Lectures 8

Solving models with occasionally binding constraints

Part II

Macroeconomics 4

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Time iteration

- Consider the Euler equation for our recursive problem:

$$\begin{aligned}
u_c\left(c_{z,i}\right)-\varphi_{z,i} &= \beta\left(1+r\right)\sum_{j=1}^{n}\Pi_{ij}u_c\left(c'_{z,i,j}\right),\\
c_{z,i} &= \left(1+r\right)\mathrm{a}_z+w\mathrm{s}_i-a'_i\left(\mathrm{a}_z\right),\\
c'_{z,i,j} &= \left(1+r\right)a'_i\left(\mathrm{a}_z\right)+w\mathrm{s}_j-a'_j\left[a'_i\left(\mathrm{a}_z\right)\right],\\
a'_i &\geq 0,\\
\varphi_{z,i}a'_i &= 0,\\
\varphi_{z,i} &\geq 0.
\end{aligned}$$

where $z \in \{1, 2, ..., m\}$ and $i, j \in [1, 2, ..., n]$.

Time iteration

- Suppose that a current guess for the policy function(s), say $a'_{k,i}(a)$, exists.

- For the moment, set $\varphi_{z,i} = 0$ for all z and i. The Euler eq. boils down to:

$$u_c\left[(1+r)\,\mathrm{a}_z + w\mathrm{s}_i - h_{k,i}(\mathrm{a}_z)\right] = \beta\,(1+r)\times \sum_{j=1}^{n}\Pi_{ij}u_c\left\{(1+r)\,h_{k,i}(\mathrm{a}_z) + w\mathrm{s}_j - a'_{k,j}\left[h_{k,i}(\mathrm{a}_z)\right]\right\}.$$

- This is a system of $m\times n$ non-linear eqs. that can be solved for the values of $h_{k,i}(\mathrm{a}_z)$, using some robust method like the *Trust Region algorithm*.

- Note that the term $a'_{k,j}\left[h_{k,i}(\mathrm{a}_z)\right]$ has to be computed via interpolation.

Time iteration

- The current guess $a'_{k,i}(a)$ is then updated in the following way:

$$\begin{cases} a'_{k+1,i}(\mathrm{a}_z) = 0 & \text{if } h_{k,i}(\mathrm{a}_z) \leq 0 \\ a'_{k+1,i}(\mathrm{a}_z) = h_{k,i}(\mathrm{a}_z) & \text{if } h_{k,i}(\mathrm{a}_z) > 0 \end{cases}, \quad \forall i, z.$$

- Convergence can be assessed with the usual stopping rule:

$$\left\| a'_{k+1,i}(a) - a'_{k,i}(a) \right\| \leq \varepsilon > 0, \quad \forall i.$$

- Note that, if needed, the multiplier $\varphi_{z,i}$ can be obtained from:

$$\varphi_{z,i} = u_c(c_{z,i}) - \beta(1+r) \sum_{j=1}^{n} \Pi_{ij} u_c\left(c'_{z,i,j}\right).$$

Time iteration

- This method is know as *Time Iteration on the Euler equation* (*TI*).
- In standard models, i.e. without borrowing constraints, under the usual regularity conditions the method converges to a solution of the Bellman equation.
- Rendhal (2013) shows that is the case also for models with occasionally binding constraints.
- He deals with problems that can be formulated as:

$$V(a,s) = \max_{a' \in \Gamma(a,s)} \left\{ F(a,a',s) + \beta \mathbb{E}\left[V(a',s') \mid s \right] \right\},$$

where $F(\cdot)$ is **bounded** over a **compact** set A, and:

$$\Gamma(a,s) = \left\{ a' \in \mathrm{A} : m_j(a,a',s) \leq 0,\ j = 1, 2, ..., r \right\}.$$

Time iteration

- Standard results show that, under some regularity conditions, the *VFI* scheme:

$$V_{n+1}(a,s) = \max_{a' \in \Gamma(a,s)} \left\{ F(a,a',s) + \beta \mathbb{E}\left[V_n(a',s') \mid s \right] \right\},$$

$$a'_{n+1}(a,s) = \arg \max_{a' \in \Gamma(a,s)} \left\{ F(a,a',s) + \beta \mathbb{E}\left[V_n(a',s') \mid s \right] \right\},$$

  will uniformly converge, respectively, to the unique fixed points V and a' for any weakly concave and bounded V_0.

Time iteration

- The previous *VFI* scheme can be written as:

$$V_{n+1}(a,s) = \min_{\boldsymbol{\mu}\geq 0} \max_{a'\in \mathrm{A}} L(a,a',s,\boldsymbol{\mu}) = \max_{a'\in \mathrm{A}} \min_{\boldsymbol{\mu}\geq 0} L(a,a',s,\boldsymbol{\mu}),$$

where:

$$L(a,a',s,\boldsymbol{\mu}) = F(a,a',s) + \beta\mathbb{E}\left[V_n(a',s') \mid s\right] - \boldsymbol{\mu}^T\mathbf{m}(a,a',s),$$

and $\boldsymbol{\mu}$ is a $r\times 1$ column vector of Lagrange multipliers.

Time iteration

- The *TI* scheme can be described as:

$$F_{a'}\left(a,a',s\right)-\mathbf{J}_{a'}\left(a,a',s\right)\boldsymbol{\mu}_{k+1}\left(a,s\right)+ \\ \beta\mathbb{E}\left[F_{a}\left[a',h_{k}\left(a',s'\right),s'\right]\mid s\right]+ \\ -\beta\mathbb{E}\left[\mathbf{J}_{a}\left(a',h_{k}\left(a',s'\right),s'\right)\boldsymbol{\mu}_{k}\left(a',s'\right)\mid s\right]=0$$

  where $a'=h_{k+1}\left(a,s\right)$, and $\mathbf{J}_x$ the Jacobian of $\mathbf{m}$ w.r.t. x.

- Rendhal (2013) proves that, if $h_k\left(a,s\right)=a'_k\left(a,s\right)$, then:

$$h_{k+1}\left(a,s\right)=a'_{k+1}\left(a,s\right).$$

- Hence, $h_k\left(a,s\right)\rightarrow a'\left(a,s\right)$ when $k\rightarrow\infty$: *TI* converges to the true solution also under occasionally binding constraints.

Fixed point iteration

- *TI* has solid theoretical foundations, but is still cumbersome to implement, and relatively slow to converge.

- A simple variant, known as *Fixed Point Iteration* (*FPI*), has some useful properties, and can be characterized as:

$$u_c\left[(1+r)\,\mathrm{a}_z + w\mathrm{s}_i - h_{k,i}(\mathrm{a}_z)\right] =$$
$$\beta(1+r)\times\sum_{j=1}^{n}\Pi_{ij}u_c\left\{(1+r)\,a'_{k,i}(\mathrm{a}_z) + w\mathrm{s}_j - a'_{k,j}\left[a'_{k,i}(\mathrm{a}_z)\right]\right\}.$$

- Note that the r.h.s. is computed using the current guess $a'_{k,i}(\mathrm{a}_z)$ only, so there is no need to solve any non-linear system: you just need to invert u_c to solve for $h_{k,i}(\mathrm{a}_z)$.

- The guess is the updated with the same rule as before:

$$\bar{a}'_{k,i}(\mathrm{a}_z) = \max\left[h_{k,i}(\mathrm{a}_z), 0\right].$$

Fixed point iteration

- There aren't theoretical results on the convergence properties of *FPI*, to the best of my knowledge.
- However, convergence can be improved, or even "forced," by using a simple **damping scheme**.
- In other words, the updating rule becomes:

$$\begin{cases} a'_{k+1,i}\left(\mathrm{a}_z\right) = 0 & \text{if } h_{k,i}\left(\mathrm{a}_z\right) \leq 0 \\ a'_{k+1,i}\left(\mathrm{a}_z\right) = \varrho \bar{a}'_{k,i}\left(\mathrm{a}_z\right) + \left(1-\varrho\right) a'_{k,i}\left(\mathrm{a}_z\right) & \text{if } h_{k,i}\left(\mathrm{a}_z\right) > 0 \end{cases},$$

for some $\varrho \in (0,1)$, say $\varrho = 0.9$.

- There are no general rules - some nasty cases require low values of ϱ to converge - but *FPI* is considerably faster, and, if it converges, then it converges to the unique solution of the *VFI* scheme.

Fixed point iteration

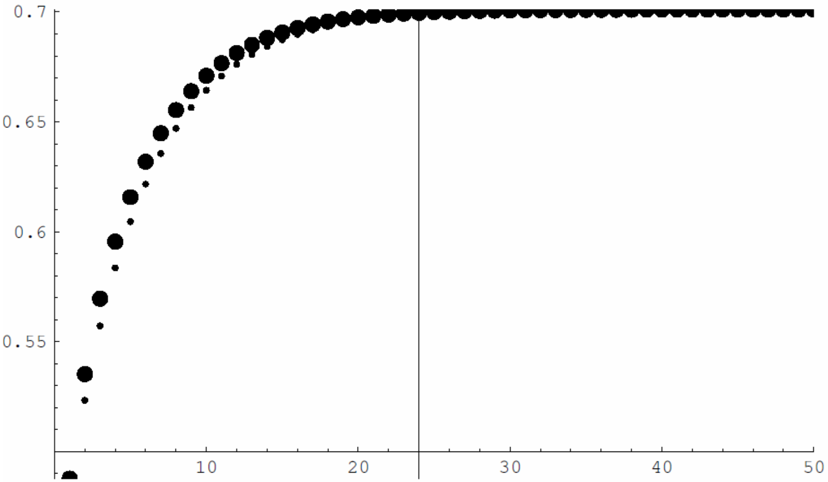

Figure 4 $c_{T-s}(\underline{k}, \underline{\theta})$ Using Time Iteration versus Fixed Point Iteration

References I

Rendhal, P. (2013). Inequality Constraints and Euler Equation based Solution Methods. *The Economic Journal*, forthcoming.