Lectures 10
Bewley models
Part II

Macroeconomics 4

A.Y. 2014-15

Calibration

- Following Huggett (1993), assume a *CES* form for the istant. utility function, $u(c) = c^{1-\mu}/(1-\mu)$, and set $\mu = 2$; furthermore, set $\beta = 0.97$, $w = 1$, and $\mathrm{b} = 1$.
- Assume that labor income follows a stationary $AR(1)$ process:

$$\ln s_{t+1} = \rho \ln s_t + \sigma \sqrt{(1-\rho^2)}\varepsilon_t$$

  where $\varepsilon_t \sim N(0,1)$, $\rho = 0.53$, and $\sigma = 0.296$.
- Finally, following Aiyagari (1994), se set $\alpha = 0.36$ and $\delta = 0.08$.

Numerical examples

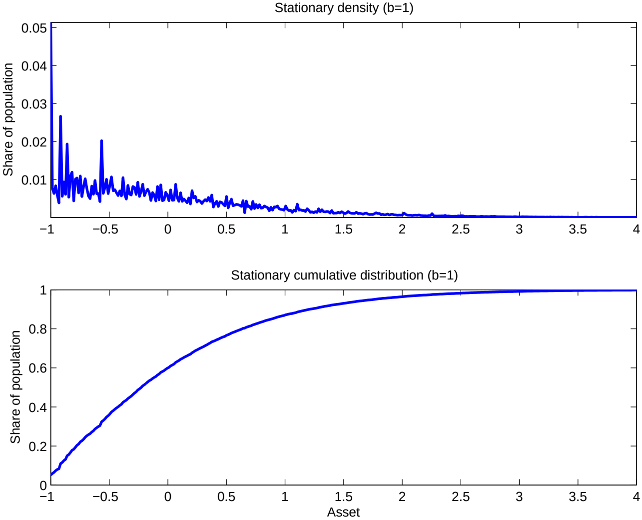

Figure: Pure credit with b = 1.

Numerical examples

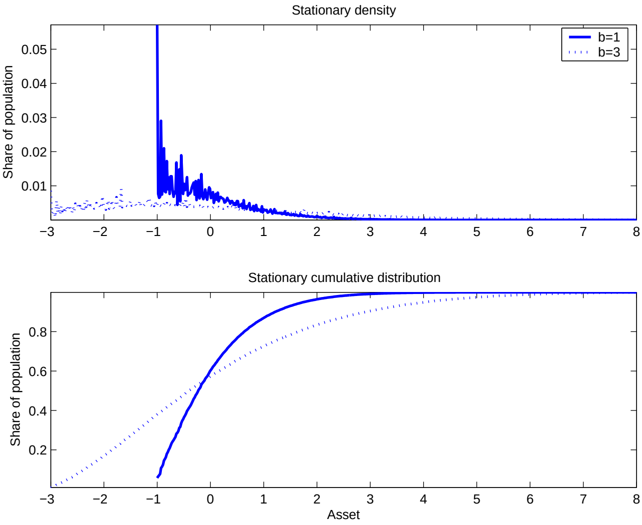

Figure: Pure credit with b = 3.

Numerical examples

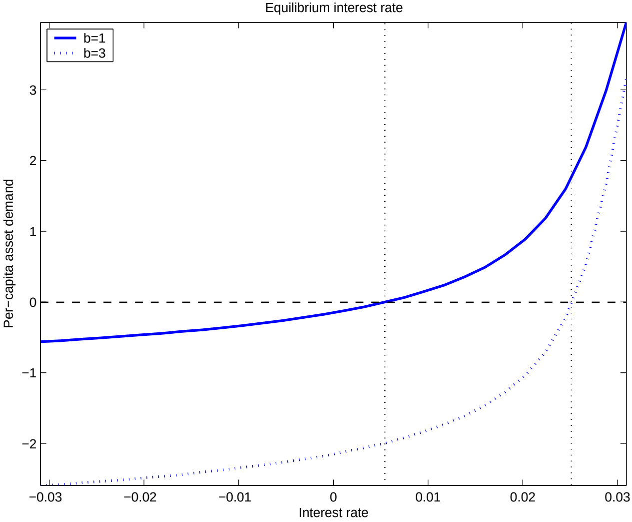

Figure: Pure credit: agg. demand for assets.

Numerical examples

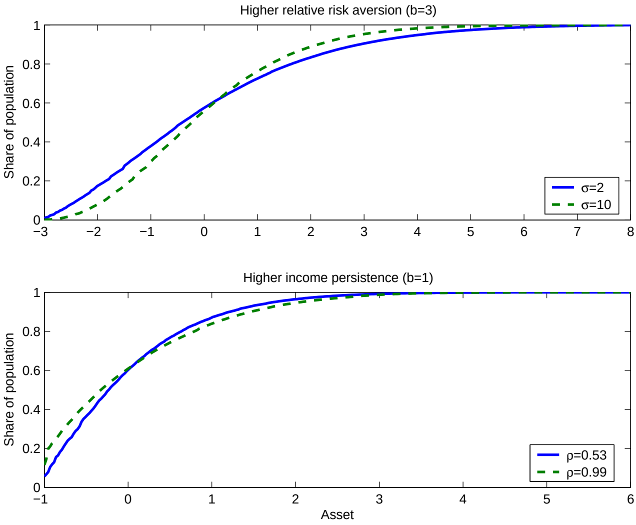

Figure: Pure credit: sensitivity analysis.

Numerical examples

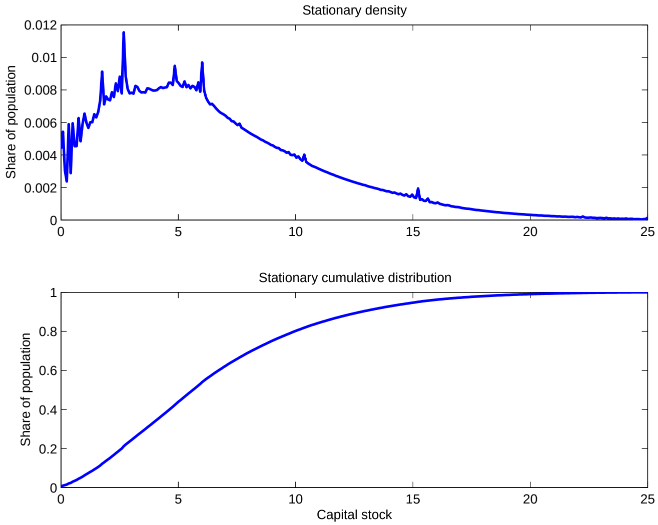

Figure: Physical capital: density.

Numerical examples

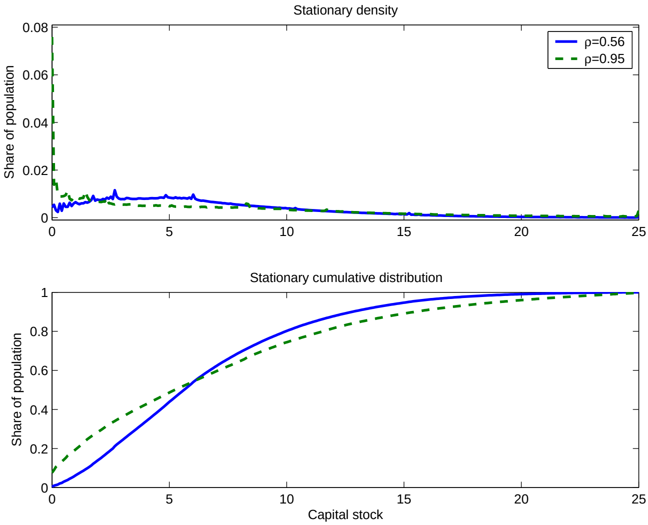

Figure: Physical capital: higher persistence.

Numerical examples

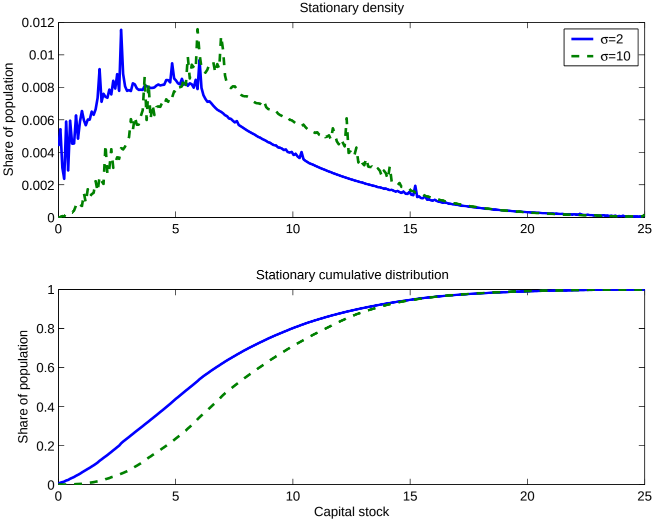

Figure: Physical capital: higher elast. of int. subst.

Numerical examples

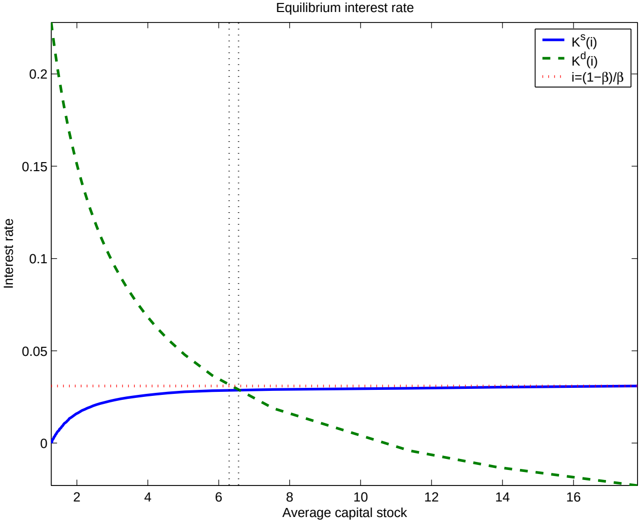

Figure: Physical capital: equilibrium.

Numerical examples

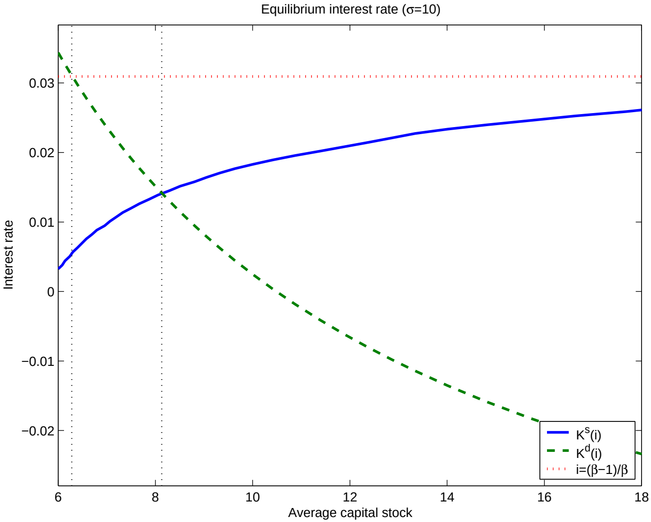

Figure: Physical capital: sensitivity analysis.

Summary of substantive findings

- The equilibrium interest rate is lower (than the time pref. rate) with incomplete markets than it is with complete markets: this is true in both Huggett (1993) and Aiyagari (1994).
  - This may potentially explain the equity premium puzzle, but the difference is quantitatively small.
- The aggregate capital stock in Aiyagari (1994) is larger than it is under complete markets, although again the difference is not quantitatively large.
- The model generates the right ranking between different types of inequality: wealth is more dispersed than income, income is more dispersed than consumption.
- The model does **NOT** generate enough inequality, if idiosincratic shocks are just modeled as shocks to labor earnings.

A fully-fledged version of Aiyagari (1994)

- Let us revert to a **general formulation** of Aiyagari (1994).
- For given sequences $\{w_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$, a generic agent solves:

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\ \text{s.t.} \quad & k_{t+1} \leq (1 - \delta + r_t) k_t + w_t s_t - c_t, \\ & k_{t+1} \geq 0. \end{aligned}$$

- As usual, c_t denotes individual consumption , k_t the individual beginning-of-period capital stock, r_t the rental rate, w_t the hourly wage, and $\delta \in (0, 1)$ the depreciation rate.
- As before, s_t is an idiosincratic shock that follows a discrete Markov chain: $s_t \in \mathcal{S} = \{\mathrm{s}_i\}_{i=1}^{n}$, where $\mathrm{s}_i > 0 \ \forall i$, and $\Pi_{i,j} = \Pi(\mathrm{s}_i, \mathrm{s}_j) = \text{prob}(s_{t+1} = \mathrm{s}_j \mid s_t = \mathrm{s}_i) > 0$.

A fully-fledged version of Aiyagari (1994)

- A Lagrangian is easily obtained:

$$L_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \times$$
$$\left\{u\left(c_t\right) + \xi_t\left[\left(1-\delta+r_t\right)k_t + w_t s_t - c_t - k_{t+1}\right] + \varphi_t k_{t+1}\right\}.$$

- The *FOC*s and slackness conditions read as:

$$\begin{aligned}
u_c\left(c_t\right) &= \xi_t, \\
\xi_t - \varphi_t &= \mathbb{E}_t\left[\beta\xi_{t+1}\left(1-\delta+r_{t+1}\right)\right], \\
k_{t+1} &\leq \left(1-\delta+r_t\right)k_t + w_t s_t - c_t, \\
k_{t+1} &\geq 0, \\
\varphi_t k_{t+1} &= 0, \\
0 &= \xi_t\left[\left(1-\delta+r_t\right)k_t + w_t s_t - c_t - k_{t+1}\right], \\
\varphi_t &\geq 0, \\
\xi_t &\geq 0.
\end{aligned}$$

A fully-fledged version of Aiyagari (1994)

- Since $\lim_{c \to 0} u_c(c) = \infty$ and $s_t > 0$ by assumption, in equilibrium $\xi_t > 0 \ \forall t$.

- Hence, the previous conditions boil down to:

$$\begin{aligned} u_c(c_t) - \varphi_t &= \mathbb{E}_t \left[\beta u_c(c_{t+1}) (1 - \delta + r_{t+1}) \right], \\ k_{t+1} &= (1 - \delta + r_t) k_t + w_t s_t - c_t, \\ k_{t+1} &\geq 0, \\ \varphi_t k_{t+1} &= 0, \\ \varphi_t &\geq 0. \end{aligned}$$

- The "Euler inequality" can be represented as:

$$\begin{cases} u_c(c_t) = \mathbb{E}_t \left[\beta u_c(c_{t+1}) (1 - \delta + r_{t+1}) \right] & \text{if} \quad k_{t+1} > 0, \\ u_c(c_t) \geq \mathbb{E}_t \left[\beta u_c(c_{t+1}) (1 - \delta + r_{t+1}) \right] & \text{if} \quad k_{t+1} = 0. \end{cases}$$

A fully-fledged version of Aiyagari (1994)

- The competitive firms are characterized by a *CRS* technology; let K_t and L_t stand for the per-capita aggregate capital stock and labor supply, respectively.
- Per-capita aggregate output is given by:

$$Y_t = f\left(K_t, L_t\right).$$

- The first-order conditions for the representative firm read as:

$$w_t = f_L\left(K_t, L_t\right),$$
$$r_t = f_K\left(K_t, L_t\right).$$

A fully-fledged version of Aiyagari (1994)

- The vector of individual state variables $\mathbf{x}_t \equiv \{k_t, s_t\}$ lies in $\mathcal{X} = [0, \infty) \times \mathcal{S}$.

- The distribution of $\mathbf{x}_t$ across agents is described by an aggregate state, the probability measure λ_t.

- More precisely, λ_t is the unconditional prob. dist. of $\{k_t, s_t\}$, defined over the Borel subset of $\mathcal{X}$:

$$\lambda_t\left(\mathrm{k}, \mathrm{s}_j\right) = \mathrm{prob}\left(k_t = \mathrm{k}, s_t = \mathrm{s}_j\right).$$

- For the *LoLN*, $\lambda_t\left(\mathbf{x}\right)$ can be interpreted as the mass of agents whose individual state vector is equal to $\mathbf{x}$.

- Being λ_t a prob. measure, the total mass of agents is one.

A fully-fledged version of Aiyagari (1994)

- Being the model fully recursive, we can recast it as a dynamic programming problem.

- The policy function for a generic agent satisfies the Euler eq. in recursive form:

$$u_c\left[c\left(\mathbf{x};\lambda\right)\right] \geq \beta \mathbb{E}\left\{u_c\left[c\left(\mathbf{x}';\lambda'\right)\right]\left[1-\delta+r\left(\lambda'\right)\right] \mid \mathbf{x};\lambda\right\},$$

  where $\mathbf{x} = \{k, s\}$, and:

$$k'\left(\mathbf{x};\lambda\right) = \left[1-\delta+r\left(\lambda\right)\right]k + w\left(\lambda\right)s - c\left(\mathbf{x};\lambda\right).$$

- Note that the policy fun. c depends upon the individual state $\mathbf{x}$ and the agg. distribution λ, while the agg. prices w and r depend upon λ only.

A fully-fledged version of Aiyagari (1994)

- The Markov chain driving s and the policy function $c(\mathbf{x};\lambda)$ *induce* a Law of Motion (LoM) for λ:

$$\lambda'(\mathrm{k},\mathrm{s_j}) = \sum_{i=1}^{n} \int \mathcal{I}(\mathrm{k},k,\mathrm{s}_i)\,\Pi_{i,j}\lambda(k,\mathrm{s}_i)\,dk$$
$$= \int_{\mathcal{X}} \mathcal{I}(\mathrm{k},k,s)\,\Pi(s,\mathrm{s}_j)\,d\lambda,$$

where:

$$\mathcal{I}(\mathrm{k},k,\mathrm{s}_i) = \begin{cases} 1 & \text{if } k'(k,\mathrm{s}_i;\lambda) = \mathrm{k} \\ 0 & \text{if } k'(k,\mathrm{s}_i;\lambda) \neq \mathrm{k} \end{cases}.$$

A fully-fledged version of Aiyagari (1994)

Definition

A *recursive equilibrium* is a policy fun. $c(\mathbf{x};\lambda)$, a couple of sequences $\{w_t, r_t\}$, and a sequence of distributions $\{\lambda_t\}$ such that:

- The policy function $c(\mathbf{x};\lambda)$ solves the individual problem.
- The factor prices $\{w_t, r_t\}$, together with $K_t = \int_{\mathcal{X}} k d\lambda_t$ and $L_t = \int_{\mathcal{X}} s d\lambda_t$, satisfy the firm's *FOC*s $\forall t \geq 0$.
- The market for the final good clears:

$$\int_{\mathcal{X}} \left[c(\mathbf{x};\lambda_t) + k'(\mathbf{x};\lambda_t)\right] d\lambda_t = (1-\delta) K_t + f(K_t, L_t), \quad \forall t \geq 0.$$

- The sequence $\{\lambda_t\}$ satisfies the induced LoM:

$$\lambda_{t+1}(\mathrm{k}, \mathrm{s_j}) = \int_{\mathcal{X}} \mathcal{I}(\mathrm{k}, k, s)\, \Pi(s, \mathrm{s}_j)\, d\lambda_t, \quad \forall t \geq 0,\ \forall \mathbf{x} \in \mathcal{X}.$$

A fully-fledged version of Aiyagari (1994)

Definition

A *stationary recursive equilibrium* is a policy function $c(\mathbf{x})$, a couple of values $\{w, r\}$, and a distribution λ such that:

- The policy function $c(\mathbf{x})$ solves the individual problem.
- The factor prices $\{w, r\}$, together with $K = \int_{\mathcal{X}} k d\lambda$ and $L = \pi' \mathcal{S}$, satisfy the firm's *FOC*s.
- The market for the final good clears:

$$\int_{\mathcal{X}} \left[c(\mathbf{x}) + k'(\mathbf{x})\right] d\lambda = (1-\delta) K + f(K, L).$$

- The distribution satisfies the induced *LoM*:

$$\lambda(\mathrm{k}, \mathrm{s_j}) = \int_X \mathcal{I}(\mathrm{k}, k, s) \Pi(s, \mathrm{s}_j) d\lambda, \quad \forall \mathbf{x} \in \mathcal{X}.$$

A fully-fledged version of Aiyagari (1994)

Algorithm: how to solve for the equilibrium

1) Given $L = \pi'\mathcal{S}$, choose an initial guess for K, say $K_0 > 0$.
   a) Given K_j, compute w_j and r_j from the firm's FOCs.
   b) Solve the agent's problem for the policy function $c_j(\mathbf{x})$.
   c) Compute the implied stationary distribution $\lambda_j(\mathbf{x})$.
   d) Compute the implied agg. capital stock, $\hat{K}_j = \int_{\mathcal{X}} k d\lambda_j$.
   e) Given $\hat{K}$, compute a new guess for K:

   $$K_{j+1} = \upsilon \hat{K}_j + (1-\upsilon) K_j$$

   where $\upsilon \in (0,1)$ is a damping parameter.
2) Iterate steps $(a)-(e)$ until convergence.

Time iteration

Algorithm: how to solve for the policy function

1) Define (only once!) a finite grid for the individual capital stock on R_+, say $\mathbf{k} = (\mathrm{k}_j)_{j=1}^{m}$, where $\mathrm{k}_1 = 0$ and $\mathrm{k}_m = \bar{\mathrm{k}} < \infty$.
2) Choose an initial guess for the optimal cons. levels at each node, i.e. n vectors $\mathbf{c}_{i,0} = (c_{i,0,j})_{j=1}^{m}$, one for each possible realization of s.
   a) Given the current guess $\mathbf{c}_{i,z}$, where z denotes the iteration, compute the implied k':
   $$\mathbf{k}'_{i,z} = \max\left(\mathbf{y}_i - \mathbf{c}_{i,z}, 0\right),$$
   where:
   $$\mathbf{y}_i = (1 - \delta + r)\,\mathbf{k} + w\mathrm{s}_i.$$
   b) Given the vectors $\mathbf{k}'_{i,z}$, compute the future optimal consumption levels $\mathbf{c}'_{q,i,z}$, where $q = 1, 2, ..., n$, via **interpolation** on $\mathbf{k}$ and $\mathbf{c}_{i,z}$.
   c) ...

Aiyagari (1994) Time iteration

Algorithm: how to solve for the policy function

2) ...

c) Compute the r.h.s. of the Euler eq., and solve for:

$$\hat{\mathbf{c}}_{i,z} = \min \left\{ u_c^{-1} \left[\beta \left(1 - \delta + r\right) \sum_{q=1}^{n} \Pi_{i,j} u_c \left(\mathbf{c}'_{q,i,z} \right) \right], \mathbf{y}_i \right\}.$$

d) Update the guess for $\mathbf{c}_{i,z}$ as follows:

$$\mathbf{c}_{i,z+1} = \upsilon \hat{\mathbf{c}}_{i,z} + \left(1 - \upsilon\right) \mathbf{c}_{i,z},$$

where $\upsilon \in (0,1)$ is a damping parameter.

3) Iterate steps $(a) - (d)$ until convergence.

References I

Aiyagari, S. R. (1994, August). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics 109*(3), 659–84.

Huggett, M. (1993, September). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control 17*(5-6), 953–69.