Lecture 4

The income fluctuations problem

Part I

Macroeconomics 4

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Preferences

- The previous Lecture kind of suggested that complete markets aren't such a good approximation of reality.
- We will now study (in partial equilibrium) the problem of a household that is subject to idiosyncratic income shocks, but cannot insure them away because of incomplete markets.
- In particular, the household will be able to smooth consumption over time and states of the world only via a safe, non-state contingent, asset.
- The only source of insurance in this framework is "self insurance."

Preferences

- The household's pref. on cons. streams can be summarized by:

$$U_0 = \sum_t^T \beta^t u\left(c_t\right)$$

  where $c_s \in R_+$ is the cons. level at date s, $\beta \in (0, 1)$ the intert. subjective disc. factor, $u \colon R_+ \to R$ the instant. utility function, and $T \leq \infty$.

- Define also the intert. discount rate as $\rho \equiv (1 - \beta) / \beta$.

- Three implicit assumptions: (i) stationarity, (ii) additive separability, (iii) time impatience.

- Assume that u is C^3, strictly increasing, and strictly concave; furthermore, impose the Inada condition, $\lim_{c \to 0} u'(c) = +\infty$.

- The last assumption implies that, in equilibrium, it will never be optimal to set $c_t = 0$.

Intratemporal budget constraint

- The household may accumulate assets through the following technology:

$$a_{t+1} = a_t + s_t,$$

  where $a_t \in R$ is the assets stock at the beginning of date t, measured in units of consumption good, and $s_t \in R$ are savings at date t (note that savings can be negative).

- Assets may be held only as consumption loans (debts); the interest rate $r > 0$ is constant over time.

- The household receives an exogenous income flow $y_t \in (0, y_{\max}]$, where $y_{\max} < +\infty$, and faces the following intratemporal budget constraint:

$$c_t + s_t \leq y_t + ra_t.$$

NPG condition

- It would be unfeasible for any household to finance its current indebtedness by continuously increasing it.

- To avoid this possibility, we impose the so-called **No-Ponzi-Games** (*NPG*) condition.

- If $T < \infty$, the *NPG* cond. simply states that $a_{T+1}/\left(1+r\right)^{T} \geq 0$; if $T \to \infty$, instead:
$$\lim_{t\to\infty} \frac{a_{t+1}}{\left(1+r\right)^{t}} \geq 0,$$
for all feasible sequences $\{a_s\}_{s=t}^{\infty}$.

- The *NPG* cond. states that the present market value of the asset stock cannot be strictly negative in the long-run: it rules out free lunches.

Intertemporal budget constraint

- Focusing on the case $T \to \infty$, and iterating on the intratemporal budget constraint, gets:

$$(1+r)\,a_0 = \sum_{t=0}^{\infty} \frac{c_t - y_t}{(1+r)^t} + \lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t}.$$

- Imposing the *NPG* cond. takes us to:

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq (1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}.$$

- The *PMV* of the consumption stream cannot be strictly greater than the *PMV* of lifetime resources: by imposing the *NPG*, the *intra*temporal budget constraint becomes an ***inter*temporal** one.

Natural borrowing limit

- The intert. budget const. can be rewritten as:

$$a_t \geq \sum_{s=t}^{\infty} \frac{c_s}{(1+r)^{s-t+1}} - \sum_{s=t}^{\infty} \frac{y_s}{(1+r)^{s-t+1}}.$$

- Being $c_t \geq 0$ for all t, this implies that:

$$a_t \geq -\sum_{s=t}^{\infty} \frac{y_s}{(1+r)^{s-t+1}}.$$

- The previous inequality summarizes the exo. borrowing const. implied by the *NPG* condition, i.e. the *natural borrowing limit*.

Natural borrowing constraint

- The *maximum* level of debt that can be repaid from date t onwards setting cons. to zero is $\sum_{s=t}^{\infty} y_s (1+r)^{-s+t-1}$.
- In general, the ex-ante *natural borrowing constraint* takes the form $a_t \geq -\mathsf{b}$ for all $t \geq 0$, where:

$$\mathsf{b} \equiv \inf_t \left[\sum_{s=t}^{\infty} \frac{y_s}{(1+r)^{s-t+1}} \right].$$

- Since $c_t = 0$ for some $t \geq 0$ will never be optimal in equilibrium, this borrowing const. will never be actually binding.

The problem in sequential form

- Hence, the household solves the following problem:

$$\max_{\{c_s, a_{s+1}\}_{s=t}^{\infty}} U_0 = \sum_{t=0}^{T} \beta^t u\left[(1+r)\, a_t + y_t - a_{t+1}\right]$$

$$\text{s.t. } a_{t+1} \geq -\mathsf{b}.$$

given r and some **deterministic** sequence $\{y_t\}_{t=0}^{T}$, where $T \leq \infty$.

The problem in recursive form

- The problem can also be written in **recursive form** (if $T \to \infty$, then $V_t = V_{t+1} = V$):

$$V_t\left(a_t, y_t\right) = \max_{a_{t+1}} u\left[\left(1+r\right)a_t + y_t - a_{t+1}\right] + \beta V_{t+1}\left(a_{t+1}, y_{t+1}\right),$$
$$\text{s.t. } a_{t+1} \geq -\mathsf{b}.$$

- Defining **"cash in hand"** as $x_t = \left(1+r\right)a_t + y_t$, we can further simplify to:

$$V_t\left(x_t\right) = \max_{x_{t+1}} u\left(x_t - \frac{x_{t+1} - y_{t+1}}{1+r}\right) + \beta V_{t+1}\left(x_{t+1}\right),$$
$$\text{s.t. } \frac{x_{t+1} - y_{t+1}}{1+r} \geq -\mathsf{b}.$$

- This version of the model will become useful later.

FOCs and TVC

- *FOCs*:

$$u_c(\hat{c}_t) = \hat{\lambda}_t,$$
$$\hat{\lambda}_{t+1}\beta(1+r) = \hat{\lambda}_t,$$
$$\hat{a}_{t+1} = (1+r)\hat{a}_t + y_t - \hat{c}_t.$$

- Assume $T \to \infty$; the *NPG* cond. and the *FOCs* jointly imply that $\lim_{t\to\infty} \beta^t \hat{\lambda}_t a_{t+1} \geq 0$ for all feasible sequences $\{a_t\}_{t=0}^{\infty}$.

- In this case, the *FOCs* together with the *TVC*:

$$\lim_{t\to\infty} \beta^t \hat{\lambda}_t \hat{a}_{t+1} = 0.$$

  are jointly necessary *and* sufficient.

- If $T < \infty$, the TVC collapses to $\hat{a}_{T+1} = 0$.

Euler equation

- Combining the FOCs, the Euler equation easily obtains:

$$\frac{u_c(c_t)}{u_c(c_{t+1})} = \beta(1+r).$$

- Being $u'' < 0$ for the strict concavity of u, the Euler eq. implies:

$$\begin{cases} \Delta c_{t+1} > 0 & \text{if} \quad \beta(1+r) > 1, \\ \Delta c_{t+1} < 0 & \text{if} \quad \beta(1+r) < 1, \\ \Delta c_{t+1} = 0 & \text{if} \quad \beta(1+r) = 1. \end{cases}$$

Permanent income

- For the TVC, the intert. budget const. holds with equality:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = (1+r)\, a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}.$$

- Assume $\beta(1+r) = 1$, so that $c_t = \bar{c}\,\forall t$. We can solve the inert. budget const. for $\bar{c}$:

$$\bar{c} = \omega_0 \equiv r a_0 + h_0,$$

where:

  - ω_t denotes **permanent (per period) income**,
  - $h_t \equiv \frac{r}{1+r} \sum_{s=t}^{T} \frac{y_s}{(1+r)^{s-t}}$ denotes the **annuity value** of the PMV of future income (a.k.a *human wealth*).

Potentially binding constraint

- Tedious calculations show that savings, i.e. the growth of assets, are negatively corr. to future income growth:

$$\Delta a_{t+1} = s_t = -\sum_{s=t}^{T} \frac{\Delta y_{s+1}}{(1+r)^{s-t+1}}.$$

- If an household expects its income to increase (decrease) in the future, it will decumulate (accumulate) assets in the present.

Potentially binding constraint

- Suppose now that the household faces a **potentially binding borrowing constraint**: $a_{t+1} \geq -\phi$, where $0 \leq \phi < b$ is exogenously given (without loss of generality, $\phi = 0$).

- The first order and slackness conditions can be combined into the following "Euler inequality":

$$\begin{cases} u_c(c_t) > \beta(1+r)\, u_c(c_{t+1}) & \text{if} \;\; a_{t+1} = 0, \\ u_c(c_t) = \beta(1+r)\, u_c(c_{t+1}) & \text{if} \;\; a_{t+1} > 0. \end{cases}$$

- From the budget constraint, $c_t \leq (1+r)\, a_t + y_t$, with equality when $a_{t+1} = 0$. Hence:

$$u_c(c_t) = \max\left\{u_c\left[(1+r)\, a_t + y_t\right], \beta(1+r)\, u_c(c_{t+1})\right\}.$$

Potentially binding constraint

- Let us focus on the case $T = \infty$, since the other one is rather trivial.
- Recall that our assumptions on y_t should guarantee that $\sum_{s=t}^{\infty} \frac{y_s}{(1+r)^{s-t}} < \infty$ for all t.
- Define M_t as:

$$M_t \equiv u_c(c_t)\left[\beta(1+r)\right]^t.$$

- The "Euler ineq." implies that $M_t \geq M_{t+1} > 0$; thus, M_t is bounded.

Potentially binding constraints

- If $\beta(1+r) > 1$, then $\lim_{t\to\infty} [\beta(1+r)]^t = \infty$.
- Being M_t bounded, necessarily $\lim_{t\to\infty} u_c(c_t) = 0$. This implies, for the Inada condition, $\lim_{t\to\infty} c_t = \infty$.
- Recall the intert. budget const.:

$$a_t \geq \sum_{s=t}^{\infty} \frac{c_s}{(1+r)^{s-t+1}} - \sum_{s=t}^{\infty} \frac{y_s}{(1+r)^{s-t+1}}.$$

- If c_t is unbounded, then a_t has to be unbounded too.

Potentially binding constraints

- If $\beta(1+r)=1$, then, from the Euler eq., $u_c(c_t) \geq u_c(c_{t+1})$. Hence, consumption is a non-decreasing sequence: $c_{t+1} \geq c_t$.

- Chamberlain and Wilson (2000), Th. 3, show that, in this case, $\lim_{t\to\infty} c_t = \bar{h}_t = \sup_t h_t$, where $\bar{h}_t$ exists for the boundedness of y_t.

- The intuition goes as follows:

  - The borrowing const. may be binding only when the household wants to transfer purchasing power from the future to the present because y_t - and consequently h_t - is expected to increase, so that $c_{t+1} > c_t$; this cannot last forever, being y_t bounded.

  - As soon as h_t is expected to remain constant or decrease over time, the incentive to borrow disappears, and $c_t = \bar{c}$ from then on.

- Note that if c_t is bounded, a_t is bounded too.

Potentially binding constraints

- Consider the recursive version of the problem, and compute the envelope condition:

$$u_c\left[c\left(x\right)\right]=V_x\left(x\right).$$

- Diff. w.r.t. x gets:

$$u_{cc}\left(c\right)c_x=V_{xx}\;\Rightarrow\;c_x=\frac{V_{xx}}{u_{cc}\left(c\right)}.$$

- Being V strictly concave and diff. under our assumptions, then $V_{xx}<0$, $u_{cc}<0$ by assumption, and therefore $c_x>0$, i.e. cons. is an increasing function of "cash in hand"

Potentially binding constraints

- Consider now the case $\beta(1+r) < 1$.
- As long as the constraint is not binding, i.e. as long as $a_{t+1} > 0$, the sequence of c_t is strictly decreasing, i.e. $c_{t+1} < c_t$, because of the Euler equation.
- Hence, being $c_x > 0$, $x_{t+1} < x_t$ too as long as $a_{t+1} > 0$; thus, we can expect the household to reach the borrowing limit in finite time.
- To easily prove it, assume a constant income profile, $y_t = \bar{y}$; we have to prove that in *finite time* $x_t \to \bar{y}$ so that $a_t \to 0$.

Potentially binding constraints

- Suppose instead that $x_t \to \bar{x} > \bar{y}$ so that $a_t > 0$ for all x_t.
- Iterating on the Euler equation, and taking the assumptions on u into account, we get that:

$$0 < u_c(c_t) = \lim_{s \to \infty} [\beta(1+r)]^s u_c[c(x_{t+s})].$$

- Being $x_{t+s} > \bar{y}$ for all s and $c_x > 0$, a contradiction emerges:

$$0 < u_c(c_t) \leq \lim_{s \to \infty} [\beta(1+r)]^s u_c[c(\bar{y})] = 0.$$

Potentially binding constraints

- Once $x_t = \bar{y}$, then $a_{t+1} = 0$ and $c_t = \bar{y}$ from then on, i.e. once the household becomes credit-constrained, it remains constrained forever.

- The intuition is straightforward: if $a_t = 0$ and $a_{t+1} > 0$ for some t, then the Euler equation implies:

$$u_c\left[c\left(\bar{y}\right)\right] = \underbrace{\beta\left(1+r\right)}_{<1} u_c\left[c\left(x_{t+1} > \bar{y}\right)\right],$$

  so that $u_c\left[c\left(\bar{y}\right)\right] < u_c\left[c\left(x_{t+1} > \bar{y}\right)\right]$.

- But if $c_x > 0$ then $c\left(x_{t+1} > \bar{y}\right) > c\left(\bar{y}\right)$; hence:

$$u_c\left[c\left(\bar{y}\right)\right] > u_c\left[c\left(x_{t+1} > \bar{y}\right)\right].$$

- A contradiction emerges!

Potentially binding constraints

- Summary of the results so far:
  - When $\beta(1+r) > 1$, consumption and assets diverge over time.
  - When $\beta(1+r) = 1$, consumption and assets remain bounded.
  - When $\beta(1+r) < 1$, consumption remains bounded and assets converge to 0.

References I

Chamberlain, G. and C. A. Wilson (2000, July). Optimal Intertemporal Consumption Under Uncertainty. *Review of Economic Dynamics 3*(3), 365–95.