Lectures 9

Bewley models

Part I

Macroeconomics 4

A.Y. 2014-15

The simplest setting

- For the sake of simplicity, consider again the "constrained assets" version of the income fluc. problem.
- Hence, discretize the state space and constrain asset holdings on the finite-dimensional grid:

$$\mathcal{A} = \{0 < \mathrm{a}_1 < \mathrm{a}_2 < ... < \mathrm{a}_m\}.$$

- As prev. discussed, the problem becomes:

$$V_i(\mathrm{a}_z) = \max_{a' \in \mathcal{A}} \left\{ u\left[(1+r)\,\mathrm{a}_z + w\mathrm{s}_i - a'\right] + \beta \sum_{j=1}^{n} \Pi_{ij} V_j(a') \right\}.$$

- The pol. function is represented by a set of $m \times m$ matrices $\mathbf{G}_i$:

$$\mathbf{G}_i(z,j) = \begin{cases} 1 & \text{if } g(\mathrm{a}_z, \mathrm{s}_i) = \mathrm{a}_j \\ 0 & \text{if } g(\mathrm{a}_z, \mathrm{s}_i) \neq \mathrm{a}_j \end{cases}.$$

The ergodic distribution

- Denote as λ_t the **uncond. prob. dist.** of $\{a_t, s_t\}$, and represent it as a $m \times n$ matrix, with non-negative elements that sum to unity, s.t.:

$$\lambda_t\left(\mathrm{a}_z, \mathrm{s}_i\right) = \operatorname{prob}\left(a_t = \mathrm{a}_z, s_t = \mathrm{s}_i\right).$$

- The Markov chain for s and the policy function $g(a, s)$ *induce* a **Law of Motion** (*LoM*) for the distribution λ_t:

$$\underbrace{\operatorname{prob}\left(a_{t+1} = \mathrm{a}_z, s_{t+1} = \mathrm{s}_i\right)}_{\text{Unconditional } t+1} =$$

$$\sum_{h=1}^{n} \sum_{j=1}^{m} \underbrace{\operatorname{prob}\left(a_{t+1} = \mathrm{a}_z | a_t = \mathrm{a}_h, s_t = \mathrm{s}_j\right)}_{\text{Policy function}} \times$$

$$\underbrace{\operatorname{prob}\left(s_{t+1} = \mathrm{s}_i | s_t = \mathrm{s}_j\right)}_{\text{Transition prob.}} \times$$

$$\underbrace{\operatorname{prob}\left(a_t = \mathrm{a}_h, s_t = \mathrm{s}_j\right)}_{\text{Unconditional } t}.$$

The ergodic distribution

- The intuition goes as follows: let us compute the probability of transiting from, say, $(\mathrm{a}_h, \mathrm{s}_j)$ to $(\mathrm{a}_z, \mathrm{s}_i)$.

- The starting point is the uncond. prob. of being in $(\mathrm{a}_h, \mathrm{s}_j)$, i.e. $\text{prob}\,(a_t = \mathrm{a}_h, s_t = \mathrm{s}_j)$.

- Then, we need to take the probability of transiting from s_j to s_i into account:
$$\text{prob}\,(s_{t+1} = \mathrm{s}_i | s_t = \mathrm{s}_j) = \Pi_{ji}.$$

- The (degenerate) prob. of transiting from a_h to a_z, given s_i, is given by the pol. function:
$$\text{prob}\,(a_{t+1} = \mathrm{a}_z | a_t = \mathrm{a}_h, s_t = \mathrm{s}_j) = \begin{cases} 1 & \text{if } g\,(\mathrm{a}_h, \mathrm{s}_j) = \mathrm{a}_z \\ 0 & \text{if } g\,(\mathrm{a}_h, \mathrm{s}_j) \neq \mathrm{a}_z \end{cases}.$$

The ergodic distribution

- The uncond. prob. of reaching state $(\mathrm{a}_z, \mathrm{s}_i)$ in the following period is the sum, over all elements of $\mathcal{A} \times \mathcal{S}$, of the prob. of transiting from any other state into $(\mathrm{a}_z, \mathrm{s}_i)$.
- The *LoM* can be compactly rewritten as:

$$\lambda_{t+1}(\mathrm{a}_z, \mathrm{s}_i) = \sum_{h=1}^{m} \sum_{j=1}^{n} \mathbf{G}_j(h, z)\, \Pi_{ji} \lambda_t(\mathrm{a}_h, \mathrm{s}_j)$$

$$= \sum_{j=1}^{n} \sum_{\{a : \mathrm{a}_z = g(a, \mathrm{s}_j)\}} \Pi_{ji} \lambda_t(a, \mathrm{s}_j).$$

The ergodic distribution

Definition

A **stationary (ergodic) distribution** is a time-invariant dist. λ that solves:

$$\lambda\left(\mathrm{a}_z, \mathrm{s}_i\right) = \sum_{h=1}^{m} \sum_{j=1}^{n} \mathbf{G}_j\left(h, z\right) \Pi_{ji} \lambda\left(\mathrm{a}_h, \mathrm{s}_j\right),$$

for $z = 1, 2, ..., m$ and $i = 1, 2, ..., n$.

- For the *Law of Large Numbers*, the ergodic dist. λ will reproduce, in the limit, the fraction of time that households spend in each state $\{\mathrm{a}_z, \mathrm{s}_i\}$.

The ergodic distribution

- The stationary *LoM* can be written in matrix notation as:

$$\text{vec}\left(\lambda\right) = \mathbf{P}^T \text{vec}\left(\lambda\right),$$

where:

$$\mathbf{P} \equiv \begin{bmatrix} \Pi_{11}\mathbf{G}_1 & \Pi_{21}\mathbf{G}_2 & \cdots & \Pi_{n1}\mathbf{G}_n \\ \Pi_{12}\mathbf{G}_1 & \Pi_{22}\mathbf{G}_2 & \cdots & \Pi_{n2}\mathbf{G}_n \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{1n}\mathbf{G}_1 & \Pi_{2n}\mathbf{G}_2 & \cdots & \Pi_{nn}\mathbf{G}_n \end{bmatrix} =$$

$$\left(\mathbf{\Pi} \otimes \mathbf{I}_m\right) \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_n \end{bmatrix}.$$

The ergodic distribution

- The $nm \times nm$ matrix $\mathbf{P}$ is right stochastic, being $\mathbf{\Pi}$ so by assumption.

- Clearly, $\operatorname{vec}(\lambda)$ can be interpreted as the ergodic distribution of a discrete Markov chain characterized by the transition matrix $\mathbf{P}$.

- Note that $\mathbf{P}$ is *sparse matrix*: the best way to numerically compute $\operatorname{vec}(\lambda)$ in this case is to iterate until convergence on (taking the sparsity into account):

$$\operatorname{vec}(\lambda_{k+1}) = \mathbf{P}^T \operatorname{vec}(\lambda_k),$$

  given a suitable initial guess $\operatorname{vec}(\lambda_0)$.

General equilibrium

- The economy is populated by a *continuum* of measure 1 of ex-ante identical households that all face an income fluctuation problem.
- Assume that:
  - Each household faces the **same** stochastic labor endowment process, which follows a stationary Markov process.
  - The income processes are pairwise uncorrelated, so that risk is purely idiosyncratic.
- The key question is: does this idiosyncratic risk disappear upon aggregation? In other words, can a **Law of Large Numbers** (*LoLN*) be applied in this case?

General equilibrium

- Uhlig (1996) provides a suitable *LoLN*:

Theorem

Let $X(i)$, $i \in [0,1]$, be a collection of identically distributed and pairwise uncorrelated random variables with common finite mean μ and variances bounded above by $\sigma^2 < \infty$.
Then X is L_2-Riemann integrable, and the integral is almost everywhere constant:

$$\text{prob}\left[\int_0^1 X(i)\, di = \mu\right] = 1.$$

Proof.

Uhlig (1996), Th. 2, p. 43. □

General equilibrium

- If a *LoLN* holds, then the stationary distribution λ measures:
  - The fraction of time that households spend in each state $\{\mathrm{a}_z, \mathrm{s}_i\}$.
  - The fraction of the total population in each state $\{\mathrm{a}_z, \mathrm{s}_i\}$.
- In other words, λ can be interpreted as the **steady-state wealth distribution**.

GE: Pure Credit

- Following Huggett (1993), assume that households have access to a loan market in which they can borrow or lend at the risk-free interest rate r; no other assets are available.
- This implies that loans are in zero net aggregate supply.

Definition

A **stationary equilibrium** is an interest rate r, a policy function $a' = g(a, s)$, and a distribution $\lambda(a, s)$, such that:

- $g(a, s)$ solves the household's problem,
- $\lambda(a, s)$ is the stationary distribution induced by Π and $g(a, s)$,
- the loan market clears:

$$\sum_{z=1}^{m} \sum_{i=1}^{n} \lambda(\mathrm{a}_z, \mathrm{s}_i) \underbrace{g(\mathrm{a}_z, \mathrm{s}_i)}_{a'} = 0.$$

GE: Pure Credit

- To compute the equilibrium, we can use the following algorithm:

Algorithm

1) Choose an initial guess for r, say $r_j > 0$ where $j = 0$.
   a) Given r_j, solve the household problem for $g_j(a, s)$ and $\lambda_j(a, s)$.
   b) Check whether the loan market clears by computing the excess demand (or supply) of loans:

$$\sum_{z=1}^{m} \sum_{i=1}^{n} \lambda_j(\mathrm{a}_z, \mathrm{s}_i)\, g_j(\mathrm{a}_z, \mathrm{s}_i) = E_j$$

   c) If $E_j > 0$, then set $r_{j+1} < r_j$. If, instead, $E_j < 0$, then set $r_{j+1} > r_j$.
2) Iterate steps $(a) - (c)$ until convergence.

GE: Physical Capital

- Aiyagari (1994) studies a more general version of the previous model.
- Households are allowed to invest in a single, homogenous, capital good, and denote k_t the household's capital holdings.
- No other assets exist: households are not allowed to borrow or lend on a loan market (in this case the borrowing constr. is redundant since $k \geq 0$ by assumption).
- k_t evolves according to the following accumulation equation:

$$k_{t+1} = (1 + r_K - \delta)\, k_t + w s_t - c_t,$$

where $r_K \equiv r + \delta$ is the rental rate.

GE: Physical Capital

- Denote $\lambda(k, s)$ the stationary distribution of capital across households.

- The agg. (per-capita) steady-state capital stock is:

$$K' = K = \sum_{z=1}^{m} \sum_{i=1}^{n} \lambda(\mathrm{k}_z, \mathrm{s}_i) \underbrace{g(\mathrm{k}_z, \mathrm{s}_i)}_{k'}.$$

- The agg. employment rate can be computed as:

$$N = \sum_{z=1}^{m} \sum_{i=1}^{n} \lambda(\mathrm{k}_z, \mathrm{s}_i)\, \mathrm{s}_i = \pi' \mathbf{S},$$

  where π is the invariant distribution associated with Π and $\mathbf{S} = [\mathrm{s}_0, \mathrm{s}_1, ..., \mathrm{s}_n]'$ the corresponding set of possible realizations.

GE: Physical Capital

- A representative firm combines K and N to produce the single cons./inv. good via the following agg. prod. function:

$$Y = F(K, N) \equiv K^{\alpha} N^{1-\alpha},$$

where $\alpha \in (0, 1)$.

- The *FOC*s for the problem of the firm imply that:

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha},$$

$$r_K = \alpha\left(\frac{K}{N}\right)^{\alpha-1}.$$

GE: Physical Capital

Definition

A **stationary equilibrium** is a policy function $k' = g(k, s)$, a distribution $\lambda(k, s)$, and a triple of positive real numbers $\{K, r_K, w\}$, such that:

- $g(k, s)$ solves the household's problem;
- $\lambda(k, s)$ is the stationary dist. induced by Π and $g(k, s)$;
- The factor prices satisfy the firm's FOCs;
- The agg. capital stock K is implied by households' decisions:

$$K = \sum_{z=1}^{m} \sum_{i=1}^{n} \lambda(\mathrm{k}_z, \mathrm{s}_i)\, g(\mathrm{k}_z, \mathrm{s}_i).$$

GE: Physical Capital

- To compute the equilibrium, we can use the following algorithm:

Algorithm

1) Choose an initial guess for K, say $K_j > 0$ where $j = 0$.
   a) Compute w_j and $r_{K,j}$ from the firm's FOCs.
   b) Given w_j and $r_{K,j}$, solve the household problem for $g_j(k,s)$ and $\lambda_j(k,s)$.
   c) Compute the aggregate capital stock:

$$\hat{K}_j = \sum_{z=1}^{m} \sum_{i=1}^{n} \lambda_j(\mathrm{k}_z, \mathrm{s}_i)\, g_j(\mathrm{k}_z, \mathrm{s}_i).$$

   d) Given a damping parameter $\kappa \in (0,1)$, compute a new estimate of K from:

$$K_{j+1} = \kappa K_j + (1-\kappa)\hat{K}_j.$$

2) Iterate steps $(a) - (d)$ until convergence.

References I

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