LECTURES 12

Aggregate uncertainty: Krusell-Smith models

Macroeconomics 4

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Aggregate uncertainty

- Until now, we maintained the assumption that the only source of randomness was idiosyncratic in nature, and washed out at the aggregate level thanks to a *LoLN*.
- Following Krusell and Smith (1998), let us extend the Aiyagari model and introduce an aggregate productivity shock, say z_t:

$$Y_t = z_t f\left(K_t, L_t\right).$$

- Suppose that z_t follows a discrete Markov chain, taking values in $\mathcal{Z} = \{\mathrm{z}_1, \mathrm{z}_2, ..., \mathrm{z}_d\}$ and characterized by the transition matrix $\mathbf{\Gamma}$.
- Note that the Markov chains for z_t and s_t can be merged into a single chain, representing the exogenous state $\{s_t, z_t\}$, if idiosyncratic and aggregate shocks are completely orthogonal.

Aggregate uncertainty

- Consider the *FOC*s for the firm:

$$w\left(z_t;\lambda_t\right) = w\left(z_t, K_t, N_t\right) = z_t f_L\left(K_t, L_t\right),$$
$$r\left(z_t;\lambda_t\right) = r\left(z_t, K_t, N_t\right) = z_t f_K\left(K_t, L_t\right).$$

- Note that the factor prices depend on the distribution λ_t, as before, but also on the agg. productivity level z_t.

- Hence, factor prices will never remain constant, not even in the long run: no stationary equilibrium in this economy!

Aggregate state variables

- Krusell and Smith (1998) posit (without proof) that k_t and λ_t are **sufficient** agg. state variables for a recursive eq. to be defined.
- Hence, our set of state variables will contain an individual endo. state, k_t, an ind. exo. state, s_t, an agg. exo. state, z_t, and a prob. measure, λ_t, so that $\mathbf{x}_t \equiv \{k_t, s_t; z_t, \lambda_t\}$.
- Note that:

$$\lambda_t\left(\mathrm{k}, \mathrm{s}_j, \mathrm{z}_i\right) = \operatorname{prob}\left(k_t = \mathrm{k}, s_t = \mathrm{s}_j, z_t = \mathrm{z}_i\right).$$

- The Markov chains driving s and z, and the policy function $c\left(\mathbf{x}\right)$ induce a LoM for λ:

$$\lambda'\left(\mathrm{k}, \mathrm{s}_j, \mathrm{z}_i\right) = \sum_{r=1}^{n}\sum_{l=1}^{d}\int \mathcal{I}\left(\mathrm{k}, k, \mathrm{s}_r, \mathrm{z}_l, \lambda\right)\Pi_{r,j}\Gamma_{l,i}\lambda\left(k, \mathrm{s}_r, \mathrm{z}_l\right)dk.$$

The individual problem

- Def. $V_{j,i}(k;\lambda) \equiv V(k, \mathrm{s}_j; \mathrm{z}_i, \lambda)$, the individual problem becomes:

$$
\begin{aligned}
V_{j,i}(k;\lambda) = \max_{\{c,k'\}} u(c) + \beta \sum_{r=1}^{n} \sum_{l=1}^{d} \Pi_{jr} \Gamma_{il} V_{r,l}(k';\lambda'), \\
\text{s.t. } k' = [1-\delta + r(\mathrm{z}_i,\lambda)] k + w(\mathrm{z}_i,\lambda) \mathrm{s}_j - c, \\
k' \geq 0, \\
\lambda' = \mathcal{H}(\mathrm{z}_i, \lambda, z').
\end{aligned}
$$

- The function $\mathcal{H}(z, \lambda, z')$ summarizes the *aggregate law of motion* as **perceived** by the individual agent.

- In order to forecast w' and r', the agent has to forecast λ': hence, she has to take λ **AND** the aggregate *LoM* into account.

Recursive equilibrium

Definition

A *recursive equilibrium* is a policy fun. $c(\mathbf{x})$, a couple of pricing fun. $w(z,\lambda)$ and $r(z,\lambda)$, and an aggregate *LoM* such that:

- $c(\mathbf{x})$ solves the individual problem, while $w(z,\lambda)$ and $r(z,\lambda)$, together with $K=\int kd\lambda$ and $L=\int sd\lambda$, satisfy the firm's *FOC*s.

- The market for the final good clears:

$$\int\left[c(\mathbf{x})+k'(\mathbf{x})\right]d\lambda=(1-\delta)K+f(K,L).$$

- The perceived aggregate LoM $\mathcal{H}(z,\lambda,z')$ is consistent with:

$$\lambda'(\mathrm{k},\mathrm{s}_j,\mathrm{z}_i)=\sum_{r=1}^{n}\sum_{l=1}^{d}\int\mathcal{I}(\mathrm{k},k,\mathrm{s}_r,\mathrm{z}_l,\lambda)\,\Pi_{r,j}\Gamma_{l,i}\lambda(k,\mathrm{s}_r,\mathrm{z}_l)\,dk,\quad\forall\mathbf{x}.$$

Recursive equilibrium

- A sequential markets equilibrium in this economy will in general exist, but it is impossible to compute directly, as it is impossible to make any claim of uniqueness of such an equilibrium.
- Given that a sequential eq. exists, there is a state space large enough such that a recursive equilibrium (recursive in that state space) exists (see Miao, 2006, if you really need to dig deeper)
- The issue is whether a recursive eq. in which the aggregate state only contains z and λ does exist.
- There is no guarantee of existence of such a recursive eq. existence can be proven if we assume uniqueness of the sequential eq., but that cannot descent from the primitives of the model.
- The analysis of this economy is purely computational as neither the existence, uniqueness, stability or qualitative features of the equilibrium can be (currently) theoretically established.

Numerical strategy

- Krusell and Smith (1998) show that this model features "**approximate aggregation**."
    - To predict future prices, agents need to forecast a small set of statistics of the asset dist. rather than the entire dist. itself.
- The key step in the solution procedure, then, is to approximate a infinite-dimensional object, the measure λ, with a finite number of its moments.
- Hence, the LoM $\mathcal{H}(z, \lambda, z')$ reduces to a function mapping the current moments to the future ones.

Numerical strategy

- This further approx. makes the individual problem become:

$$
\begin{aligned}
V_{j,i}\left(k;\mathcal{M}\right) = \max_{\{c,k'\}} \; & u\left(c\right) + \beta \sum_{r=1}^{n}\sum_{l=1}^{d} \Pi_{jr}\Gamma_{il} V_{r,l}\left(k';\mathcal{M}'\right), \\
\text{s.t. } & k' = \left[1-\delta + r\left(\mathrm{z}_i,\mathcal{M}\right)\right] k + w\left(\mathrm{z}_i,\mathcal{M}\right) \mathrm{s}_j - c, \\
& k' \geq 0, \\
& \mathcal{M}' = \bar{\mathcal{H}}_M\left(\mathrm{z}_i,\mathcal{M}\right).
\end{aligned}
$$

  where $\mathcal{M} \equiv \{m_1, m_2, ..., m_M\}$.

- Note that agents are now **boundedly rational** in the sense that moments of higher order than M may help to more accurately forecast the first M moments tomorrow.

Numerical strategy

- Krusell and Smith (1998) show that, **for their model**, the first moment - the mean - is sufficient to get a fairly accurate result.

- Furthermore, they choose a very simple log-linear functional form for $\bar{\mathcal{H}}$, so that the problem becomes:

$$
\begin{aligned}
V_{j,i}(k;K) = \max_{\{c,k'\}} u(c) + \beta \sum_{r=1}^{n} \sum_{l=1}^{d} \Pi_{jr}\Gamma_{il} V_{r,l}(k';K'), \\
\text{s.t. } k' = [1-\delta + r(\mathrm{z}_i, K)]\,k + w(\mathrm{z}_i, K)\,\mathrm{s}_j - c. \\
k' \geq 0, \\
\log(K') = \bar{\mathcal{H}}_1(\mathrm{z}_i, K) = \varsigma_i + \varrho_i \log(K).
\end{aligned}
$$

- Note that the parameters ς_i and ϱ_i depend on the current realization of z.

Numerical strategy

Algorithm: how to solve a K-S model

1) Guess parameters ς_i and ϱ_i.

2) Solve the individual problem for $c_{j,i}(k;K)$, given $\bar{\mathcal{H}}_1$.

3) Simulate the economy for a large number of T periods for a large number N of agents (say $T = 11000$ and $N = 5000$).

4) Aggregate to find the implied sequence of aggregate capital stocks, $K_t = \frac{1}{N}\sum_{i=1}^{N} k_{i,t}$.

5) Run the regressions: $\log(K') = \hat{\varsigma}_i + \hat{\varrho}_i \log(K)$, for $i = 1, 2, ..., d$.

6) If $\{\hat{\varsigma}_i, \hat{\varrho}_i\} \approx \{\varsigma_i, \varrho_i\}$, the R^2 is high, and the var. of regression errors is small, stop, otherwise update the guess and iterate again.

Quasi-aggregation

- Suppose all agents, for all pricing functions $r(z, K)$ and $w(z, K)$, have linear savings functions with the same Marginal Propensity to Save (MPS), so that:

$$k'(k, s; z, K) = \nu_z + \xi_z s + \iota_z k.$$

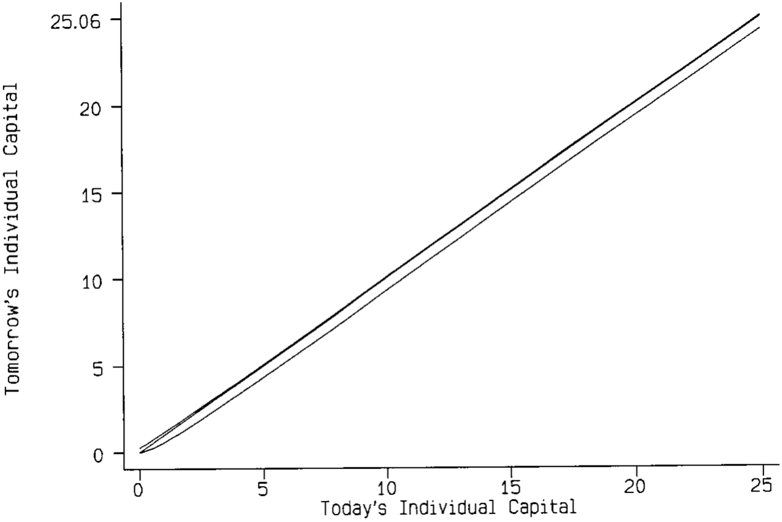
- Then, the aggregate capital stocks is:

$$K'(z) = \int k'(k, s; z, K)\, d\lambda = \nu_z + \xi_z L + \iota_z K.$$

- Exact aggregation obtains and the first moment of the wealth distribution, K, is in fact a sufficient statistic for forecasting K'.

- In Krusell and Smith (1998) the savings functions are almost linear with same slope.

Quasi-aggregation



Quasi-aggregation

- The only exceptions are unlucky agents with little assets which are liquidity constrained and have a low (zero) *MPS*.
- However, since these (few) agents hold a negligible fraction of aggregate wealth, they don't matter for the aggregate capital dynamics.
- All other agents have almost identical *MPS*, thus individual saving decisions *almost exactly* aggregate.
- The current aggregate capital stock is *almost* a sufficient statistic when forecasting K': quasi-aggregation obtains.
- The key question is why individual savings functions are almost linear in k at just about all current k levels.

Quasi-aggregation

- From Figure 2 of Krusell and Smith (1998) we see that the slope of k' when plotted against k is roughly equal to 1 for all but very low asset levels.

- Consider the income fluctuations problem, and recall that under certainty equivalence, and if $\beta(1+r)=1$, we had:

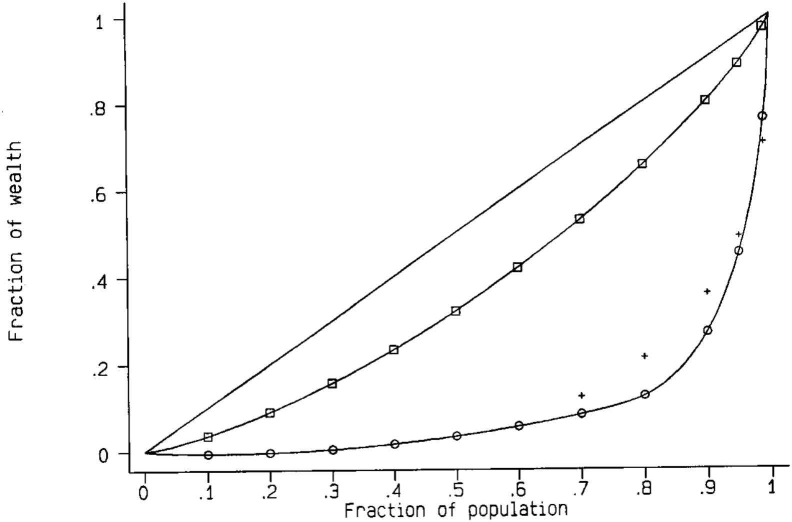
$$k_{t+1} = k_t + \Xi(s_t).$$

- In the *K-S* economy agents are prudent and face liquidity constraints, but almost act as if they are certainty equivalence consumers.

Quasi-aggregation

- Possible explanations:
  - Agents are prudent, but not all that much. A $\sigma = 1$ is at the lower end of the empirical estimates for risk aversion.
  - The unconditional standard deviation of individual income is roughly 0.2, at the lower end of the estimates used by Aiyagari.
  - Probably most important, large negative income shocks are infrequent and not very persistent, so that they don't force a large departure in behavior from certainty equivalence.

A final glimpse at K-S results



References I

Krusell, P. and A. A. Smith, Jr. (1998, October). Income and Wealth Heterogeneity in the Macroeconomy. *Journal of Political Economy 106*(5), 867–96.

Miao, J. (2006, May). Competitive equilibria of economies with a continuum of consumers and aggregate shocks. *Journal of Economic Theory 127*(1), 274–298.