Lecture 3:
Heterogeneity under complete markets

Macroeconomics 4

A.Y. 2014-15

A neoclassical growth model

- The economy is composed of N households and a single Representative Firm (RF).

- Each household maximizes an intert. utility function of the form:

$$U_i = \sum_{t=0}^{\infty} \beta^t u\left(c_{i,t}\right),$$

  where c_i is the household's cons. of a homogenous good, and $\beta \in (0, 1)$ the intert. discount factor.

- The instant. utility function $u(\cdot)$ is of the *HARA* class (see also Lecture 2); three possible functional forms:
  - $u(c) = \sigma^{-1}(\alpha + c)^{\sigma}$, with $(\alpha + c) \geq 0$, $\sigma < 1$, $\sigma \neq 0$, and $\alpha \in \mathbb{R}$.
  - $u(c) = (1 - \beta)\ln(\alpha + c)$, with $(\alpha + c) \geq 0$, and $\alpha \in \mathbb{R}$.
  - $u(c) = -\alpha \exp(-\eta c)$, with $\alpha > 0$ and $\eta > 0$.

A neoclassical growth model

- The *RF* produces the homogenous good using only physical capital, via the following production function:

$$y_t = f\left(k_t\right),$$

  where:

  - y_t denotes per capita output,
  - k_t the per capita stock of capital at the beginning of period t,
  - $f > 0$, $f' > 0$, and $f'' < 0$.

- **Markets are complete.**

- The price of period-t consumption in terms of period-0 consumption is p_t, so that $p_0 \equiv 1$.

A neoclassical growth model

- The optimization problem for the RF is:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t d_t$$
$$\text{s.t. } d_t = f(k_t) + (1-\delta) k_t - k_{t+1},$$
$$k_0 > 0.$$

  where:
  - d_t denotes dividends, i.e. per capita distributed profits of the firm in period t,
  - $\delta \in (0,1)$ denotes the depreciation rate on capital.

A neoclassical growth model

- The optimization problem for household i can be stated as:

$$\max_{\{c_{i,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_{i,t}\right)$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t c_{i,t} \leq w_{i,0} \equiv s_{i,0} \sum_{t=0}^{\infty} p_t \left(N d_t\right),$$

  where:

  - $w_{i,0}$ is the household's initial wealth,
  - $s_{i,0}$ is the share of the RF owned by household in period 0.

- Assume that $s_{i,0}$ is large enough so that the optimization problem for household i admits an interior solution.

A neoclassical growth model

- Define the period-t wealth of household i measured in terms of period-t consumption as:

$$w_{i,t} = s_{i,t} \sum_{z=t}^{\infty} \left(\frac{p_z}{p_t} \right) N d_z.$$

- Given the *HARA* form of the utility function, it can be shown that consumption of household i in period t is an *affine function* of wealth in period t:

$$c_{i,t} = a\left({}_{\mathrm{t}}\mathrm{p}\right) + b\left({}_{\mathrm{t}}\mathrm{p}\right) w_{i,t},$$

where ${}_{\mathrm{t}}\mathrm{p} \equiv \{p_z\}_{z=t}^{\infty}$.

A neoclassical growth model

- Under log utility, ie. if $u(c) = (1-\beta)\ln(\alpha + c)$, with $c \geq -\alpha$, it turns out that:

$$\begin{aligned} a({}_t\mathrm{p}) &= \alpha\left[(1-\beta)\sum_{z=t}^{\infty}\frac{p_z}{p_t} - 1\right], \\ b({}_t\mathrm{p}) &= 1-\beta. \end{aligned}$$

- See Chatterjee (1994) for more details on the functions $a(\cdot)$ and $b(\cdot)$ in the other two cases.

A neoclassical growth model

- A competitive equilibrium in this environment is a sequence $\{p_t\}_{t=0}^{\infty} \geq 0$ such that the optimal choices of all households and the firm satisfy market clearing:

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N} c_{i,t} + k_{t+1} &= f\left(k_t\right) + \left(1-\delta\right) k_t, \;\; \forall t \geq 0, \\ \sum_{i=1}^{N} s_{i,t} &= 1. \end{aligned}$$

- The objective of this analysis is to study the competitive evolution of the wealth share vector $\mathrm{s}_t \equiv \left(s_{1,t}, s_{2,t}, ..., s_{N,t}\right)$.

Wealth shares

- From the budget constraint, a household's growth rate of wealth is given by:
$$\frac{w_{i,t+1}}{w_{i,t}} = \frac{p_t}{p_{t+1}} \left(1 - \frac{c_{i,t}}{w_{i,t}}\right).$$

- The evolution of a household's wealth share depends on its rate of accumulation relative to the rate of accumulation of per capita wealth.

- The fundamental equation governing the evolution of the wealth share is:
$$s_{i,t+1} = \frac{w_{i,t+1}}{w_{i,t}} \frac{w_t}{w_{t+1}} s_{i,t},$$
where $w_t \equiv \frac{1}{N} \sum_{i=1}^{N} w_{i,t}$.

Aggregate dynamics

- Linear Engel curves (*homothetic preferences*, ie. of the Gorman form) imply that:

$$c_t \equiv \frac{1}{N} \sum_{i=1}^{N} c_{i,t} = a\left({}_{t}\mathrm{p}\right) + b\left({}_{t}\mathrm{p}\right) w_t,$$

so that per capita consumption is the desired consumption of a household with per capita wealth.

- Thus, the competitive quantities can be recovered from the following *social planning problem* featuring a representative household (i.e. a standard *RCK* model):

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left[f\left(k_t\right) + \left(1-\delta\right) k_t - k_{t+1}\right],$$
$$\text{s.t. } k_0 > 0.$$

Aggregate dynamics

- Equilibrium prices can be recovered from the following recursion:

$$\hat{p}_0 \equiv 1,$$
$$\frac{\hat{p}_t}{\hat{p}_{t+1}} = f'\left(\hat{k}_{t+1}\right) + 1 - \delta,$$

where a hat identifies the optimal (as well as competitive) path.

Steady-state indeterminacy

- As before, assume log utility for the sake of simplicity.
- The steady state allocation and prices are characterized by the following conditions (note that $p_t/p_{t+1} = 1/\beta$):

$$c_i = s_i \left[f\left(k^*\right) - \delta k^*\right], \quad i = 1, 2, ..., N,$$
$$f'\left(k^*\right) = \frac{1-\beta}{\beta} + \delta,$$
$$\sum_{i=1}^{N} s_i = 1.$$

- The unknowns to be solved for are $\{c_i, s_i\}_{i=1}^{N}$ and k^*: there are $2N + 1$ unknowns, and $N + 2$ equations!
- The steady-state distribution of wealth, ie. the vector of $N - 1$ wealth shares s_i, is simply *ex-ante* indeterminate ...

Steady-state indeterminacy

- More precisely, from an ex-ante point of view, there exists a **continuum of steady-state wealth distributions**, with dimension $N-1$.
- However, given an initial distribution $\{s_{i,0}\}_{i=1}^{N}$, the equilibrium wealth distribution $\{s_{i,t}\}_{i=1}^{N}$ is uniquely determined in every period t, and therefore in steady state too.
- In other words, the steady-state distribution becomes determined once we condition on a given initial distribution, since the **equilibrium path** is uniquely determined.
- Under complete markets, our environment predics the evolution of the wealth distribution, but does not offer a theory of the intial or final distributions themselves.

Evolution of the wealth distribution

- We will now introduce the concept of *Lorenz dominance*: if a distribution Lorenz-dominates another one, then it implies less inequality.

Definition

Let all households be ordered according to increasing wealth. The vector s_t Lorenz-dominates the vector s_{t+1} if

$$\sum_{i=1}^{k} s_{i,t+1} \leq \sum_{i=1}^{k} s_{i,t}$$

for all $k \in [1, N]$, with strict inequality holding for some k.

Evolution of the wealth distribution

- The linearity of Engel curves implies that:,

$$\frac{c_{i,t}}{w_{i,t}} < (\geq) \frac{c_t}{w_t} \;\Leftrightarrow\; a\left({}_{\mathrm{t}}\mathrm{p}\right)\left(w_{i,t} - w_t\right) > (\leq)\, 0.$$

- Recall that:

$$\frac{w_{i,t+1}}{w_{i,t}} = \frac{p_t}{p_{t+1}}\left(1 - \frac{c_{i,t}}{w_{i,t}}\right).$$

- Thus, whether $s_{i,t+1}$ increases depends on the size of $w_{i,t}$ relative to w_t and on the sign of $a\left({}_{\mathrm{t}}\mathrm{p}\right)$. It turns out that:

$$\begin{cases} a\left({}_{\mathrm{t}}\mathrm{p}\right) > 0 & \Leftrightarrow\; \alpha\left(k_t - k^*\right) > 0, \\ a\left({}_{\mathrm{t}}\mathrm{p}\right) = 0 & \Leftrightarrow\; \alpha\left(k_t - k^*\right) = 0, \\ a\left({}_{\mathrm{t}}\mathrm{p}\right) < 0 & \Leftrightarrow\; \alpha\left(k_t - k^*\right) < 0, \end{cases}$$

  where k^* denotes the (per capita) steady-state capital stock.

Evolution of the wealth distribution

Theorem

$$\begin{cases} \alpha\left(k_t - k^*\right) > 0 & \Leftrightarrow \ \mathrm{s}_t \ \textit{Lorenz-dominates}\ \mathrm{s}_{t+1}, \\ \alpha\left(k_t - k^*\right) = 0 & \Leftrightarrow \ \mathrm{s}_t = \mathrm{s}_{t+1}, \\ \alpha\left(k_t - k^*\right) < 0 & \Leftrightarrow \ \mathrm{s}_{t+1} \ \textit{Lorenz-dominates}\ \mathrm{s}_t. \end{cases}$$

Proof.

See Chatterjee (1994), p. 104. □

- Recall that α is a preference parameter, assumed to be any real number in case of *CRRA* and log utility, and strictly positive in case of exponential utility.

Evolution of the wealth distribution

- The configuration of greatest interest is one where $\alpha < 0$ and $k_0 < k^*$: this is the case of economic growth in the presence of a **subsistence consumption level** $-\alpha$.

- In this situation **an increasing level of economic well being is accompanied by a worsening of the distribution of wealth**.
    - Essentially, households who are poor and consume close to $-\alpha$ find it difficult to further reduce their consumption and accumulate capital.
    - In contrast, rich households take advantage of the higher rates of return prevailing in the early stages of growth and accumulate wealth rapidly.

Comparative dynamics

Theorem

Consider two economies which are identical in all respects in period t except that s_t^1 Lorenz-dominates s_t^2. In this case, s_z^1 will Lorenz-dominate s_z^2 for all $z > t$.

Proof.

See Chatterjee (1994), p. 109. $\square$

- The ranking of economies with respect to the dist. of wealth is not affected by time, provided the initial distributions are Lorenz-comparable and the economies are identical in all other respects.
- The result holds independently of whether the distributions in the two economies are changing over time.

The role of market structure

- The complete market assumption plays a key role in these results.
- Consider the case of no credit or equity markets, so that each household invests in its privately owned firm.
- Assume that all households have more than enough resources to sustain minimum consumption, and have access to the same technology.
- Under these circumstances, **each household will eventually converge to the same capital stock and the long-run distribution of wealth would be perfectly equal**.
- This convergence is due to the higher marginal return to capital faced by households with low levels of initial capital.

The role of market structure

- Complete markets, instead, imply Pareto efficiency, which in turn implies constant ratios of marginal utilities across households.

- For simplicity, consider again the log case: if $\alpha = 0$, constant marginal utility ratios imply constant consumption ratios, since:

$$\frac{u'(c_{i,t})}{u'(c_{j,t})} = \frac{c_{j,t}}{c_{i,t}} = \frac{\lambda_i}{\lambda_j}.$$

- If $\alpha \neq 0$, however, this has not to be the case:

$$\frac{u'(c_{i,t})}{u'(c_{j,t})} = \frac{c_{j,t} + \alpha}{c_{i,t} + \alpha} = \frac{\lambda_i}{\lambda_j}.$$

Extensions

- Obiols-Homs and Urrutia (2005) study a slightly modified version of the previous model, and get quite an interesting result.
- They assume log utility with a minimum consumption requirement and a "Cobb-Douglas" production function.
- Furthermore, they introduce inelastic labor supply, and move the ownership of capital from firms to households: this allows them to distinguish between **lifetime wealth**, the subject of Chatterjee (1994), and **asset holdings**.

Extensions

- Each household solves the following problem:

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t} - \bar{c})$$
$$\text{s.t.} \quad c_{i,t} + k_{i,t+1} = R_t k_{i,t} + w_t,$$
$$c_{i,t} \geq \bar{c},$$
$$k_{i,0} > 0.$$

where R_t and w_t are, respectively, the gross real rental rate and the wage rate.

Extensions

- Define the lifetime wealth of household i as:

$$\omega_{i,t} = R_t \left(a_{i,t} + \sum_{j=0}^{\infty} \frac{w_{t+j}}{\prod_{z=0}^{j} R_{t+z}} \right).$$

- Not surprisingly, it turns out that consumption is linear in lifetime wealth:

$$c_{i,t} = B_t + (1-\beta)\,\omega_{it},$$

where:

$$B_t \equiv \bar{c} \sum_{j=0}^{\infty} \frac{\beta R_{t+1-j} - 1}{\prod_{z=0}^{j} R_{t+1+z}}.$$

Extensions

- The dynamics of household's assets holdings is characterized by:

$$a_{i,t+1} = \beta R_t a_{i,t} + D_t,$$

  where D_t is a common component that depends on current and future factor prices, and on $\bar{c}$.

- Assume now that $\bar{c} = 0$ or that $\bar{c}$ is "not too big" and the initial capital stock is "large enough" (see Obiols-Homs and Urrutia (2005) for details).

Extensions

- Under the previous assumptions:

Theorem

In any transition from below, $k_{t+1}/k_t > \beta R_t$ for all t, where k_t denotes the aggregate per capita stock of capital, and the coefficient of variation (standard dev./mean) in assets across households monotonically ***decreases*** *over time.*

Proof.

See Obiols-Homs and Urrutia (2005), p. 390. $\square$

Extensions

- The intuition goes as follows: assume $\bar{c} = 0$, and note that households share the same elast. of int. substitution, thus the same desired rate of growth of consumption (if $\bar{c} = 0$!):

$$c_{i,t+1}/c_{i,t} = \beta R_t, \ \ \forall i.$$

- But we know from the previous Theorem that $k_{t+1}/k_t > \beta R_t$, i.e. $k_{t+1}/k_t > c_{i,t+1}/c_{i,t}$ for all i.

- Being $c_{i,t}$ linear in lifetime wealth, we have that $k_{t+1}/k_t > \omega_{t+1}/\omega_t$, where ω_t is agg. wealth.

- Wealth is a weighted average of agg. capital and the PV of labor income, which is equal across households:

$$\omega_t = R_t \left(k_t + \sum_{j=0}^{\infty} \frac{w_{t+j}}{\prod_{z=0}^{j} R_{t+z}} \right).$$

Extensions

- Evidently, labor income must grow at a lower rate than ω_t.
- "Poor" households have a larger share of labor income in their lifetime wealth portfolio that "rich" households.
- Thus, "poor" agents accumulate assets at a faster rate than rich agents, because they need to save more to match the rate of growth of ω_t.
- This explains intuitively the convergence in the distribution of assets.

Extensions

- Chatterjee (1994) in a similar environment shows that the inequality in the distribution of *lifetime wealth* remains constant when $\bar{c} = 0$, and increases when $\bar{c} > 0$.
- Obiols-Homs and Urrutia (2005) show that, under the same conditions and at the same time, the inequality in the distribution of *assets* can actually decrease.
- **This should warn against interpreting changes in the distribution of assets as having implications for consumption inequality or welfare.**
- Furthermore, this suggests also that some of the implication of the complete markets assumptions as far as the evolution of inequality is concerned seem likely at odds with empirical evidence.

Some disturbing results

- Carroll and Young (2009) study another variation of the neoclassical growth model under complete markets.
- They assume that households supply labor inelastically, but are heterogeneous with respect to their labor productivity, denoted ε.
- This heterogeneity is permanent, i.e. the population is composed by as a finite set of "types" and their corresponding measure ψ_i.
- There is the usual representative firm, producing the homogenous consumption good using capital and labor.
- There is also a government, that imposes **progressive income taxes** and pays the revenues back lump sum.

Some disturbing results

- The maximization problem solved by a type-i household should be familiar by now:

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_{i,t}\right)$$
$$\text{s.t.} \quad c_{i,t} + k_{i,t+1} = k_{i,t} + y_{i,t} - \tau\left(y_{i,t}\right) + TR_t,$$
$$y_{i,t} = w_t \varepsilon_i + r_t k_{i,t},$$
$$k_{i,0} > 0.$$

where $\tau\left(y_{i,t}\right)$ is the total tax function, such that $\tau\left(y\right) \leq y$ for all y with equality only if $y = 0$.

Some disturbing results

- Factor prices are determined competitively (the production function satisfies all the standard assumptions):

$$
\begin{aligned}
w_t &= F_N\left(K_t, N\right), \\
r_t &= F_K\left(K_t, N\right) - \delta.
\end{aligned}
$$

  where:

$$
\begin{aligned}
K_t &= \sum_i k_{i,t}\psi_i, \\
N &= \sum_i \varepsilon_i \psi_i.
\end{aligned}
$$

- The government budget constraint holds in each period t:

$$
TR_t = \sum_i \tau\left(y_{i,t}\right)\psi_i.
$$

Some disturbing results

- In steady state, the Euler equation can be rewritten as:

$$\tau_y\left(y_{i,t}\right) = \frac{1 + r - \beta^{-1}}{r} = \phi,$$

where $r = F_K\left(K, N\right) - \delta$.

- Evidently, there can be only one marginal tax, ϕ, which all household face in the long run.
- If $\tau\left(y\right) = \tau y$, i.e. under prop. taxation, then the income and wealth distributions are again **ex-ante indeterminate**:
    - the Euler equation pins down the agg. capital stock, not the distribution of capital across households;
    - it's the initial distribution $\{k_{i,0}\}$ that pins down the actual equilibrium paths.

Some disturbing results

Theorem

If $\tau_y(y)$ is strictly increasing, i.e. if the tax function is a ***marginal-rate progressive*** *one, then the long-run income dist. is* ***degenerate****, i.e. $y_i = y$ for all i.*

Proof.

When $\tau_y(y)$ is strictly increasing, there is a unique income level associated with ϕ. $\square$

Some disturbing results

Theorem

If $\tau_y(y)$ is strictly increasing, k and ε are ***negatively correlated****, while labor income and asset income are* ***perfectly*** *negatively correlated.*

Proof.

Consider two households i and j, and let $\varepsilon_i > \varepsilon_j$. The degenerate income distribution implies that $y_i = y_j$, which in turn implies that $w_t\varepsilon_i + r_t k_{i,t} = w_t\varepsilon_j + r_t k_{j,t}$; hence:

$$\varepsilon_i - \varepsilon_j = -\frac{r}{w}\left(k_i - k_j\right).$$

If $\varepsilon_i > \varepsilon_j$, then $\varepsilon_i w > \varepsilon_j w$. Since $y_i = y_j$, $rk_i < rk_j$. Thus, εw and rk have a correlation of -1. $\square$

Some disturbing results

Table 1
SCF statistics.

Correlation (S.E. $\times 10^3$)	1992	1995	1998	2001	2004
$\rho(y,k)$	0.33 (0.091)	0.46 (0.079)	0.61 (0.062)	0.53 (0.070)	0.59 (0.062)
$\rho(w\varepsilon h, rk)$	0.24 (0.096)	0.43 (0.082)	0.14 (0.097)	0.20 (0.093)	0.15 (0.092)
$\rho(y, w\varepsilon)$	0.08 (0.102)	0.53 (0.072)	0.06 (0.098)	0.18 (0.094)	0.17 (0.092)

- Carroll and Young (2009) point out that their results are:
  - robust to the introduction of elastic labor supply, exogenous borrowing limits, and preference heterogeneity;
  - grossly inconsistent with the data.
- The model predicts a zero correlation between income and wealth, and a perfectly negative correlation between capital income and labor income. Empirical evidence for the *US* is reported above.
- These findings suggest that for questions related to income and wealth inequality, predictions based upon the complete market assumption are unlikely to correspond well to the data.

References I

Carroll, D. R. and E. R. Young (2009, July). The Stationary Distribution of Wealth under Progressive Taxation. *Review of Economic Dynamics 12*(3), 469–478.

Chatterjee, S. (1994, May). Transitional dynamics and the distribution of wealth in a neoclassical growth model. *Journal of Public Economics 54*(1), 97–119.

Obiols-Homs, F. and C. Urrutia (2005, February). Transitional dynamics and the distribution of assets. *Economic Theory 25*(2), 381–400.