Normative RH

Theorem

Consider an economy with $N < \infty$ commodities, H households, and a convex aggregate production possibilities set Y.
Suppose that preferences of each household are of the Gorman form, so that the economy admits a RH, and that each household has a positive demand for each commodity. Then:

- *Any feasible allocation that maximizes the utility of the RH is Pareto optimal.*
- *Moreover, if $a_i(\mathrm{p}) = a_i$ for all p and all households, then any Pareto optimal allocation maximizes the utility of the RH.*

Proof.

See Acemoglu (2008), Th 5.3, p. 154. □

The Representative Firm

Theorem

Consider a competitive production economy with $N < \infty$ commodities and a countable set $\mathcal{F}$ of firms, each with a production possibilities set $Y_f \subset \mathbb{R}^N$.
Let $p \in \mathbb{R}_+^N$ be the price vector and denote the set of profit-maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}_f(p) \subset Y_f$.
Then there exists a ***representative firm*** *with production possibilities set $Y \subset \mathbb{R}^N$ and a set of profit-maximizing net supplies $\hat{Y}(p) \subset Y$ such that for any p, $\hat{y} \in \hat{Y}(p)$ iif $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}_f$ for some $\hat{y}_f \in \hat{Y}_f(p)$ for each $f \in \mathcal{F}$.*

Proof.

See Acemoglu (2008), Th 5.4, p. 158. □

The Representative Firm

- The previous Theorem implies that when there are no externalities and markets are competitive focusing on a *Representative Firm* (RF) - that takes prices as given! - is without loss of generality.
- Why is there such a striking difference with the demand side?
- The answer is related to income effects:
  - Changes in prices create income effects which affect households differently.
  - A RH exists only when those income effects can be ignored, as in the Gorman case.
  - Since there are no income effects in producer theory, the RF assumption can be made without loss of generality.

A dynamic economy (*without* idiosincratic risk)

- Rubinstein (1974) extends Gorman's result to a dynamic economy where individuals consume out of wealth.
- Consider a competitive economy in which each household solves an intertemporal consumption-savings problem and a portfolio allocation problem.
- Every period current wealth w_t is consumed or invested in a portfolio of a risk-free and a risky security with respective gross returns $R_{f,t}$ and $R_{s,t}$.
- Let α_t denote the portfolio share of the risk-free asset, and β the subjective time discount factor.

A dynamic economy (*without* idiosincratic risk)

- Assume that the inst. utility function is of the *Hyperbolic Absolute Risk Aversion* (HARA) class, characterized by linear **risk tolerance** (the reciprocal of absolute risk aversion):

$$-u'(c)/u''(c) = \rho + \gamma c,$$

where $\rho \geq 0$ and γ are fixed parameters.

- This class has only three members that are compatible with the requirements $u'(c) > 0$ and $u''(c) < 0$:
  - If $\gamma \neq 0, 1$, the *Constant Relative Risk Aversion* (CRRA) utility:

$$u(c) = \frac{(\rho + \gamma c)^{1-\frac{1}{\gamma}}}{\gamma - 1},$$

with $\gamma > 0$ if $\rho = 0$ and $c < \rho/\gamma$ if $\gamma = -1, -1/2, -1/3, ...$

  - If $\gamma = 1$, log utility: $u(c) = \log(\rho + c)$,
  - If $\gamma = 0$ and $\rho > 0$, exponential utility: $u(c) = -\rho \exp(-c/\rho)$.

A dynamic economy (*without* idiosincratic risk)

- Individuals solve the following problem:

$$\max_{\{c_t,\alpha_t\}} \ \mathbb{E}\left[\sum_{t=1}^{T}\beta^t u\left(c_t\right)\right]$$
$$\text{s.t. } w_{t+1} = \left(w_t - c_t\right)\left[\alpha_t R_{f,t} + \left(1-\alpha_t\right)R_{s,t}\right].$$

- If all households have the same initial resources w_0, discount factor β, and utility function $u(c)$, then the equilibrium rates of return are determined as if there existed only identical "composite households" with initial resources w_0, discount factor β, and utility function $u(c)$.

- In case your were wondering, this means the same α_t for all households, i.e. the same portfolio composition!

A dynamic economy (*without* idiosincratic risk)

- Consider now the following (alternative) homogeneity conditions:
  - All households have the same discount factor β and the same $\gamma \neq 0$ (log or CRRA utility with potentially different w_0 and ρ).
  - All households have the same $\gamma = 0$ (exponential utility with potentially different β, ρ, and w_0).
  - All households have the same w_0, $\rho = 0$, and $\gamma = 1$ (log utility with potentially different β).
- In those cases all equilibrium rates of return are determined as if there exists identical composite households with the following characteristics:
  - Resources: $w_0 = \sum_{i=1}^{I} w_{0,i}/I$.
  - Preferences: $\rho = \sum_{i=1}^{I} \rho_i/I$ and γ.
  - Discount factor: $\frac{1-\beta}{\beta} = \prod_{i=1}^{I} \left(\frac{1-\beta_i}{\beta_i}\right)^{\frac{\rho_i}{\sum_i \rho_i}}$ if $\rho > 0$ or $\beta = \sum_{i=1}^{I} \beta_i/I$ if $\rho = 0$.

A dynamic economy (*without* idiosincratic risk)

- An important consequence of these results is that in cases (i) and (ii), in equilibrium, rates of return are insensitive to the distribution of resources among households.
- This is because the aggregate demand functions (for consumption and assets) depend only on total wealth, and not on its distribution.
- Hence, demand aggregation obtains, and therefore we can construct a RH.
- Note that demand aggregation requires households to have the same curvature parameter γ; however identical curvature is not enough, more conditions have to be added on top.

A dynamic economy (*with* idiosincratic risk)

- Rubinstein (1974) abstracts from *idiosincratic uncertainty*.
- Constantinides (1982) shows that, under **complete markets** and much weaker conditions, one can replace heterogeneous households with a planner who maximizes a weighted sum of households' utilities.
- In turn, the central planner can be replaced by a composite consumer who maximizes a utility function of aggregate consumption.
- As we will see, however, he does not get demand aggregation.

A dynamic economy (*with* idiosincratic risk)

- Consider a perfectly competitive economy with production as in Debreu (1959), with M households, N firms, and L commodities.

- Commodities can be thought of as date-event labeled goods, allowing us to map these results into an intertemporal economy with uncertainty.

- Household m is endowed with $w_{m,l} \geq 0$ units of commodity l, and $\theta_{m,n} \geq 0$ shares of firm n, where $\sum_{m=1}^{M} \theta_{m,n} = 1$ for all n.
  - Note that endowments can be interpreted as exogenous and idiosincratic income processes.

- Let the vectors C_m and Y_n denote, respectively, the consumption possibilities set of household m and the production possibilities set of firm n.

A dynamic economy (*with* idiosincratic risk)

- An equilibrium is a set of:
  - optimal consumption plans, $\mathbf{c}^* \equiv \{\mathrm{c}_m^*\}_{m=1}^M$,
  - optimal production plans, $\mathbf{y}^* \equiv \{\mathrm{y}_n^*\}_{n=1}^N$,
  - market-clearing prices, $\mathrm{p}^* \equiv \{p_l^*\}_{l=1}^L$.
- In equilibrium:
  - households maximize utility,
  - firms maximize profits,
  - markets clear.
- Under standard assumptions, an equilibrium exists and is *Pareto optimal*.

A dynamic economy (*with* idiosincratic risk)

- We know that interior allocations of given resources are Walrasian equilibria if and only if they maximize a utilitarian social welfare function (a weighted sum of utilities) on the set of feasible allocations: this is a way to characterize *Pareto optima.*

- Hence, optimality implies that there exists a set $\{\lambda_m\}_{m=1}^{M} \geq 0$ such that $\mathbf{c}^*$ and $\mathbf{y}^*$ solve the following problem (P1):

$$\begin{aligned}
\max_{\{\mathbf{c},\mathbf{y}\}} \quad & \sum_{m=1}^{M} \lambda_m U_m(\mathrm{c}_m) \\
\text{s.t.} \quad & \mathrm{c}_m \in \mathrm{C}_m, \ \ \forall m, \\
& \mathrm{y}_n \in \mathrm{Y}_n, \ \ \forall n, \\
& \sum_{m=1}^{M} (c_{m,l} - w_{m,l}) = \sum_{n=1}^{N} y_{n,l}, \ \ \forall l.
\end{aligned}$$

A dynamic economy (*with* idiosincratic risk)

- Define **aggregate consumption** as $\mathrm{z} \equiv \{z_l\}_{l=1}^{L}$, where $z_l \equiv \sum_{m=1}^{M} c_{m,l}$.
- For a given z, consider the problem (P2) of efficiently allocating it across consumers:

$$
\begin{aligned}
U(\mathrm{z}) \equiv \max_{\mathbf{c}} \quad & \sum_{m=1}^{M} \lambda_m U_m(\mathrm{c}_m) \\
\text{s.t.} \quad & \mathrm{c}_m \in \mathrm{C}_m, \ \forall m, \\
& \sum_{m=1}^{M} c_{m,l} = z_l, \ \forall l.
\end{aligned}
$$

A dynamic economy (*with* idiosincratic risk)

- Define the **total endowment** of commodity l as $w_l \equiv \sum_{m=1}^{M} w_{m,l}$.
- Finally, consider the optimal production decision (P3):

$$\max_{\{\mathrm{z},\mathbf{y}\}} \; U(\mathrm{z})$$

$$\text{s.t.} \quad \mathrm{y}_n \in \mathrm{Y}_n, \;\; \forall n,$$

$$z_l = \sum_{n=1}^{N} y_{n,l} + w_l, \;\; \forall l.$$

A dynamic economy (*with* idiosincratic risk)

Theorem

a) *The solution to P3 is* $\mathbf{y}^*$, *and* $z_l^* = \sum_{n=1}^{N} y_{n,l}^* + w_l$.

b) $U(\mathrm{z})$ *is increasing and concave in* z.

c) *The solution to P2 is* $\mathbf{c}^*$.

d) *Given* $\{\lambda_m\}_{m=1}^{M}$, *if the households are replaced by a RH with utility* $U(\mathrm{z})$, *endowments* $\{w_l\}_{l=1}^{L}$, *and shares* $\{\theta_n\}_{n=1}^{N} = 1$, *then the set* $\{\mathrm{z}^*, \mathbf{y}^*, \mathrm{p}^*\}$, *where* $\mathrm{z}^* = \{z_l^*\}_{l=1}^{L}$, *is an equilibrium.*

Proof.

See Constantinides (1982), Lemma 1. □

A dynamic economy (*with* idiosincratic risk)

- The existence of a RH does not imply demand aggregation, for two reasons:
  - Composite demand depends on the weights $\{\lambda_m\}$, and thus on the distribution of endowments.
  - The RH is defined at equilibrium prices and there is no presumption that its demand curve is identical to the aggregate demand function.
- Hence, the usefulness of these results hinges on:
  - the degree to which markets are complete,
  - whether we want to allow for idiosincratic risk and heterogeneous preferences,
  - whether or not we need demand aggregation.

Recent extensions

- Ogaki (2003) generalizes the results in Constantinides (1982), and assumes that households have:
  - rational expectations (i.e. common beliefs),
  - time-additive and time-separable von Neumann-Morgenstern intertemporal utility functions,
  - time-invariant intratemporal utility functions and identical intertemporal discount factors.
- Under those assumptions, under complete markets a RH exists, and intraperiod demand aggregation applies.

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