LECTURES 5

The income fluctuations problem

Part II

Macroeconomics 4

A.Y. 2014-15

Uncertainty

- Assume now that $\{y_t\}_{t=0}^{\infty}$ follows some generic stochastic process.
- For the moment, let's abstract from borrowing constraints and impose the *NPG* condition only:

$$\mathbb{E}_0\left[\lim_{t\to\infty}\frac{a_{t+1}}{(1+r)^t}\right]\geq 0.$$

- The Euler equation becomes the following:

$$\mathbb{E}_t\left[u_c\left(c_{t+1}\right)\right]=\frac{u_c\left(c_t\right)}{\beta\left(1+r\right)}.$$

- As Bob Hall pointed out, since u_c follows a univ. first-order Markov process, no other variable should Granger-cause it.
- If $\beta\left(1+r\right)=1$, then u_c follows a **random walk**:

$$\mathbb{E}_t\left[u_c\left(c_{t+1}\right)\right]=u_c\left(c_t\right).$$

Permanent Income Hypothesis

- Before going on, let's quickly discuss Modigliani's and Friedman's **Permanent Income Hypothesis** (*PIH*).

Definition

PIH states that the households saves (dissaves) in anticipation of possible future decreases (increases) in labor - or more precisely *non-financial* - income.

- In general, the household save also for precautionary reasons, possibly induced by **prudence** ($u_{ccc} > 0$).

- Furthermore, the household possibly dissaves because of impatience, i.e. when $\beta (1 + r) < 1$.

Permanent Income Hypothesis

- Assume quadratic utility: $u(c_t) = c_t - \frac{\alpha}{2} c_t^2$, for some $\alpha > 0$. Furthermore, set $\beta(1+r) = 1$.
  - These assumptions kill the demand for precautionary savings and rule impatience out.
- The Euler equation implies that c_t follows a random walk:

$$\mathbb{E}_t(c_{t+1}) = c_t.$$

- In general, $\mathbb{E}_t(c_{t+s}) = c_t$ for all $s \geq 1$, so that consumption is actually a **martingale**.

Permanent Income Hypothesis

- It is straightforward to show that:

$$c_t = ra_t + h_t,$$

where $h_t = \frac{r}{1+r} \sum_{s=t}^{\infty} \frac{\mathbb{E}_t(y_s)}{(1+r)^{s-t}}$.

- Note that the volatility of income does not play any role in the previous decision rule: moments higher than the first of the income process do not matter! This is known as **certainty equivalence**.

- Tedious calculations confirm that:

$$\Delta c_{t+1} = \frac{r}{1+r} \sum_{s=t+1}^{\infty} \frac{\mathbb{E}_{t+1}(y_s) - \mathbb{E}_t(y_s)}{(1+r)^{s-t-1}}.$$

- Under the *PIH*, Δc_{t+1} is proportional to the revision in expected income due to the new pieces of information accruing over time.

Permanent Income Hypothesis

- Suppose now that income follows an $AR(1)$ process, i.e. $y_t = \rho y_{t-1} + \varepsilon_t$, where is a zero-mean *iid* innovation.
- In this case, it turns out that:

$$\Delta c_{t+1} = \frac{r}{1+r-\rho}\varepsilon_{t+1}.$$

  - If $\rho = 0$, then $\Delta c_{t+1} = \frac{r}{1+r}\varepsilon_{t+1}$: the household expects the income shock to be fully temporary so it only consumes its annuity value.
  - If $\rho = 1$, then $\Delta c_{t+1} = \varepsilon_{t+1}$: the household consumes all of the income shock, as it expects it to be permanent.
- The *Marginal Propensity to Consume* (MPC) out of an income shock is evidently increasing in the persistence of the shocks themselves.

Permanent Income Hypothesis

- Assume that the income process is the sum of two orthogonal components, a permanent one, $z_t = z_{t-1} + \varepsilon_t$, and a transitory one, ϵ_t, so that $y_t = z_t + \epsilon_t$.
- In this case, the household will respond weakly to transitory shocks and one for one to permanent ones:

$$\Delta c_{t+1} = \frac{r}{1+r} \epsilon_{t+1} + \varepsilon_{t+1}.$$

- Let var_i denote cross-sectional variance; then:

$$\text{var}_i \left(\Delta c_t \right) = \left(\frac{r}{1+r} \right)^2 \text{var}_i \left(\epsilon_t \right) + \text{var}_i \left(\varepsilon_t \right) \approx \text{var}_i \left(\varepsilon_t \right),$$
$$\text{var}_i \left(\Delta y_t \right) = 2\text{var}_i \left(\epsilon_t \right) + \text{var}_i \left(\varepsilon_t \right).$$

- With cross-sectional data on cons. and income one can separately identify the var. of underlying structural income shocks.

PIH and CARA utility

- Wang (2003) shows that the *PIH* describes the optimal cons. rule also in a model characterized by:
  - *CARA* utility:
    $$u(c) = -\frac{\exp(-\theta c)}{\theta},$$
    where $\theta > 0$.
  - Stationary $AR(1)$ exo. income:
    $$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma \varepsilon_t,$$
    where $|\phi_1| < 1$, $\sigma > 0$ and ε_t is *iid*, normally dist. with zero mean and unit var.

PIH and CARA utility

- In particular, he proves that:

$$c_t = ra_t + h_t - \Gamma(r),$$

where $\Gamma(r)$ depends on the parameters θ, β, ϕ_0, ϕ_1, and σ.

- As in the quadratic utility case, the *MPC* out of financial wealth and out of human wealth are equal.

- Wang (2003) shows that, in *general equilibrium*, both the demand for precautionary savings and dissavings due to impatience are constant, and exactly equal, so that they effectively cancel out, and the household behaves in accordance with *PIH*.

PIH and borrowing constraints

- Assume again quadratic utility; we can show that:

$$\Delta a_{t+1} = -\sum_{s=t}^{\infty} \frac{\mathbb{E}_t\left(\Delta y_{s+1}\right)}{\left(1+r\right)^{s-t}}.$$

- If y_t follows a random walk, then $\mathbb{E}_t\left(\Delta y_{s+1}\right) = 0$ for all $s \geq t$, and $\Delta a_{t+1} = 0$. In this case, if the household starts above the borrowing constraint, the latter will never be binding.

- If y_t follows an *iid* process, $y_t = \varepsilon_t$, then $\mathbb{E}_t\left(\Delta y_{t+s}\right) = -\varepsilon_t$ for $s = 1$ and $\mathbb{E}_t\left(\Delta y_{t+s}\right) = 0$ for $s > 1$, so that $\Delta a_{t+1} = \varepsilon_t$. In this case, wealth follows a random walk, and any constraint on asset holdings will be binding with probability one.

- In general, **borrowing constraints cannot be ignored**!

PIH and borrowing constraints

- Consider now the "Euler inequality" under quadratic utility and a potentially binding credit constraint:

$$-\alpha c_t + 1 = \max\left\{-\alpha\left[(1+r)\,a_t + y_t\right] + 1, -\alpha\mathbb{E}_t\left(c_{t+1}\right) + 1\right\}.$$

- This can be rewritten as:

$$\begin{aligned} c_t &= \min\left[(1+r)\,a_t + y_t, \mathbb{E}_t\left(c_{t+1}\right)\right] \\ &= \min\left((1+r)\,a_t + y_t, \mathbb{E}_t\left\{\min\left[(1+r)\,a_{t+1} + y_{t+1}, \mathbb{E}_{t+1}\left(c_{t+2}\right)\right]\right\}\right), \end{aligned}$$

  and so forth, so that $c_t \leq \mathbb{E}_t\left(c_{t+1}\right)$.

- Thus, if the solution to the *PIH* problem is such that $a_t > 0$ for all t with probability one, then the presence of borrowing constraints does not affect the solution.

PIH and borrowing constraints

- In the absence of borrowing constraints, $c_t = \mathbb{E}_t(c_{t+s})\ \forall s \geq 1$.
- Suppose that, for some realization of y_{t+1} with positive probability, the household will be borrowing constrained in period $t+1$, so that:

$$c_{t+1} = (1+r)\,a_{t+1} + y_{t+1} < \mathbb{E}_{t+1}(c_{t+2}).$$

- For the *Law of Iterated Expectations*, the previous "Euler inequality" implies:

$$c_t = \mathbb{E}_t \left\{ \min\left[(1+r)\,a_{t+1} + y_{t+1}, \mathbb{E}_t(c_{t+2}) \right] \right\},$$

- Hence, $c_t < \mathbb{E}_t(c_{t+2})$!
- **Even if the liquidity constraint is not binding in period t, future potentially binding borrowing constraints affect current consumption choices.**

PIH and borrowing constraints

- Suppose now that the variance of future income increases, say for $t+1$, making more low realizations of y_{t+1} possible or likely.
- If the set of y_{t+1} realizations for which the borrowing constraint binds expands, then, for given $\mathbb{E}_t\left(c_{t+2}\right)$, the term:

  $$\mathbb{E}_t\left\{\min\left[\left(1+r\right)a_{t+1}+y_{t+1},\mathbb{E}_t\left(c_{t+2}\right)\right]\right\}$$

  declines, and so does c_t, if the household is unconstrained in period t.
- **Savings increase as a consequence of risk aversion: precautionary savings emerge even without prudence, i.e. without a precautionary savings motive, because of the existence of borrowing constraints.**

Borrowing constraints under uncertainty

- Let us now finally consider the case with stochastic income and potentially binding borrowing constraints.
- The "Euler inequality" becomes the following:

$$\begin{cases} u_c(c_t) > \beta(1+r)\mathbb{E}_t[u_c(c_{t+1})] & \text{if } a_{t+1} = 0, \\ u_c(c_t) = \beta(1+r)\mathbb{E}_t[u_c(c_{t+1})] & \text{if } a_{t+1} > 0. \end{cases}$$

- Again, define $M_t \equiv u_c(c_t)[\beta(1+r)]^t > 0$; the "Euler inequality" implies:

$$M_t \geq \mathbb{E}_t[M_{t+1}].$$

- In other words, it implies that M_t follows a **supermartingale**.

Borrowing constraints under uncertainty

Theorem

Doob's supermartingale convergence theorem. *Let M_t follow a supermartingale. Then:*

$$P\left(\lim_{t\to\infty} M_t = \bar{M}\right) = 1$$

where $\bar{M}$ is a non-negative random variable such that $E\left(\bar{M}\right) < +\infty$.

- Loosely speaking, the above limit is (almost surely) finite.

Borrowing constraints under uncertainty

- Consider the case $\beta(1+r) > 1$.
- $M_t = u_c(c_t)[\beta(1+r)]^t$ converges to a finite limit, but evidently $\lim_{t\to\infty}[\beta(1+r)]^t = \infty$. Thus, necessarily $P[u_c(c_t) \to 0] = 1$.
- Being $u_c > 0$, this obviously implies that $P(c_t \to \infty) = 1$.
- Since income and debt are bounded, this can only be achieved with $P(a_t \to \infty) = 1$.
- The **asset space is therefore unbounded**, as in the case under certainty.

Borrowing constraints under uncertainty

- Consider now the case $\beta(1+r) = 1$, so that $M_t = u_c(c_t)$.

Theorem

If the exogenous income process is "sufficiently stochastic," i.e. if there is a $\varphi > 0$ such that:

$$var_t \left(\sum_{s=t}^{\infty} \frac{y_s}{(1+i)^{s-t}} \right) \geq \varphi,$$

for all $t \geq 0$, and the istant. utility function u is ***bounded****, then:*

$$P\left(\lim_{t\to\infty} c_t = \infty \right) = 1.$$

Proof.

Chamberlain and Wilson (2000), Corollary 2, p. 381. □

Borrowing constraints under uncertainty

Corollary

Being income and debt bounded, $P\left(\lim_{t\to\infty} c_t = \infty\right) = 1$ *implies:*

$$P\left(\lim_{t\to\infty} a_t = \infty\right) = 1.$$

- Chamberlain and Wilson (2000) allow the interest r to follow a stochastic process, too.
- Sotomayor (1984) proves the previous results assuming an *iid* exo. income process and **unbounded** istant. utility.

Borrowing constraints under uncertainty

- We will now develop some intuition by assuming that y_t follows an *iid* process, so that the "cash in hand" trick applies.
- Using the envelope condition and the "Euler inequality" we show that V_x is a supermartingale too:

$$V_x(x_t) \geq \mathbb{E}_t \left[V_x(x_{t+1}) \right].$$

- Hence, V_x converges to a non-negative random variable. If this limit is strictly positive, then $P(x_t \to \bar{x} < \infty) = 1$.
- Consumption is strictly increasing in x_t: thus, if $P(x_t \to \bar{x} < \infty) = 1$ then $P(c_t \to \bar{c} < \infty) = 1$.

Borrowing constraints under uncertainty

- However, consider the budget constraint:

$$x_{t+1} - (1 + r)(x_t - c_t) = y_{t+1}.$$

- The left hand side would converge a.s. to $\bar{x} - (1 + r)(\bar{x} - \bar{c})$, the right hand side would not: a contradiction!

- Hence, it has to be that $V_x(x_t) \to 0$ and $x_t \to \infty$ a.s.

- If $u_{ccc} > 0$, i.e. if the household shows prudence, the intuition is simple; from the "Euler ineq.":

$$u_c(c_t) \geq \mathbb{E}_t[u_c(c_{t+1})] > u_c[\mathbb{E}_t(c_{t+1})],$$

  where the last step is due to *Jensen's inequality*.

- Concavity implies $c_t < \mathbb{E}_t(c_{t+1})$: c_t will increase over time.

Borrowing constraints under uncertainty

- Finally, consider the case $\beta\,(1+r) < 1$.
- Theoretical results in this case are unfortunately very limited: some general propositions are available for the *iid* case only.
- Assume that y_t follows an *iid* process with minimum income level $y_1 \geq 0$ and maximum income level y_N.

Theorem

If y_t is iid, then the following holds:
(i) $0 < c_x(x) \leq 1$, i.e. c is strictly increasing in x,
(ii) $a'(x) = 0$ or $0 < a'_x(x) < 1$, a' is zero or strictly increasing in x.

- Hence, $a'(x) \geq 0$ and $x' = a' + y' \geq y_1$ so that y_1 is a lower bound on the state space for x.

Borrowing constraints under uncertainty

Theorem

There exists $\bar{x} \geq y_1$ s.t. for all $x \leq \bar{x}$ we have $c(x) = x$ and $a'(x) = 0$.

Proof.

Suppose, to the contrary, that $a'(x) > 0$ for all $x \leq \bar{x}$; then:

$$V_x(x) = \beta(1+r)\mathbb{E}\left[V_x(x')\right] \leq \beta(1+r)V_x(y_1) < V_x(y_1).$$

If $x = y_1$ this yields a contradiction. □

- Hence, there is a cutoff level of x below which the household becomes borrowing constrained.

Borrowing constraints under uncertainty

Theorem

Suppose that u_c has the property that there exists a finite ϱ s.t.:

$$\lim_{c \to \infty} \log_c u_c(c) = \varrho.$$

Then there exists a $\tilde{x}$ s.t. $x' = a'(x) + y_N \leq x$ for all $x \geq \tilde{x}$.

Proof.

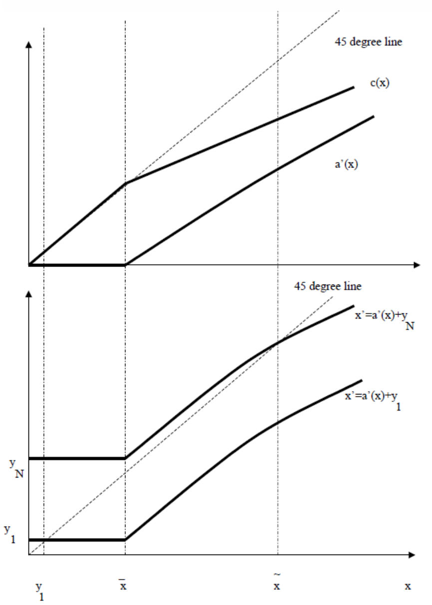
See Schechtman and Escudero (1977), Theorems 3.8 and 3.9. □

- Note that, for CRRA utility functions, $\log_c c^{-\sigma} = -\sigma \log_c c = -\sigma$, so the prev. proposition applies.
- However, it does not apply to CARA functions: $\log_c e^{-c} = ... = -\infty$.

Borrowing constraints under uncertainty

- If the previous Theorem holds, x stays in the bounded set $X = [y_1, \tilde{x}]$: this is extremely important from a numerical point of view!
- Furthermore, $c(x) = x$ for $x \leq \bar{x}$ and $c(x) < x$ for $x > \bar{x}$, so that $a'(x) > 0$.
- The following slides provides a graphical representation of this results.

Borrowing constraints under uncertainty



Borrowing constraints under uncertainty

- When y_t is correlated over time, the "cash in hand" trick does not work anymore, and not much can be said in general about the properties of $c(a, y)$ and $a'(a, y)$.
- Huggett (1993) proves that:
  - $c(a, y)$ is strictly increasing in a,
  - $a'(a, y)$ is constant at the borrowing limit or strictly increasing,
  - a cutoff value $\bar{a}(y)$ that depends on income exists.
- It turns out to be very difficult to prove the existence of an upper bound of the state space.
  - Huggett (1993) proves it for the following special case: income takes only two states, $y \in \{y_L, y_H\}$ with $0 < y_L < y_H$, $\pi(H \mid H) \geq \pi(H \mid L)$, and *CRRA* utility.

Borrowing constraints under uncertainty

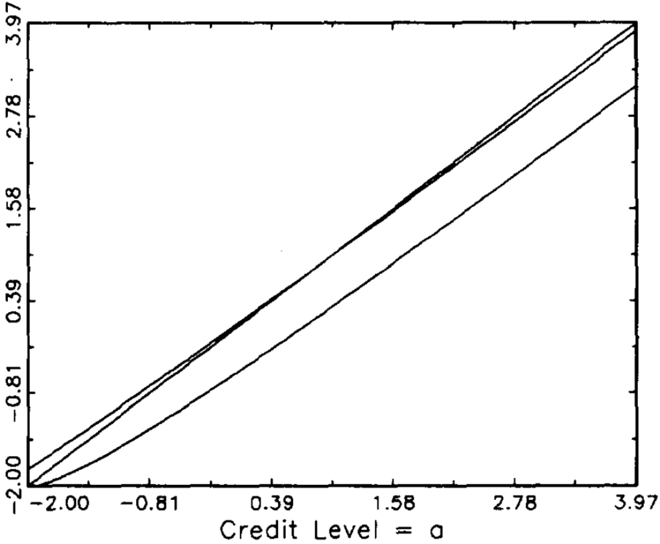


Fig. 1. Optimal decision rule.

Borrowing constraints under uncertainty

- Summary of the results so far:
  - When $\beta(1+r) > 1$, consumption and assets diverge over time, almost surely.
  - When $\beta(1+r) = 1$, consumption and assets are expected to diverge over time, almost surely; we can formally prove it if utility is bounded or incomes shocks are *iid*.
  - When $\beta(1+r) < 1$, consumption and assets may remain bounded, but we can prove it only in special cases.

References I

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