Intertemporal Asset Pricing and the Marginal Utility of Wealth

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Abstract

We consider the general class of discrete-time, finite-horizon intertemporal asset pricing models in which preferences for consumption at the intermediate dates *are allowed to be state-dependent, satiated, non-convex and discontinuous*, and the information structure is *not required* to be generated by a Markov process of state variables. We supply a generalized definition of marginal utility of wealth based on the Fréchet differential of the value operator that maps time t wealth into maximum conditional remaining utility. We show that in this general case all state-price densities/stochastic discount factors are fully characterized by the marginal utility of wealth of optimizing agents even if their preferences for intermediate consumption are highly irregular. Our result requires only the strict monotonicity of preferences for terminal wealth and the existence of a portfolio with positive and bounded gross returns. We also relate our generalized notion of marginal utility of wealth to the equivalent martingale measures/risk-neutral probabilities commonly employed in derivative asset pricing theory. We supply an example in which our characterization holds while the standard representation of state-price densities in terms of marginal utilities of optimal consumption fails.

Keywords: arbitrage, viability, linear pricing rules, optimal portfolio-consumption problems, marginal utility of wealth.

JEL classification numbers: G11, G12, G13, G14, C6.

1 Introduction

We consider the general class of discrete-time, finite-horizon intertemporal asset pricing models in which i) preferences for consumption at the intermediate dates *are allowed to be state-dependent, satiated, non-convex and discontinuous*,[1] and ii) the information structure is *not required* to be generated by a Markov process of state variables. For this general class of models we supply a generalized definition of marginal utility of wealth based on the Fréchet differential of the value operator that maps time t wealth into maximum conditional remaining utility. The main contribution of the paper (Theorems 2 and 3) is to show that in this very general case all linear pricing rules (as well as all state-price densities/stochastic discount factors) are fully characterized by the marginal utility of wealth of optimizing agents even when their preferences for intermediate consumption are highly irregular. Our result requires only the strict monotonicity of preferences for terminal wealth and the existence of a portfolio with positive and bounded gross returns.

To motivate our approach we first recall that Harrison and Kreps (1979) define a discrete-time security market *viable* if some agent, with preferences continuous, concave and strictly increasing over terminal wealth, attains an optimal portfolio given the securities prices. They show that viability is equivalent to the existence of continuous, strictly positive linear pricing rules. In our first result (Theorem 1) we extend Harrison and Kreps' result by introducing preferences for intermediate consumption, and allowing them to be satiated, non-concave and discontinuous. We show that linear pricing rules exist as long as there is an optimizing individual in this much larger class of agents. Properties of the intermediate preferences are therefore irrelevant for the existence of linear pricing rules.

In the first part of our paper agents' preference are described quite generally by means of complete and transitive preferences relations. The asset pricing literature, however, typically imposes enough structure on preferences so that a link between linear pricing rules and the marginal utility of consumption can be established via the Euler equation

[1] Habit formation is a natural example that leads to the possibility of state dependency and satiation in an asset pricing model (for a theoretical analysis see Ryder and Heal, 1973, Abel, 1990, Chapman, 1998, and Campbell and Cochrane, 1999). First-order risk aversion a-là Epstein and Zin (1990) is a motivation for the possibility of non-smooth preferences (see in particular Segal and Spivak, 1997, and the references therein, for the analysis of the link between first-order risk-aversion and non-differentiability). For a discussion on satiation and equilibrium see Polemarchakis, H., Siconolfi, P.(1993), Allouch, N., Le Van, C. (2008, 2009) and Sato (2010).

(see, for instance, Cochrane, 2001, and Duffie, 2001). For this reason, in the second part of the paper we restrict our attention to time additive, but possibly state-dependent and non time-separable preferences. More specifically, we allow the period utility at date t to depend on past consumption and on the realized state of the world. This setting is large enough to accommodate both external and internal habit formation preferences (see Ryder and Heal, 1973, Abel, 1990, Campbell and Cochrane, 1999, and Costantinides, 1990). Since our Theorem 1 shows that the existence of linear pricing rules is independent from preferences for intermediate consumption, we also allow intermediate period utilities to be non-smooth, non-increasing, and non-concave. As for the utility of terminal period, we only require it to be strictly increasing. In this framework the asset pricing relation that links state price densities to the marginal utility of consumption via the Euler equation fails, since the marginal utility of optimal consumption is not required to be well-defined.

Our main contribution is to show that an explicit link between asset prices and preferences holds also in our general class of models as long as the notion of marginal utility of wealth is suitably generalized. Our approach goes as follows. Since we do not require the information structure to be generated by a Markov process, the time t maximum remaining utility given a wealth level $W(t)$ needs to be defined as an *essential supremum* over all budget-feasible portfolio-consumption pairs. In our more general case, therefore, the value function is itself a random variable, and hence the standard notion of marginal utility of wealth as the partial derivative of the value function with respect to wealth may lose any meaning. To deal with this general case we require the value function to be *Fréchet-differentiable*, namely, we require the existence of a linear continuous *operator* that approximates the value function in a neighborhood of the optimal wealth.[2] A generalized notion of marginal utility of wealth follows then by taking the expectation of this linear approximation, and applying a Riesz-representation argument to the linear functional hence obtained.

Our generalized notion of marginal utility of wealth allows us to extend a fundamental property of standard asset pricing theory to our general class of models. In the standard, Markovian asset pricing literature with well-behaved time-separable preferences, the Euler equation together with the envelope condition imply that the asset prices weighted by the

[2] The Frèchet-differentiability requirement of the value function is imposed for example by Machina (1982) in his classical analysis of expected utility without the independence axiom.

marginal utility of optimal wealth are martingales under the physical probability measure. Employing our generalized notion of marginal utility of wealth, and without the need of any envelope-type argument, in Proposition 2 we extend this result to our general class of models. With these findings in hand, we are able to establish two central results of our paper, Theorem 2 and Corollary 1, in which we show that for the general class of asset pricing models under scrutiny the entire set of linear pricing rules/state-price densities is in one-to-one correspondence with the set of marginal utilities of wealth of optimizing agents.

Our notion of marginal utility of wealth is the natural extension of the standard one to non-Markov asset pricing models. To substantiate this claim, in Subsection 4.2 we assume that the information structure is generated by a Markov state-variable. We still allow, however, the intermediate period utilities to depend on past consumption and to display satiation, non-convexities and discontinuities. In this case, the state-dependency of the period utilities of consumption from the state of nature manifests itself only through the realizations of the state variables. Given the past consumption, therefore, the maximum remaining utility is a deterministic function of the wealth level and the state variable. Proposition 3 shows that if the value function is both Fréchet-differentiable, and admits partial derivatives with respect to the realizations of wealth, then our generalized notion of marginal utility of wealth does coincide with the standard one.

In Proposition 4, moreover, we provide a parsimonious condition under which, in a Markov framework, the standard marginal utility of wealth is a state-price density regardless of the Fréchet-differentiability of the value function. In particular, we show that, regardless of intermediate period utilities, if the terminal utility is both strictly increasing and concave in terminal wealth and the value function is differentiable in wealth, then asset prices weighted by the marginal utility of optimal wealth given the optimal past consumption are martingales under the physical probability. An important implication of this result is that the class of agents whose marginal utilities of optimal wealth characterize all linear pricing rule/state-price densities may be enlarged. In Theorem 3, in fact, we show that the marginal utilities of wealth of optimizing agents with smooth value functions characterize all linear pricing rules as long as one of the following two conditions holds: the mapping of current wealth into maximum remaining utility is Fréchet-differentiable, or the terminal utility is concave in terminal consumption.

Equivalent martingale measures (risk-neutral probabilities), the probabilistic counterparts of the linear pricing rules/state-price densities of a security market, constitute a fundamental tool of derivative asset pricing theory. In our final results, Theorem 4 and Corollary 2, we revert to our general framework and bridge our generalized notion of marginal utility of wealth with the notion of equivalent martingale measure.

The layout of the paper is as follows. In the next section we introduce the basic notation and assumptions. In Section 3 we extend Harrison and Kreps (1979, Theorem 1) to the case of satiated, non convex, and discontinuous preferences for intermediate consumption. In Section 4, with time-additive but possibly non time-separable and state-dependent preferences with non-regular intermediate period utilities, we provide our generalized notion of marginal utility of wealth and characterize the linear pricing rules/state-price densities in terms of marginal utilities of wealth of optimizing agents. In Subsection 4.2 we discuss the special case of Markov information structure, while in Subsection 4.3 we provide an example with marginal utility of optimal wealth well-defined at all dates, even though the utility of consumption before the terminal date is satiated and discontinuous. In Section 5 we link our generalized notion of marginal utility of wealth to the equivalent martingale measures. Section 6 concludes. All proofs are in the Appendix.

2 The basic model: assumptions and definitions

We consider a frictionless security market in which J assets are traded over the investment horizon $\mathcal{T} = \{0, 1, \ldots, T\}$. Asset prices and cash-flows are denominated in units of the single good consumed in the economy. We assume that investors can freely dispose of the good. To describe the stochastic evolution of asset prices and cash-flows we take as given a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t=0}^{T})$,[3] and denote by $d_j(t)$ the $\mathcal{F}_t-$measurable cash flow distributed by asset j at date t and by $S_j(t)$ the $\mathcal{F}_t-$measurable date t price of asset j *net of the current cash flow*, for $j = 1, \ldots, J$, where J is the number of assets. Given $p \in [1, +\infty]$, we assume that $S_j(t), d_j(t) \in L^p(\Omega, \mathcal{F}_t, P)$ for all t.[4] Without loss of generality, we assume that the assets distribute no cash flow at date 0 and a liquidating one at date T, that is $d_j(0) = S_j(T) = 0$ almost surely. We stress the fact that, aside from

[3] As usual, we assume that $\mathcal{F}$ is augmented with $P-$null sets, $\mathcal{F}_0$ is the trivial sigma-algebra $\{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.

[4] If $p = +\infty$ we endow L^∞ with the Mackey topology relative to L^1 (see Kreps (1981) and the references therein) .

Subsection 4.2 that deals with the special case of an information structure generated by a Markov process, throughout the paper we impose no specific assumptions on the stochastic evolution of prices and dividends.

We describe intertemporal trading by means of sequences $\theta = \{\theta(t)\}_{t=0}^{T-1}$ of J-dimensional, $\mathcal{F}_t-$measurable random variables, that is $\theta(t) = \{\theta_1(t), \theta_2(t), \ldots, \theta_J(t)\}$, where $\theta_j(t)$ represents the position (in number of units) in assets j taken at date t and liquidated at date $t+1$. We call any such θ a *dynamic investment strategy.*

We denote by $V_\theta(t)$ the date t value of a dynamic investment strategy, defined as the cost of establishing the positions in the J assets at their net-of-cash-flow prices, if t precedes the last trading date, and, at T, as the payoff from the final liquidation of θ. Formally:

$$V_\theta(t) = \begin{cases} \theta(t) \cdot S(t) & t < T \\ \theta(T-1) \cdot d(T) & t = T. \end{cases}$$

In what follows, we refer to the sequence $V_\theta = \{V_\theta(t)\}_{t=0}^{T}$ as to the *value process* of the dynamic investment strategy θ.

At any date t, a dynamic investment strategy θ produces a cash flow $x_\theta(t)$, generated by the difference between the resources obtained from liquidating the positions taken at $t-1$ at the cum-cash flow prices $S(t)+d(t)$, and the cost to establish the new positions at the net-of-cash flow prices $S(t)$. The cash-flow $x_\theta(t)$ is therefore related to the value $V_\theta(t)$ as follows:

$$x_\theta(t) = \begin{cases} -V_\theta(0) & t = 0 \\ \theta(t-1) \cdot [S(t) + d(t)] - V_\theta(t) & t = 1, \ldots, T-1 \\ V_\theta(T) & t = T. \end{cases} \tag{1}$$

Henceforth, we call the sequence $x_\theta = \{x_\theta(t)\}_{t=0}^{T}$ the *cash-flow process* of θ.

Definition 1 *We call* ***admissible*** *any dynamic investment strategy θ such that $V_\theta(t)$, $x_\theta(t) \in L^p(\Omega, \mathcal{F}_t, P)$ for $t = 0, 1, \ldots, T$. We denote with Θ the set of all admissible dynamic investment strategies.*

A dynamic investment strategy is called *self-financing* when its cash flow is null at all intermediate dates, that is, $x_\theta(t) = 0$ for all $t = 1, \ldots, T-1$; in other words, the cost to establish the new positions $\theta(t)$ at the net-of-cash flow prices $S(t)$ is exactly matched by the value obtained from liquidating the positions $\theta(t-1)$ at the cum-cash-flow prices $S(t)+d(t)$. We denote with Θ^{SF} the set of all *admissible* self-financing dynamic investment strategies.

Particularly important among the self-financing strategies are those whose value process is almost surely strictly positive at all dates, and hence can be used as a new unit of account, or numeraire. More precisely:

Definition 2 *A* ***numeraire*** *is the value process of any admissible self-financing strategy $\theta^{SF} \in \Theta^{SF}$ such that*

$$P\left(V_{\theta^{SF}}(t) > 0\right) = 1, \qquad\qquad t = 0, \ldots, T.$$

We collect in the set Θ^N the self-financing strategies that can be used as numeraires.

We make the following assumption:

Condition 1 *There exists a numeraire with bounded returns, namely $\theta^{BN} \in \Theta^N$ such that*

$$\frac{V_{\theta^{BN}}(t+1)}{V_{\theta^{BN}}(t)} \in L^{\infty}(\Omega, F_{t+1}, P), \quad t = 0, \ldots, T-1$$

We denote with Θ^{BN} the set of all numeraires with bounded returns.

Any cash flow $x(t) \in L^p(\Omega, \mathcal{F}_t, P)$, $t = 0, 1, \ldots, \ T-1$ invested in a numeraire with bounded returns is still an admissible investment strategy.[5] In other words, the boundedness requirement for the returns of the numeraires implies that wealth can be transferred forward in an admissible way. Such assumption is satisfied in Harrison and Kreps (1979), who require that the numeraire is bounded above and away from zero, to ensure that the space of contingent claims that can be priced via no arbitrage is not affected by a change of numeraire (see also Section 5).

In the rest of the paper, we assume that Condition 1 holds when not otherwise specified.

[5] More specifically, while the strategy $\frac{x(t)}{V_{\overline{\theta}^N}(t)}\overline{\theta}^N(t)$, $t = 0, 1, \ldots, T-1$ is still admissible if $\overline{\theta}^N$ has bounded returns, it may be inadmissible if the returns of $\overline{\theta}^N$ are unbounded.

3 Viability with non-convex and non-increasing preferences for intermediate consumption

In multiperiod security markets with a risk-free asset and trading restricted to self-financing strategies, Harrison and Kreps (1979) show that viability for agents who care only about consumption at the terminal date is equivalent to the existence of linear pricing rules in the L^2 framework. Kreps (1981) extends this result to a general topological vector space maintaining the regularity assumptions on preferences. In this section, we extend Harrison and Kreps' result to security markets in which existence of a risk-free asset is replaced by the weaker Condition 1, agents trade and consume at all dates and, most importantly, preferences for intermediate consumption are allowed to be highly non-regular, i.e., are allowed to display satiation, non convexities and discontinuities.

We consider a general class $\mathcal{A}$ of agents, each identified by an *initial endowment* $e_0 \geq 0$ of the single consumption good and a *complete and transitive preference relation* $\succeq$ on the set $\mathcal{C} = \prod_{t=0}^{T} L^p(\Omega, \mathcal{F}_t, P)$ of consumption sequences $c = (c(0), c(1), \ldots, c(T))$, with $c(t) \in L^p(\Omega, \mathcal{F}_t, P)$ for all t.

In choosing the optimal intertemporal consumption and asset allocation, each agent $(e_0, \succeq)$ in $\mathcal{A}$ faces the budget constraint

$$B(e_0) = \{c \in \mathcal{C} \mid c(0) \leq x_\theta(0) + e_0, \; c(t) \leq x_\theta(t) \; \forall \; t > 0 \quad \textit{for some } \theta \in \Theta\}$$

so that $c^* \in B(e_0)$ is an optimal intertemporal consumption sequence for the agent $(e_0, \succeq)$ if and only if

$$c^* \succeq c \qquad \forall \; c \in B(e_0). \tag{2}$$

The preferences of each agent in $\mathcal{A}$ are supposed to be strictly increasing, convex, and continuous *only at the optimum* and *only* with respect to the level of *final consumption*, $c(T)$, but are otherwise unrestricted. More formally, we assume that each optimizing agent in $\mathcal{A}$ with optimal intertemporal consumption c^* has preferences satisfying the following three requirements:

1. *Monotonicity at the optimum in terminal consumption.* For every $c(T)$, $c'(T) \in L^p(\Omega, \mathcal{F}_T, P)$ such that $c'(T) \geq c(T)$ and $P[c'(T) > c(T)] > 0$, then

$$(c^*(0), \ldots, c^*(T-1), c'(T)) \succ (c^*(0), \ldots, c^*(T-1), c(T)).$$

2. *Convexity at the optimum in terminal consumption.* For any $c(T) \in L^p(\Omega, \mathcal{F}_T, P)$ the set

$$\{c' = (c^*(0), \ldots, c^*(T-1), c'(T)) \in \mathcal{C} : c' \succeq (c^*(0), \ldots, c^*(T-1), c(T))\}$$

is convex.

3. *Continuity at the optimum in terminal consumption.* For any $c(T) \in L^p(\Omega, \mathcal{F}_T, P)$ the sets

$$\{c' = (c^*(0), \ldots, c^*(T-1), c'(T)) \in \mathcal{C} : c' \succeq (c^*(0), \ldots, c^*(T-1), c(T))\}$$

and

$$\{c' = (c^*(0), \ldots, c^*(T-1), c'(T))) \in \mathcal{C} : (c^*(0), \ldots, c^*(T-1), c(T)) \succeq c'\}$$

are both closed with respect to the product topology of $\mathcal{C}$.

The following definition adapts the notion of viability of Harrison and Kreps (1979) to our framework.

Definition 3 *The security market is viable with respect to $\mathcal{A}$ if there exists an optimal solution to the consumption-portfolio problem* (2) *of some agent $(e_0, \succeq)$ in $\mathcal{A}$.*

To formalize the concept of linear pricing rule in our framework, we denote with $X = \prod_{t=1}^{T} L^p(\Omega, \mathcal{F}_t, P)$ the linear space of all future cash flows, and endow this space with the product topology. Moreover, we denote with X^+ the positive cone of X, that is, the set of sequences $x = \{x(t)\}_{t=1}^{T} \in X$ of future cash-flows such that $x(t) \geq 0$ almost surely for all $t > 0$, and $P\left[x(t) > 0\right] > 0$ for some $t > 0$. Finally, we denote with M the subspace of X that contains all sequences of future cash flows generated by dynamic trading, that is $m = \{m(t)\}_{t=1}^{T} \in M$ means that there exists $\theta \in \Theta$ such that $m(t) = x_\theta(t)$ for $t = 1, \ldots, T$.

Definition 4 *A linear pricing rule for the security market is a linear functional $\psi : X \to \Re$ continuous, strictly positive on X^+, and satisfying the law of one price, that is for all $m \in M$ and for all $\theta \in \Theta$ such that $m(t) = x_\theta(t)$, $t = 1, \ldots, T$ then $\psi(m) = V_\theta(0)$. We denote with Ψ the set of all such linear pricing rules.*

It is useful to recall that, by the Riesz representation theorem, the existence of linear pricing rules is equivalent to the existence of $\rho = \{\rho(t)\}_{t=1}^T \in \prod_{t=1}^T L^q_{++}(\Omega, \mathcal{F}_t, P)$, with q satisfying $p^{-1} + q^{-1} = 1$, such that[6]

$$\sum_{t=1}^{T} E[\rho(t)x_\theta(t)] = V_\theta(0) \qquad \forall\, \theta \in \Theta. \tag{3}$$

In our first result we characterize viability in our framework in terms of linear pricing rules.

Theorem 1 *The security market is viable if and only if the set Ψ is non-empty.*

If preferences for intermediate consumption were continuous, concave and monotone, the fact that viability implies the existence of linear pricing functionals could be readily established as in Kreps (1981, Theorem 1). In our case, however, since we have relaxed the regularity assumptions on intermediate preferences, we cannot invoke Kreps's argument to separate the set of strictly preferred consumption sequences from the budget set. We use instead the results of Stricker (1990), who extends Theorem 3 in Kreps (1981), by establishing the equivalence between *No Free Lunch* and the extension property of the linear pricing rules when the commodity space X is a real linear space, endowed with a locally convex, Hausdorff topology, and the set M of cash-flows is a subspace of X.[7]

Specifically, we first show that the existence of a Free Lunch is equivalent to the existence of a Self Financing Free Lunch (Lemma 1).We then show that if the market is viable according to our Definition 3, then the budget constraint of the optimizing agent is binding at the optimum, even though our intermediate preferences do not satisfy Kreps' usual assumptions (Lemma 2). This allows us to adapt the proof of Theorem 2 in Kreps (1981) to show that viability implies No Self Financing Free Lunch, which in turn implies No Free Lunch. As a consequence, viability implies No Free Lunch. Applying Stricker's' extension of Theorem 3 in Kreps (1981), we can conclude that viability implies the extension property of the linear pricing rule, i.e., the set Ψ is non-empty.

[6] In case $p = +\infty$ with the Mackey topology Riesz representation theorem holds with $\rho \in L^1$.

[7] See also Schachermayer (2004) and the reference therein.

4 Linear pricing and the marginal utility of wealth

In the previous section we have shown that viability is equivalent to the existence of linear pricing rules. In this section we want to characterize the set of these linear pricing rules in terms of the marginal utility of optimal intertemporal wealth in the case when agents' preferences have a time-additive vonNeumann-Morgenstern representation. In particular, the period utilities of our agents are allowed to depend on both the individual's past and present consumptions and on the state. In this way our model is able to accommodate habit formation models of both the internal and external type. Indeed, the explicit dependence of the utilities on individual past consumptions clearly includes internal habit preferences (see Ryder and Heal (1973) and Costantinides (1990) among the various references). On the other hand, external habit preferences, which are typically modeled by letting the period utilities depend on aggregate consumptions, are captured by our model, due to the fact that period utilities are allowed to explicitly depend on the state (see for instance Abel (1990), Campbell and Cochrane (1999), Grishenko (2010)).

In our general setting, (see, for instance, Duffie (2001)): given an optimizing agent with smooth time-additive, state-independent, strictly increasing and concave vonNeumann-Morgenstern preferences, a linear pricing rule is defined via the gradient of the utility function computed at the optimal consumption. The novelty in our approach is that, in coherence with the previous section, we require agents' preferences to be smooth, and increasing at the terminal date only and allow them to be intertemporally dependent and state-dependent at any date. In this case the marginal utility of optimal consumption may fail to be well-defined; we use instead the marginal utility of optimal wealth. Since we do not impose any a-priori restriction on the stochastic evolution of prices and dividends, the value function is a random variable, which may be not represented as a real function of random variables as, for instance, when the information is generated by a Markov process of state variables. To define the marginal utility of wealth, therefore, we require the value function to take values in L^1 in a neighborhood of the optimal wealth, and we define the marginal utility of optimal wealth by means of its Fréchet differential.

This general case is analyzed in the first subsection, while in the second subsection we discuss the special case where the information is generated by a Markov process of state variables (see for instance Veronesi (2004) and the references therein). In this second case, the maximum remaining utility at time t is a deterministic function of the wealth level and the marginal utility of optimal intertemporal wealth can be defined via usual differentiation. We conclude this section with an example that clarifies the extent of our contribution.

4.1 The general case

We consider the class of agents with initial endowment e_0 whose preferences $U(c)$ over consumption sequences $c \in \mathcal{C}$ take the following form

$$U(c) = \sum_{t=0}^{T} \int_{\Omega} u_t(\gamma(t,\omega),\omega) dP(\omega) = \sum_{t=0}^{T} E\left[u_t\left(\gamma(t)\right]\right. \tag{4}$$

where for all $t < T$, $\gamma(t) = (c(0), c(1), \ldots, c(t))$ is the collection of consumptions up to time t and the period utilities $u_t : \Re^{t+1} \times \Omega \to \Re$ are assumed to satisfy the following conditions:[8]

(i) for all t, the function $u_t(\gamma,\omega) : \Re^{t+1} \times \Omega \to \Re$ is measurable with respect to the product σ-algebra $\mathcal{B}(\Re^{t+1}) \otimes \mathcal{F}_t$ (where $\mathcal{B}(\Re^{t+1})$ denotes the Borel σ-algebra)[9];

(ii) for all $c \in B(e_0)$, the integrals in (4) $\int_{\Omega} u_t(\gamma(t,\omega),\omega)dP(\omega)$ are well defined[10] and either are finite or take the value $-\infty$; as a consequence, $U(c) < +\infty$ for all $c \in B(e_0)$.

(iii) if U is the utility of an optimizing agent and $\{\gamma^*(t)\}_{t=0}^{T}$ the stream of optimal consumptions, the function $u_T(\gamma^*(T-1)(\omega), \cdot, \omega)$: $\Re \to \Re$ is real-valued and strictly increasing for almost every ω.

To a sequence of past consumptions $\gamma(t-1)$ and a $\mathcal{F}_t$- measurable random variable W which represents the current level of wealth of an agent, we associate a random variable $H(t, \gamma(t-1), W)$, defined as follows:[11]

[8] For a discussion of period utilities that depend directly also on the state of nature ω see for instance Berrier, Rogers and Tehranchi (2007).

[9] This condition guarantees that for every $\mathcal{F}_t$-measurable random vector (γ, c), the function $u_t\left(\gamma(\omega), c(\omega), \omega\right)$ (defined on Ω with values in $\Re$) is $\mathcal{F}_t$-measurable.

[10] As usual for a random variable Z the integral $\int_{\Omega} Z(\omega)dP(\omega)$ is well defined and finite if both $\int_{\Omega} Z^+(\omega)dP(\omega) < +\infty$ and $\int_{\Omega} Z^-(\omega)dP(\omega) < +\infty$. We set $\int_{\Omega} Z(\omega)dP(\omega) = -\infty$ if $\int_{\Omega} Z^-(\omega)dP(\omega) = +\infty$ and $\int_{\Omega} Z^+(\omega)dP(\omega) < +\infty$. We set $\int_{\Omega} Z(\omega)dP(\omega) = +\infty$ if $\int_{\Omega} Z^+(\omega)dP(\omega) = +\infty$ and $\int_{\Omega} Z^-(\omega)dP(\omega) < +\infty$. Otherwise the integral is not defined.

[11] We recall that for any set Φ of random variables, there exists a random variable φ^*, called the *essential supremum* of Φ and denoted as $\varphi^* = \operatorname{ess\,sup}_{\varphi\in\Phi} \varphi$, such that: (i) $\varphi^* \geq \varphi$; (ii) any other random variable $\tilde{\varphi}$ such that $\tilde{\varphi} \geq \varphi$ for all $\varphi \in \Phi$ satisfies $\tilde{\varphi} \geq \varphi^*$ P-a.s.

$$
\begin{aligned}
H(t,\gamma(t-1),W) \equiv \operatorname*{ess\,sup}_{(c,\theta)\in\mathcal{C}\times\Theta} \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1),c(t),\dots,c(s))\right] \\
s.t. \begin{cases} c(t)+V_\theta(t)\leq W & \\ c(s)\leq x_\theta(s) & s=t+1,\dots,T \end{cases}
\end{aligned}
\tag{5}
$$

for $t = 0, 1, \dots, T$, where $E_t[\ \cdot\ ]$ denotes the conditional expectation with respect to $\mathcal{F}_t$. We assume that the integrals $E\left[u_s(\gamma(t-1),c(t),\dots,c(s))\right]$ (and hence the conditional expectations in (5)) are well defined, and, for all consumptions satisfying the budget constraint at time t, are either finite or take the value $-\infty$ (in which case we set $E_t\left[u_s(\gamma(t-1),c(t),\dots,c(s))\right] = -\infty$). In the Appendix (Proposition 5) we show that $H(t,\gamma(t-1),W)$ is a well-defined $\mathcal{F}_t$-measurable random variable and recall some of its properties which will be useful.

The random variable $H(t,\gamma(t-1),W)$ represents the maximum "remaining utility" at time t given the wealth level W and the past consumption $\gamma(t-1)$. The notion of maximum remaining utility is standard in the Markov framework, where every random variable is a deterministic function of the state variables. However, since we do not impose any specific on the stochastic evolution of prices and dividends, the classical definition of maximum remaining utility is not suitable in our framework: indeed, the usual notion of pointwise supremum may not be the right concept when we work with random variables (which are defined almost surely). We have instead to use the generalization of this notion for random variables, that is the essential supremum: the economic interpretation of the maximum remaining utility is however the usual one.[12]

To any optimal consumption-portfolio choice (c^*,θ^*) for an agent with preferences as in (4) and initial endowment e_0 we associate the optimal intertemporal wealth $W^* = \{W^*(t)\}_{t=0}^{T}$ generated by θ^*, that is

$$
W^*(t) = \begin{cases} e_0 & t=0 \\ \theta^*(t-1)\cdot[S(t)+d(t)], & t=1,\dots,T. \end{cases}
$$

Given the optimal past consumption $\gamma^*(t-1)$ and the optimal intertemporal wealth $W^*(t)$, we consider the time t-maximum remaining utility $H(t,\gamma^*(t-1),W^*(t))$, defined as

[12] Cetin and Rogers (2007) use the essential supremum to define the value function in a non-Markov model of optimal portfolio choice with liquidity constraints. In their case, however, there is no intertemporal consumption and the utility of terminal wealth is required to be state-independent.

above. Note that $H(T, \gamma^*(T-1), W^*(T)) \equiv u_T\left(\gamma^*(T-1), W^*(T)\right)$ and $H(0, e_0) = U(c^*)$. Moreover, since the optimal portfolio-consumption pair satisfies the budget constraint in (5), then $H(t, \gamma^*(t-1), W^*(t)) > -\infty$. Equality (6) in Proposition 1 will imply that $H(t, \gamma^*(t-1), W^*(t)) < +\infty$.

As a first step, we need the value function to be well defined and finite in a neighborhood of the optimal wealth. To this aim, given the optimal past consumption $\gamma^*(T-1)$ we require the terminal period utility u_T to map an L^p-neighborhood of the optimal terminal consumption into L^1: this ensures that for all $t = 0, 1, \ldots, T-1$ the maximum remaining utility given the optimal past consumption $\gamma^*(t-1)$ and a level of wealth $W(t)$ in a neighborhood of $W^*(t)$ is not $-\infty$. Indeed, assume that at some intermediate date t the wealth level of the agent is $W(t) = W^*(t) + X$ for some slack X. Then the agent can invest X at date t in the numeraire θ^{BN}, follow the optimal strategy θ^* and consume c^* up to time $T-1$. Such portfolio-consumption pair satisfies the budget constraint. Moreover $u_s(\gamma^*(s))$ is integrable for all $s \leq T-1$. At the terminal date T the agent consumes the terminal cash-flow $x_{\theta^*}(T) + X\frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)}$. Since the return of the numeraire $\frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)}$ is bounded, thanks to our hypothesis on the terminal utility u_T, we can choose the neighborhood of $W^*(t)$ in such a way that $u_T\left(\gamma^*(T-1), x_{\theta^*}(T) + X\frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)}\right)$ is integrable. This shows that the maximum remaining utility is not $-\infty$ in a neighborhood of the optimal wealth.

In this general framework, although we do not require the information structure to be Markovian, nonetheless the maximum remaining utility defined in (5) satisfies at the optimum the dynamic programming principle as the following proposition shows:

Proposition 1 *The maximum remaining utility has the following properties for any* $t = 0, 1, \ldots, T-1$*:*

1. *Given the stream of optimal past consumptions* $\gamma^*(T-1)$ *and the optimal level of wealth* $W^*(t)$,

$$H\left(t, \gamma^*(t-1), W^*(t)\right) = u_t(\gamma^*(t)) + E_t\left[H\left(t+1, \gamma^*(t), W^*(t+1)\right)\right]; \qquad (6)$$

2. *For any* $\varepsilon \in \Re$ *and for any self-financing strategy* θ^{SF} *such that*

$$H\left(t+1, \gamma^*(t), W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1)\right) \in L^1(\Omega, \mathcal{F}_{t+1}, P)$$

we have

$$H\left(t, \gamma^*(t-1), W^*(t) + \varepsilon V_{\theta^{SF}}(t)\right) \geq u_t(\gamma^*(t)) + \tag{7}$$
$$+E_t\left[H\left(t+1, \gamma^*(t), W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1)\right)\right]$$

The recursive equality (6) shows that the dynamic programming equation holds at the optimum even in the non-Markov case. To understand inequality (7), suppose to perturb the optimal time t wealth level with a self-financing portfolio. Then an optimizing agent has two choices: either to maximize the utility given the perturbed wealth or to consume the optimal consumption $c^*(t)$ at time t, to invest in the perturbed optimal strategy $\theta^*(t) + \varepsilon\theta^{SF}(t)$ up to time $t+1$ and then to maximize the $t+1$ utility given the remaining wealth $W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1)$. Inequality (7) shows that, given the information at time t, this second choice is never optimal on average whatever values the self-financing portfolio may assume (although it may very well be negative on sets of positive probability). These properties will be exploited to establish Proposition 2, which constitutes a fundamental building block for our main result, Theorem 2.

In order to define the marginal utility of optimal wealth, we need to define in a suitable way the differential of the maximum remaining utility H. Since, for any given t, the function $H(t, \gamma(t-1), W)$ defined in (5) associates to any $W \in L^p(\Omega, \mathcal{F}_t, P)$ *a random variable* that cannot be represented in general as a deterministic function of the wealth level W, the $\omega - by - \omega$ derivative of H with respect to the wealth may fail to exist (and even if it existed, it might fail to generate a state-price density). A sensible definition of marginal utility of wealth in this general framework can in fact be obtained by borrowing the notion of *Fréchet-differentiability* from functional analysis. More precisely, we assume that for any t, given the past consumptions $\gamma^*(t-1)$, the function $H(t, \gamma^*(t-1), \cdot)$ maps an $L^p(\Omega, \mathcal{F}_t, P)$-neighborhood of the optimal wealth $W^*(t)$ into $L^1(\Omega, \mathcal{F}_t, P)$ and is *Fréchet-differentiable* in $W^*(t)$. For convenience of the reader, we recall that the function $H(t, \gamma^*(t-1), \cdot)$ is said to be Fréchet-differentiable at a wealth level W, if $H(t, \cdot)$ maps an $L^p(\Omega, \mathcal{F}_t, P)$-neighborhood of W into $L^1(\Omega, \mathcal{F}_t, P)$ and if there exists a linear continuous operator $\mathcal{D}(t, W) : L^p(\Omega, \mathcal{F}_t, P) \to L^1(\Omega, \mathcal{F}_t, P)$ such that

$$\lim_{\|Y\|_{L^p} \to 0} \frac{\|H(t, \gamma^*(t-1), W + Y) - H(t, \gamma^*(t-1), W) - \mathcal{D}(t, W)(Y)\|_{L^1}}{\|Y\|_{L^p}} = 0. \tag{8}$$

In this case we define the linear and continuous functional $\mathcal{E} : L^p(\Omega, \mathcal{F}_t, P) \to \Re$ via

$$\mathcal{E}(Y) = E\left[\mathcal{D}(t, W)(Y)\right]$$

for all $Y \in L^p(\Omega, \mathcal{F}_t, P)$. By the Riesz representation theorem, there exists a unique $H_W(t, \gamma^*(t-1), W) \in L^q(\Omega, \mathcal{F}_t, P)$ such that

$$\mathcal{E}(Y) = E\left[\mathcal{D}(t, W)(Y)\right] = E\left[H_W(t, \gamma^*(t-1), W) \cdot Y\right] \qquad \text{for all } Y \in L^p(\Omega, \mathcal{F}_t, P) \tag{9}$$

This leads to the following definition:

Definition 5 *The time t maximum remaining utility $H(t, \gamma^*(t-1), \cdot)$ admits time t marginal utility of wealth for the wealth level $W \in L^p(\Omega, \mathcal{F}_t, P)$ if $H(t, \gamma^*(t-1), \cdot)$ is Fréchet-differentiable at W. In this case we call time t marginal utility of the wealth W the unique $H_W(t, \gamma^*(t-1), W) \in L^q(\Omega, \mathcal{F}_t, P)$ satisfying* (9)*.*

In the next subsection we show that such definition coincides with the standard one in the Markovian setting (see Section 4.2).

In order to characterize the set of linear pricing rules in terms of marginal utility of wealth in our general framework, we restrict our attention to the following class of optimizing agents.

Definition 6 *We denote by $\mathcal{A}^*$ the class of optimizing agents with preferences as in* (4)*, initial endowment e_0, optimal consumption stream $\{\gamma^*(t)\}_{t=0}^T$ and optimal intertemporal wealth $\{W^*(t)\}_{t=0}^T$, such that:*

1. *their maximum remaining utility $H(t, \gamma^*(t-1), \cdot)$ maps an $L^p(\Omega, \mathcal{F}_t, P)$-neighborhood of the optimal wealth $W^*(t)$ into $L^1(\Omega, \mathcal{F}_t, P)$ for all $t = 0, 1, \ldots, T$;*

2. *$H(t, \gamma^*(t-1), \cdot)$ admits time t marginal utility of wealth at the optimum $W^*(t)$ for all $t = 0, 1, \ldots, T$ in the sense of Definition* 5.

3. *the terminal marginal utility of optimal wealth is strictly positive almost surely, namely,*

$$P\left(H_W(T, \gamma^*(T-1), W^*(T)) > 0\right) = 1.$$

It will become apparent that, as a consequence of Theorem 2, $\mathcal{A}^*$ is non-empty when there exist linear pricing functionals.

For sake of notation, in what follows we denote $H_W^*(t) = H_W(t, \gamma^*(t-1), W^*(t))$. The intertemporal marginal utilities of optimal wealth have the following important property, that allows us to interpret the process $\{H_W^*(t)\}_{t=0}^T$ as a state-price density and is fundamental in the proof of Theorem 2:

Proposition 2 *For every self-financing admissible strategy θ^{SF}, the process $\left(H_W^*(t)V_{\theta^{SF}}(t)\right)_{0\le t\le T}$ is a martingale, that is*

$$H_W^*(t)\,V_{\theta^{SF}}(t) = E_t\left[H_W^*(t+1)\,V_{\theta^{SF}}(t+1)\right] \tag{10}$$

for all $t = 0, \ldots, T-1$.

We are now ready to state our main result that characterizes the entire set of linear pricing rules that support viability in our security market in terms of marginal utility of optimal intertemporal wealth.

Theorem 2 *A functional $\psi : X \to \Re$ is a linear pricing rule, that is $\psi \in \Psi$, if and only if there exists an optimizing agent in $\mathcal{A}^*$ such that*

$$\psi(x) = \frac{1}{H_W^*(0)} E\left[\sum_{t=1}^{T} H_W^*(t)x(t)\right], \quad \textit{for all } x \in X. \tag{11}$$

This result can be also rephrased in terms of the relations between marginal utility of optimal intertemporal wealth and state price densities, whose definition we recall hereafter for ease of the reader.

Definition 7 *An adapted process $\{\pi(t)\}_{0\le t\le T}$ with $\pi(t) \in L^q_{++}(\Omega, \mathcal{F}_t, P)$ for all t, is a **state price density** (state-price deflator) if*

$$S(t) = \frac{1}{\pi(t)} E_t\left[\pi(t+1)(S(t+1) + d(t+1))\right] \qquad \textit{for } t = 0, \ldots, T-1.$$

From Theorem 2 it is easy to see that the state price densities are marginal utilities of optimal intertemporal wealth for some optimizing agent as in Theorem 2. This fact follows by comparing the definition of state price densities to the representation of the linear pricing functionals in terms of marginal utility of optimal intertemporal wealth in Theorem 2.

Corollary 1 *An adapted process* $\{\pi(t)\}_{0\leq t\leq T}$ *with* $\pi(t) \in L^q_{++}(\Omega, \mathcal{F}_t, P)$ *for all* t*, is a state price density if and only if there exists an optimizing agent in* $\mathcal{A}^*$ *such that*

$$\pi(t) = H^*_W(t)$$

for $t = 0, \ldots, T$.

It is interesting to compare our approach to the classical one in which the information structure is generated by a Markov process and the period utilities, although they may very well depend on the realization of the Markov process, do not depend on past consumptions and are strictly increasing, convex and differentiable in current consumption. In this standard case, it is well known (e.g. Duffie, 2001) that, denoting by $u'_t(c^*(t))$ the well defined marginal utility of optimal time t consumption, any functional $\psi : X \to \Re$ can be written in the form

$$\psi(x) = \frac{1}{u'_0(c^*(0))} E\left[\sum_{t=1}^{T} u'_t(c^*(t))x(t)\right], \quad \text{for all } x \in X.$$

Under suitable assumptions the representation of the linear functional in terms of marginal utilities of optimal wealth as in (11) is then a consequence of a standard envelope argument by which $H^*_W(t) = u'_t(c^*(t))$. In our general framework, the marginal utilities of consumption may fail to exist and hence the representation of the linear functional in terms of marginal utilities of wealth as in (11) clearly cannot be obtained via an envelope argument.[13] This is why we provide a generalized definition of marginal utilities of wealth and restrict our attention to the class of optimizing agents who have well-defined marginal utilities of optimal wealth in this generalized sense.

Is our approach in fact more general? To give a positive answer one must produce an example that fits in our framework and in which $H^*_W(t)$ is well-defined while $u'_t(c^*(t))$ is not. This is exactly what we do at the end of next subsection, where we provide a Markovian example in which the marginal utility of optimal wealth is well-defined at all times, even though the utility over consumption before the terminal date is satiated, and discontinuous at the optimum.

[13] The standard envelope argument that links the marginal utility of consumption to the marginal utility of wealth may fail even when the former is well defined if trading is subject to frictions. For a classical example of such an occurrence see for instance Grossman and Laroque (1990). An extension of the envelope theorem in L^p spaces is provided in [4].

4.2 The case of a Markovian information structure

In this section we want to compare our approach to the classical one, showing in particular that the generalized definition of marginal utility of wealth coincides with the standard definition, when the information structure is assumed to be Markovian. We assume now that the filtration $\mathcal{F}$ is generated by an $\Re^D$-valued Markov process $\mathcal{Z} = \{z(t)\}_{t=0}^T$ of state variables. In this case, investors' preferences depend on the state of nature through $z(t)$. More precisely, for any investor with preference (4) there exists a sequence of functions $\{\tilde{u}_t\}_{t=0}^T$, with $\tilde{u}_t : \Re^{t+1} \times \Re^D \to \Re$ such that

$$u_t(c_0, \ldots, c_t, \omega) = \tilde{u}_t(c_0, \ldots, c_t, z(t)(\omega))$$

and the terminal period utility $\tilde{u}_T(c_0, \ldots, c_T, z)$ is strictly increasing in c_T for all $(c_0, \ldots, c_{T-1}) \in \Re^T, z \in \Re^D$. Since the budget constraint is binding at the optimum, the maximum remaining utility defined in (5), given the stream of optimal past consumptions, reduces to

$$\begin{aligned} H(t, \gamma^*(t-1), W(t)) \;\; &\equiv \;\; \operatorname*{ess\,sup}_{\theta \in \Theta} \Big\{ \tilde{u}_t(\gamma^*(t-1), W(t) - V_\theta(t), z(t)) \; + \\ &\quad + \sum_{s=t+1}^{T} E_t \left[\tilde{u}_s(\gamma^*(t-1), x_\theta(t), \ldots, x_\theta(s), z(s)) \right] \Big\} \end{aligned}$$

for any $W(t) \in L^p(\Omega, \mathcal{F}_t, P)$. As a consequence of the Markov assumption, the maximum remaining utility given the optimal past consumptions is a deterministic function of the wealth level $W(t)$ and the state variables $z(t)$. Formally, for any agent in $\mathcal{A}^*$, for any $t = 0, 1, \ldots, T$ there exists a function $h(t, \cdot, \cdot, \cdot) : \Re^t \times \Re \times \Re^D \to \Re$ such that the agent's time t-maximum remaining utility takes the form

$$H(t, \gamma^*(t-1), W(t))\,(\omega) = h(t, \gamma^*(t-1)\,(\omega)\,, W(t)\,(\omega)\,, z(t)\,(\omega)).$$

If $h(t, \gamma, \cdot, z)$ is *differentiable* on the set $\{w \in \Re : h(t, \gamma, w, z) < +\infty\}$ for any $t = 0, 1, \ldots, T$, $\gamma \in \Re^t, z \in \Re^D$, then the marginal utility of optimal wealth coincides with the derivative of h at the optimum and, as a consequence, Theorem 2 can be restated in terms of h. Indeed, letting

$$h_w^*(t, z(t))(\omega) = \frac{\partial}{\partial w} h(t, \gamma^*(t-1)(\omega), w, z(t)(\omega))\Big|_{w = W^*(t)(\omega)}$$

the following result holds.

Proposition 3 *Given an agent in* $\mathcal{A}^*$, *let* $h(t, \gamma, \cdot, z)$ *be differentiable on the set* $\{w \in \Re : h(t, \gamma, w, z) < +\infty\}$ *for any* $t = 0, 1, \ldots, T$, $\gamma \in \Re^t, z \in \Re^D$. *Then*

$$h_w^*(t, z(t)) \in L^q(\Omega, \mathcal{F}_t, P) \text{ for any } t,$$

and

$$H_W^*(t) = h_w^*(t, z(t))$$

for all $t = 0, 1, \ldots, T$, *where* $H_W^*(t)$ *is the marginal utility of optimal wealth defined in* (9).

In words, our last result shows that the generalized definition of marginal utility of wealth introduced in the previous section coincides with the standard one when preferences do not depend explicitly on the state ω. It is now interesting to compare our approach, in which we assume both the Fréchet differentiability of the value function interpreted as an operator that maps random variable into random variables, and the differentiability with respect to wealth levels of the value function interpreted as a function of the state variables, with the approach employed in the standard asset pricing literature. In this standard literature, indeed, the requirement that the value function be Fréchet differentiable is redundant, and this is so because all period utilities are assumed to be strictly increasing, concave and continuous. Under these stronger assumptions on the period utilities, in fact, the value function is itself concave and continuous and hence, as a consequence that can be readily proved, it is itself Fréchet differentiable. In our approach, instead, we relax all assumptions on the intermediate period utilities, as well as the requirement that the terminal utility be concave. This is exactly why we need to impose from the outset the Fréchet differentiability of the value function.

A natural question that arises in light of the previous discussion is then to determine a parsimonious set of conditions that imply that the marginal utility of wealth is still a state-price density without requiring the value function to be Fréchet differentiable. This question is answered by our next result, which shows that, regardless of the behavior of the intermediate period utilities, as long as the terminal utility is strictly increasing and concave in terminal consumption and the value functions $h(t, \gamma, \cdot, z)$ are all differentiable in the wealth variable, then the marginal utility of wealth satisfies the martingale property in Proposition 2 and hence it constitutes a state-price density.

Proposition 4 *Given an optimizing agent with preferences as in* (4) *in a Markovian framework, assume that* $h(t, \gamma^*(t-1), W, z(t))$ *is integrable in a* L^p*-neighborhood of* $W^*(t)$*. Let* $h(t, \gamma, \cdot, z)$ *be differentiable on the set* $\{w \in \Re : h(t, \gamma, w, z) < +\infty\}$ *for any* $t = 0, 1, \ldots, T, \gamma \in \Re^t, z \in \Re^D$ *and assume that the terminal period-utility* $\tilde{u}_T(\gamma, \cdot, z)$ *is concave for any* $\gamma \in \Re^T, z \in \Re^D$*. Then, for every self-financing admissible strategy* θ^{SF}*, the process* $\left(h^*_w(t, z(t))\, V_{\theta^{SF}}(t)\right)_{0 \le t \le T}$ *is a martingale. Moreover, if* $h^*_w(T, z(T)) \in L^q(\Omega, \mathcal{F}_T, P)$*, then* $h^*_w(t, z(t)) \in L^q(\Omega, \mathcal{F}_t, P)$ *for all* $t = 0, \ldots, T$*.*

When the information structure is generated by a Markov process of state variables, therefore, we are able to define a larger class $\mathcal{A}^M$ of optimizing agents who satisfy the assumptions of either Proposition 3 or Proposition 4:

Definition 8 *We denote by* $\mathcal{A}^M$ *the class of optimizing agents with preferences as in* (4)*, initial endowment* e_0*, optimal consumption stream* $\{\gamma^*(t)\}_{t=0}^T$ *and optimal intertemporal wealth* $\{W^*(t)\}_{t=0}^T$*, such that one of the following two conditions holds:*

(i) the agent is in $\mathcal{A}^*$ *and the functions* $h(t, \gamma, \cdot, z)$ *are differentiable for any* $\gamma \in \Re^t, z \in \Re^D, t = 0, 1, \ldots, T$;

(ii) the agent's terminal utility $\tilde{u}_T(\gamma, \cdot, z)$ *is concave for any* $\gamma \in \Re^T$, $z \in \Re^D$*; his maximum remaining utility given the optimal past consumption* $h(t, \gamma^*(t-1), \cdot, z(t))$ *maps an* $L^p(\Omega, \mathcal{F}_t, P)$*-neighborhood of the optimal wealth* $W^*(t)$ *into* $L^1(\Omega, \mathcal{F}_t, P)$ *for all* $t = 0, 1, \ldots, T$*; the functions* $h(t, \gamma, \cdot, z)$ *are differentiable for any* $\gamma \in \Re^t, z \in \Re^D, t = 0, 1, \ldots, T$ *and* $h^*_w(T, z(T)) \in L^q_{++}(\Omega, \mathcal{F}_T, P)$*.*

Given the class $\mathcal{A}^M$ of optimizing agents, we can finally state the following more general version of Theorem 2 that holds in the current Markov setting.

Theorem 3 *In the Markov setting, a functional* $\psi : X \to \Re$ *is a linear pricing rule, that is* $\psi \in \Psi$*, if and only if there exists an agent in* $\mathcal{A}^M$ *such that*

$$\psi(x) = \left(h^*_w(0, z(0))\right)^{-1} E\left[\sum_{t=1}^{T} h^*_w(t, z(t))\, x(t)\right], \quad \textit{for all } x \in X.$$

Summing up, we proved that in the Markov case, where the maximum remaining utility is a deterministic function of the past consumption stream, the wealth level and the state variables, the characterization of linear pricing rules in terms of marginal utilities of wealth holds by taking as marginal utilities of wealth the (standard) derivative of the maximum remaining utility given the optimal past consumption, with respect to the wealth level, given that the maximum remaining utilities are differentiable (as deterministic functions). In particular, for an agent in $\mathcal{A}^*$, this derivative, evaluated in the optimal wealth, coincides with the marginal utility of wealth in the general case as defined in (9). If the agent's maximum remaining utilities are not Fréchet differentiable at the optimum but the terminal utility is concave, the derivatives of the maximum remaining utilities with respect to the wealth level evaluated at the optimal wealth are nonetheless state-price densities and, therefore, may be used to characterize a linear pricing rule.

4.3 An example

To conclude this section, we substantiate our results by providing a Markovian example in which an agent with satiated, non concave and discontinuous utility over intermediate consumption is nonetheless able to solve his optimal consumption-portfolio problem. Moreover the value function associated to the optimal problem is differentiable with respect to wealth. This example shows therefore that our results are indeed non-empty since it displays a case in which the linear pricing functionals are indeed defined by optimizing agents with highly irregular preferences for intermediate consumptions.[14]

Consider the probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = [0,1]$, $\mathcal{F}_2 = \mathcal{B}([0,1])$, that is the Borel $\sigma-$algebra on the interval $[0,1]$, and P is the Lebesgue measure. We assume $\mathcal{T} = \{0,1,2\}$ and consider the state variables $\mathcal{Z} = \{z(t)\}_{t=0}^{T}$, defined as $z(0) = 1$, $z(1)(\omega) = \mathbb{I}_{[0,1/2]}(\omega)$, $z(2)(\omega) = \omega$. The information available to investors, $\mathcal{F}$, is the one generated by $\mathcal{Z}$. In this market a unique security is traded, whose price process can be described as follows:

$$S(0) = 1 \qquad S(1)(\omega) = \begin{cases} s & \text{if } \omega \in [0,1/2] \\ \overline{s} & \text{if } \omega \in]1/2,1] \end{cases} \qquad S(2)(\omega) = R\omega.$$

with s, $\overline{s}$, $R > 0$. The agent has the following utilities:

$$\tilde{u}_0(c) = \beta R c$$

$$\tilde{u}_1(c,z) = v(c) \cdot z + \overline{v}(c) \cdot (1-z)\,.$$

[14] The detailed computations for this example can be found in the Appendix.

$$\tilde{u}_2(c,z) = 3\beta cz$$

where $\beta > 0$, $\beta R < \min\left(4s, \frac{2}{7}\overline{s}\right)$ and

$$v(c) = \begin{cases} c & c \leq 1 \\ 1-c & c > 1 \end{cases} \qquad \overline{v}(c) = \begin{cases} c^3 & c \leq 1 \\ 1-c^2 & c > 1 \end{cases}$$

We remark that the 1-period utility u_1 is non-concave, discontinuous (hence, *a fortiori* not differentiable) and exhibits satiation. Given any initial endowment e_0, the optimal consumption-portfolio problem for our agent is

$$\sup_{(c,\theta)} U(c) = \sup_{(c,\theta)} E\left[\tilde{u}_0(c(0)) + \tilde{u}_1(c(1), z(1)) + \tilde{u}_2(c(2), z(2))\right]$$

$$s.t. \begin{cases} c(0) + \theta(0) \leq e_0 \\ c(1) + \theta(1)S(1) \leq \theta(0)S(1) \\ c(2) \leq \theta(1)S(2) \end{cases}$$

The optimal consumption-portfolio pairs of the agent are

$$\begin{aligned} c^*(0) &= e_0 - \theta^*(0) & \theta^*(0) &\in \Re \\ c^*(1) &= 1 & \theta^*(1) &= \theta^*(0) - \tfrac{1}{S(1)} \\ c^*(2) &= \left(\theta^*(0) - \tfrac{1}{S(1)}\right) S(2) \end{aligned}$$

Clearly the time 1 period utility is not differentiable at the optimal consumption $c^*(1) = 1$. Since the setting is Markovian, the maximum remaining utilities of wealth have the form

$$H(t, W(t)) = h(t, W(t), z(t)).$$

In particular, we have

$$h(0,w,z) = \beta R w + 1 - \tfrac{\beta R}{8s} - \tfrac{7\beta R}{8\overline{s}}$$

$$h(1,w,z) = \left[1 - \tfrac{\beta R}{4s} + \tfrac{\beta R}{4s}\, w\right] \cdot z + \left[1 - \tfrac{7\beta R}{4\overline{s}} + \tfrac{7\beta R}{4\overline{s}} w\right] \cdot (1-z)$$

$$h(2,w,z) = 3\beta wz.$$

Observe that the value functions are all differentiable with respect to the level of wealth.

The marginal utilities of optimal wealth are given by:

$$H_W(0, e_0) = h_w(0, e(0), z(0)) = \beta R$$

$$H_W(1, W(1)) = h_w(0, W(1), z(1)) = \tfrac{\beta R}{4s} \cdot z(1) + \tfrac{7\beta R}{4\overline{s}} \cdot (1 - z(1))$$

$$H_W(2, W(2),) = h_w(0, W(2), z(2)) = 3\beta z(2)$$

As a consequence of Theorem 2 a linear pricing rule can be defined by:

$$\begin{aligned}\psi(x) &= \frac{1}{\beta R} E\left[\frac{\beta R}{4}\left[\frac{1}{s}\cdot z(1)+\frac{7}{\overline{s}}\cdot(1-z(1))\right]x(1)+3\beta z(2)x(2)\right] \\ &= \frac{1}{R} E\left[\frac{R}{4}\left[\frac{1}{s}z(1)+\frac{7}{\overline{s}}(1-z(1))\right]x(1)+3z(2)x(2)\right]\end{aligned}$$

5 Equivalent martingale measures and the marginal utility of wealth

The purpose of this final section is to relate the marginal utility of wealth via the linear pricing functionals to the equivalent martingale measures. Once again, we follow the lead of Harrison and Kreps (1979, Theorem 2) who show that in the case of a multi-period market where intermediate consumption is not allowed and there exists a risk-free asset linear pricing functionals and equivalent martingale measures are in on-to-one correspondence. Our first step is to extend their result (Theorem 2) to our framework with intermediate consumption. To this end we recall the definition of equivalent martingale measure with respect to any numeraire: we remark that in this section we do not require that the numeraire satisfies Condition 1.

Definition 9 *Let N be a numeraire. A probability measure $Q \sim P$ is an* ***Equivalent Martingale Measure*** *associated to the numeraire N if the density $L(t) = E^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]$ satisfies*

$$\frac{L(t)}{N(t)} \in L^q(\Omega, \mathcal{F}_t, P) \quad \textit{for } t = 1, \ldots, T \tag{12}$$

for q such that $\frac{1}{p}+\frac{1}{q}=1$, and

$$\frac{S(t)}{N(t)} = E_t^Q\left[\frac{S(t+1)+d(t+1)}{N(t+1)}\right] \tag{13}$$

for $t = 0, \ldots, T-1$, where $E_t^Q[\cdot]$ denotes the conditional expectation under Q with respect to $\mathcal{F}_t$. We denote with $\mathcal{Q}_N$ the set of equivalent martingale measures associated to the numeraire N.

In the presence of a risk free asset, Harrison and Kreps (1979) require the density of the equivalent martingale measures to be P-square-integrable. In this way, any discounted P-square-integrable contingent claim has finite expectation under the equivalent martingale measures. However, they also point out that in a market where the numeraire is not riskless, one must be careful to guarantee that the transition from the original market to the "discounted" market is neutral, that is the space of contingent claims does not change and the agents' preferences remain continuous. Hence, to preserve the space of contingent claims and the continuity of agents' preferences, Harrison and Kreps suggest to change the norm for the space of discounted contingent claims.

Since we allow for intermediate cash-flows, we need to take into account not only the space of contingent claims, but the entire set of discounted cash-flows. By extending Harrison and Kreps' argument to our framework, we define the following space:

$$\begin{aligned} X'_N &= \left\{ x' = \frac{x}{N} = \left(\frac{x(1)}{N(1)}, \ldots, \frac{x(T)}{N(T)} \right) \text{ for } x \in X \right\} = \\ &= \left\{ x' : x'N = \big(x'(1)N(1), \ldots, x'(T)N(T) \big) \in X \right\} \end{aligned}$$

endowed with the norm

$$\left\| x' \right\|_N = \left\| x'N \right\|_X$$

We see then that the density L of any equivalent martingale measure (given the numeraire N) belongs to the dual space[15] of X'_N that is

$$L \in \left\{ \rho' = (\rho'(0), \ldots, \rho'(T)) : \frac{\rho'(t)}{N(t)} \in L^q(\Omega, \mathcal{F}_t, P) \text{ for } t = 1, \ldots, T \right\}$$

This justifies condition (12) in the definition of Equivalent Martingale Measure.

Theorem 4 *Let N be a numeraire. Then there is a one-to-one correspondence between the set of equivalent martingale measures $\mathcal{Q}_N$ and the set of linear pricing functionals Ψ. The correspondence is given by*

$$Q(B) = \frac{1}{N(0)} \psi\left(0, \ldots, 0, I_B N(T)\right) \text{ for any } B \in \mathcal{F}_T \tag{14}$$

and

$$\psi(x) = E^Q \left[\sum_{t=1}^{T} x(t) \frac{N(0)}{N(t)} \right] \text{ for any } x \in X \tag{15}$$

[15] With respect to the norm $\|\cdot\|_N$

Theorem 4 extends Theorem 2 in Harrison and Kreps (1979) to the case of intermediate consumption. Since agents are not restricted to self-financing strategies, they may liquidate their portfolio at any date. Hence, they are not forced to transfer their wealth to the terminal date, by investing in the numeraire. This is why Condition 1 is unnecessary for our result.

We remark that Theorem 4 does not depend on the choice of the numeraire, in the sense that for every numeraire N a one-to-one correspondence can be defined between $\mathcal{Q}_N$ and Ψ. Moreover, given two different numeraires N_1 and N_2, Theorem 4 allows to construct a one-to-one correspondence between $\mathcal{Q}_{N_1}$ and $\mathcal{Q}_{N_2}$. It is easy to recognize that the obtained correspondence is the classical formula for the change of numeraire (see for instance, Proposition 5.18 in Föllmer, H. and A. Schied (2002))

Recalling the representation of linear pricing functionals in terms of equivalent martingale measures given in Theorem 4, we are now able to derive from Theorem 2 the following characterization of equivalent martingale measures in terms of marginal utility of optimal intertemporal wealth.

Corollary 2 *Let $\theta^{BN} \in \Theta^{BN}$. Then there is a one-to-one correspondence between equivalent martingale measures Q and the marginal utility of optimal wealth of optimizing agents in $\mathcal{A}^*$. The correspondence is given by*

$$\frac{dQ}{dP} = \frac{H_W^*(T)}{H_W^*(0)} \frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(0)}$$

and

$$H_W^*(t) = \frac{1}{V_{\theta^{BN}}(t)} E_t \left[\frac{dQ}{dP} \right] \qquad t = 0. \dots, T$$

(more precisely, $H_W^*(t)$ is defined up to the constant $H_W^*(0) V_{\theta^{BN}}(0)$).

6 Conclusions

In a security market with discrete-time trading we extend Harrison and Kreps (1979) characterization of viability in terms of linear pricing rules to the case of intertemporal consumption and preferences for intermediate consumption that may exhibit satiation, non-convexity and discontinuity. Also, we relax the assumption of existence of a risk-free investment opportunity. More importantly, when agents' preferences are represented by time-additive possibly state-dependent and non time-separable utilities, we prove that the set of linear pricing rules is characterized in terms of agents' marginal utilities of optimal intertemporal wealth given the past optimal consumption even if the marginal utility of consumption fails to exist, and hence the envelope condition equating the marginal utility of wealth to the marginal utility of consumption cannot be invoked. Our results do not require that the information structure is generated by a Markov process of state-variable, and allow for intermediate preferences to be non-smooth, non-increasing, non-convex and discontinuous.

One issue that deserves further investigation is the possibility of weakening the requirement imposed by Condition 1. To put it more simply, Condition 1 is sufficient for our results to hold, but it is still an open question whether it is also necessary. Developing a weaker condition to replace Condition 1 is therefore an interesting topic for future work.

Another fruitful line of future research is the possibility of extending our results to preferences that allow for separation of risk aversion and intertemporal substitution (see among others Kreps and Porteus, 1978, and Epstein and Zin, 1990). This preferences, however, are not time-additive and hence the techniques presented in this paper should be suitably extended to account for this feature. We aim to address this issue in a different paper.

Appendix

A Propositions and Proofs

We collect herefollows proofs and technical propositions used throughout the paper.

Proof of Theorem 1. (If part)

Given a linear pricing rule ψ, exploit (3) to define an agent with $e_0 = 0$ and preferences described by the utility index

$$U(c) = c(0) + \sum_{t=1}^{T} E[\rho(t)c(t)]$$

This agent is in the class $\mathcal{A}$ and is strictly non satiated at all t, so that his budget set can be restricted to

$$B(0) = \{c \in \mathcal{C} \mid c(t) = x_\theta(t) \; \forall \; t \quad \textit{for some } \theta \in \Theta\}$$

Since by definition $x_\theta(0) = -V_\theta(0)$, from (3) we have that for any $c \in B(0)$

$$U(c) = x_\theta(0) + \sum_{t=1}^{T} E[\rho(t)x_\theta(t)] = -V_\theta(0) + V_\theta(0) = 0,$$

that is any consumption sequence that satisfies the budget constraint with equality is optimal, and hence the market is viable. ■

In order to prove the *only if* part, we first need some definitions and lemmas:

Definition 10 *A **Free Lunch (FL)** is a net* $(x_{\theta^\alpha}, y^\alpha) \in M \times X$ *and* $y \in X^+$ *such that*

$$\begin{aligned} &x_{\theta^\alpha}(t) \geq y^\alpha(t) \textit{ for all } \alpha, t \\ &y^\alpha(t) \to y(t) \textit{ in } L^p \\ &\liminf_\alpha V_{\theta^\alpha}(0) \leq 0. \end{aligned}$$

*A **Self-Financing Free Lunch (SFFL)** is a free lunch such that* θ *is self-financing and* $y^\alpha(t) = 0$ *for* $t \leq T-1$*, namely, it is a net* $(x_{\theta^\alpha}, y^\alpha) \in M \times X$ *and* $y \in X^+$ *such that*

$$\begin{aligned} &x_{\theta^\alpha}(t) = y^\alpha(t) = y(t) = 0 \textit{ for all } \alpha \textit{ and } t \leq T-1 \\ &x_{\theta^\alpha}(T) \geq y^\alpha(T) \\ &y^\alpha(T) \to y(T) \textit{ in } L^p \\ &\liminf_\alpha V_{\theta^\alpha}(0) \leq 0. \end{aligned}$$

Lemma 1 *There exists a Free Lunch if and only if there exists a Self-Financing Free Lunch.*

Proof. Sufficiency is obvious. To prove necessity, let $(x_{\theta^\alpha}, y^\alpha) \in M \times X$, $y \in X^+$ be a FL. Let moreover $V_{\theta^{BN}}$, with $\theta^{BN} \in \Theta^{BN}$, be a numeraire with bounded returns.
We define:

$$\Gamma^\alpha(t) = \sum_{s=1}^{t} \frac{x_{\theta^\alpha}(s)}{V_{\theta^{BN}}(s)}$$

We can also write recursively $\Gamma^\alpha(t) = \Gamma^\alpha(t-1) + \frac{x_{\theta^\alpha}(t)}{V_{\theta^{BN}}(t)}$

Consider the strategy:

$$\tilde{\theta}^\alpha(t) = \begin{cases} \theta^\alpha(0), & \text{for } t = 0 \\ \theta^\alpha(t) + \Gamma^\alpha(t)\theta^{BN}(t), & \text{for } 1 \leq t \leq T-1 \end{cases}$$

Then $V_{\tilde{\theta}^\alpha}(0) = V_{\theta^\alpha}(0)$, hence $\liminf_\alpha V_{\tilde{\theta}^\alpha}(0) \leq 0$. The cashflow at time t, with $1 \leq t \leq T-1$ is:

$$\begin{aligned} x_{\tilde{\theta}^\alpha}(t) &= \tilde{\theta^\alpha}(t-1)\cdot[S(t)+d(t)] - \tilde{\theta}^\alpha(t)\cdot S(t) \\ &= x_\theta^\alpha(t) + (\Gamma^\alpha(t-1) - \Gamma^\alpha(t))\, V_{\theta^{BN}}(t) = 0 \end{aligned}$$

since the numeraire is self-financing. At the final date, we have:

$$x_{\tilde{\theta}^\alpha}(T) = x_{\theta^\alpha}(T) + \sum_{s=1}^{T-1} \frac{x_{\theta^\alpha}(s)}{V_{\theta^{BN}}(s)} V_{\theta^{BN}}(T)$$

which is in $L^p(\Omega, \mathcal{F}_T, P)$ since the numeraire has bounded returns. Therefore $\tilde{\theta}^\alpha$ is an admissible self-financing strategy.

Define now $\tilde{y}^\alpha$ as:

$$\tilde{y}^\alpha(t) = \begin{cases} 0 & \text{for } t \leq T-1 \\ y^\alpha(T) + \sum_{s=1}^{T-1} \frac{y^\alpha(s)}{V_{\theta^{BN}}(s)} V_{\theta^{BN}}(T), & \text{for } t = T \end{cases}$$

and $\tilde{y}$ as

$$\tilde{y}(t) = \begin{cases} 0 & \text{for } t \leq T-1 \\ y(T) + \sum_{s=1}^{T-1} \frac{y(s)}{V_{\theta^{BN}}(s)} V_{\theta^{BN}}(T), & \text{for } t = T \end{cases}$$

Since $\theta^{BN} \in \Theta^{BN}$, then $P\left[V_{\theta^{BN}}(T) > 0\right] = 1$, which implies that $x_{\tilde{\theta}^\alpha}(T) \geq \tilde{y}^\alpha(T)$ and $P[\tilde{y}(T) > 0] = 1$, that is, $y \in X^+$. Finally

$$\|\tilde{y}^\alpha(T) - \tilde{y}(T)\|_{L^p} \leq \|y^\alpha(T) - y(T)\|_{L^p} + \sum_{s=1}^{T-1} \|y^\alpha(s) - y(s)\|_{L^p} \cdot \left\| \frac{V_{\theta^{BN}}(s)}{V_{\theta^{BN}}(T)} \right\|_{L^\infty}$$

which shows that $y^\alpha(T) \to y(T)$ in L^p. Therefore $\left(x_{\tilde{\theta}^\alpha}, \tilde{y}^\alpha\right)$, $\tilde{y}$ is a SFFL. ■

Lemma 2 *If the market is viable, then the budget constraint of the optimizing agent is binding at the optimum.*

Proof. Assume that the market is viable, that is for some agent $(e_0, \succeq)$ in $\mathcal{A}$ there exist a consumption sequence $c^* \in \widehat{\mathcal{C}}$ and a dynamic investment strategy $\theta^* \in \Theta$ such that $c^* \in B(e_0)$ and $c^* \succeq c$ for all $c \in B(e_0)$. let $V_{\theta^{BN}}$, with $\theta^{BN} \in \Theta^{BN}$, be a numeraire with bounded returns. Without loss of generality, we can assume $V_{\theta^{BN}}(0) = 1$. From the monotonicity of the preferences with respect to the terminal consumption, it follows that the budget constraint at time T is binding, that is $c^*(T) = \theta^*(T-1)d(T)$. Suppose now that at some date $\bar{t} < T$, the constraint is not binding, that is either $c^*(0) < x_{\theta^*}(0) + e_0$ if $\bar{t} = 0$ or $P\left[c^*(\bar{t}) < x_{\theta^*}(\bar{t})\right] > 0$ if $\bar{t} > 0$. Define then

$$\Lambda(\bar{t}) = \begin{cases} x_{\theta^*}(0) + e_0 - c^*(0), & \bar{t} = 0, \\ \dfrac{x_{\theta^*}(\bar{t}) - c^*(\bar{t})}{V_{\theta^{BN}}(\bar{t})}, & \bar{t} > 0. \end{cases}$$

Since $\theta^{BN} \in \Theta^{BN}$, then $P\left[V_{\theta^{BN}}(\bar{t}) > 0\right] = 1$ and hence $P\left[\Lambda(\bar{t}) > 0\right] = P\left[c^*(\bar{t}) < x_{\theta^*}(\bar{t})\right] > 0$. Use now $\Lambda(\bar{t})$ to define the strategy $\tilde{\theta}$ as follows:

$$\tilde{\theta}(t) = \begin{cases} \theta^*(t), & \text{for } t < \bar{t} \\ \theta^*(t) + \Lambda(\bar{t})\theta^{BN}(t), & \text{for } t \geq \bar{t} \end{cases}$$

Since $x_{\theta^*}(\bar{t}), c^*(\bar{t}) \in L^p(\Omega, \mathcal{F}_{\bar{t}}, P)$ and $V_{\theta^{BN}}$ has bounded returns then $\Lambda(\bar{t})V_{\theta^{BN}}(t) \in L^p(\Omega, \mathcal{F}_t, P)$ for all $\bar{t} \leq t < T$. As a consequence the dynamic investment strategy $\tilde{\theta}$ is admissible. Define now the consumption sequence $\tilde{c}$ as follows:

$$\tilde{c}(t) = \begin{cases} c^*(t), & \text{for } t < T \\ c^*(T) + \Lambda(\bar{t})V_{\theta^{BN}}(T), & \text{for } t = T \end{cases}$$

It is readily checked that $\tilde{c}$ is budget-feasible. Moreover $\tilde{c} \succ c^*$ since $\tilde{c}(t) = c^*(t)$ at all dates $t < T$ while at the terminal date $\tilde{c}(T) \geq c^*(T)$ with $P\left[\tilde{c}(T) > c^*(T)\right] = P\left[\Lambda(\bar{t}) > 0\right] > 0$. Since this contradicts the optimality of (c^*, θ^*), the fact that the budget constraint is binding at the optimum is proved. ■

Lemma 3 *If the market is viable, then it admits no free lunches.*

Proof. Thanks to Lemma 2 and the non-emptyness of Θ^N, one can easily adapt the proof of Theorem 2 in Kreps (1981) to show that if the market is viable, then there exist No Self-Financing Free Lunches. Lemma 1 allows to conclude. ■

Proof of Theorem 1. (Only if part) Since viability implies No Free Lunch, we can invoke the generalization of Theorem 3 in Kreps (1981) due to Stricker (1990) to deduce that the extension property holds. We need such an extension because Kreps' theorem requires the separability of the space, an assumption that may be not satisfied in our case.

Stricker (1990) showed however that this assumption is superfluous for the L^p-spaces. (see Schachermayer (1994, 2001) for other references and proofs). ■

Proposition 5 *Let (U, e_0) be an agent as in Section 4.1. Fix $t \in \{0, \ldots, T-1\}$, and a sequence of past consumptions $\gamma(t-1) = (\bar{c}(s))_{0 \le s \le t-1}$, and let $W \in L^p(\Omega, \mathcal{F}_t, P)$ be such that for any future consumption sequence $(c(s))_{t \le s \le T}$ such that $c(t) + V_\theta(t) \le W$ and $c(s) \le x_\theta(s)$ for $s = t+1, \ldots, T$ for some admissible strategy θ, either $u_s(\bar{\gamma}(t-1), c(t), \ldots, c(s))$ is integrable or $E\left[u_s(\bar{\gamma}(t-1), c(t), \ldots, c(s))\right] = -\infty$.*

Denote with $\Phi(t, W)$ the set of random variables on $(\Omega, \mathcal{F}_t, P)$ defined as follows

$$\Phi(t, \gamma(t-1), W) = \left\{ \varphi = \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), c(t), \ldots, c(s))\right] : \quad \begin{array}{l} c(t) + V_\theta(t) \le W \\ c(s) \le x_\theta(s) \text{ for } s = t+1, \ldots, T \\ \text{for some admissible strategy } \theta \end{array} \right\}$$

Then:

(i) the set $\Phi(t, \gamma(t-1), W)$ is directed upwards, namely: for $\varphi, \tilde{\varphi} \in \Phi(t, \gamma(t-1), W)$, there exists $\psi \in \Phi(t, \gamma(t-1), W)$ such that $\psi \ge \max(\varphi, \tilde{\varphi})$

(ii) the random variable

$$H(t, \gamma(t-1), W) = \operatorname*{ess\ sup}_{\varphi \in \Phi(t, \gamma(t-1), W)} \varphi$$

is well-defined ($H(t, \gamma(t-1), W) = -\infty$ if $\Phi(t, \gamma(t-1), W) = \emptyset$ or if $\varphi = -\infty$ for all $\varphi \in \Phi(t, \gamma(t-1), W)$); moreover, there exists an increasing sequence $(\varphi_n)_{n \ge 1}$ in $\Phi(t, \gamma(t-1), W)$ such that $H = \lim_n \varphi_n$ P-almost surely

(iii) if $H(t, \gamma(t-1), W)$ is integrable, then, for $s < t$:

$$E_s\left[H(t, \gamma(t-1), W)\right] = \operatorname*{ess\ sup}_{\varphi \in \Phi(t, \gamma(t-1), W)} E_s[\varphi]$$

Proof. Let $\varphi, \tilde{\varphi} \in \Phi(t, \gamma(t-1), W)$: if $P(\varphi \ge \tilde{\varphi})$ is either 0 or 1 (that is $\max(\varphi, \tilde{\varphi}) = \varphi$ or $\max(\varphi, \tilde{\varphi}) = \tilde{\varphi}$) then the claim is trivial.

If this is not the case, then the event $A = \{\varphi \geq \tilde{\varphi}\}$ satisfies $0 < P(A) < 1$. Note that $A \in \mathcal{F}_t$. Let (c, θ) and $(\tilde{c}, \tilde{\theta})$ be such that

$$\varphi = \textstyle\sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), c(t), \ldots, c(s))\right] \qquad \tilde{\varphi} = \textstyle\sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))\right]$$

and

$$c(t) + V_\theta(t) \leq W \qquad \tilde{c}(t) + V_{\tilde{\theta}}(t) \leq W$$

$$c(s) \leq x_\theta(s) \qquad \tilde{c}(s) \leq x_{\tilde{\theta}}(s) \qquad \text{for } t+1 \leq s \leq T$$

The consumption-portfolio choice $(\hat{c}, \hat{\theta})$ defined as

$$\hat{c}(s) = c(s)\mathbb{I}_A + \tilde{c}(s)\mathbb{I}_{A^c} \qquad \hat{\theta}(s) = \theta(s)\mathbb{I}_A + \tilde{\theta}(s)\mathbb{I}_{A^c} \qquad t \leq s \leq T$$

is admissible and satisfies the budget constraints, which entails that $\psi = \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), \hat{c}(t), \ldots, \hat{c}(s))\right]$ belongs to $\Phi(t, \gamma(t-1), W)$. Moreover,

$$\begin{aligned}
\psi &= \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), \hat{c}(t), \ldots, \hat{c}(s))\right] \\
&= \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), c(t)\mathbb{I}_A + \tilde{c}(t)\mathbb{I}_{A^c}, \ldots, c(s)\mathbb{I}_A + \tilde{c}(s)\mathbb{I}_{A^c})\right] \\
&= \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), c(t), \ldots, c(s))\mathbb{I}_A + u_s(\gamma(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))\mathbb{I}_{A^c}\right] \\
&= \sum_{s=t}^{T} E_t\left[u_s(\gamma(t-1), c(t), \ldots, c(s))\right]\mathbb{I}_A + E_t\left[u_s(\gamma(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))\right]\mathbb{I}_{A^c} \\
&= \varphi\mathbb{I}_A + \tilde{\varphi}\mathbb{I}_{A^c} = \max(\varphi, \tilde{\varphi}).
\end{aligned}$$

Hence (i) is proved.

Claim (ii) follows immediately from Theorem A.18 (b) in [13].

To prove (iii), we first observe that $H(t, \gamma(t-1), W) \geq \varphi$ for all $\varphi \in \Phi(t, \gamma(t-1), W)$, hence $E_s\left[H(t, \gamma(t-1), W)\right] \geq E_s[\varphi]$ for all $\varphi \in \Phi(t, \gamma(t-1), W)$, and this implies

$$E_s\left[H(t, \gamma(t-1), W)\right] \geq \operatorname*{ess\,sup}_{\varphi \in \Phi(t, \gamma(t-1), W)} E_s[\varphi]. \tag{16}$$

Moreover, from (ii), we have that there exists an increasing sequence $(\varphi_n)_{n\geq 1} \in \Phi(t, \gamma(t-1), W)$ which converges to $H(t, \gamma(t-1), W)$ P-a.s. and, of course,

$$\operatorname*{ess\,sup}_{\varphi \in \Phi(t, \gamma(t-1), W)} E_s[\varphi] \geq \sup_n E_s\left[\varphi_n\right].$$

If H is integrable, $\sup_n E[\varphi_n] \leq E[H(t, \gamma(t-1), W)] < +\infty$. Therefore, we can apply Beppo Levi's theorem (monotone convergence) for conditional expectations to obtain that

$$\sup_n E_s[\varphi_n] = \lim_n E_s[\varphi_n] = E_s[H(t, \gamma(t-1), W)] \qquad P\text{-a.s.}$$

From this, it follows that inequality (16) is in fact an equality. ■

Proof of Proposition 1.

1. To prove (6), we show that for $t = 0, \ldots, T-1$:

$$H(t, \gamma^*(t-1), W^*(t)) = u_t(\gamma^*(t)) + \sum_{s=t+1}^{T} E_t[u_s(\gamma^*(s))]. \tag{17}$$

We proceed by induction on t. At the initial stage, $t = 0$, the claim holds trivially, since $H(0, e_0) = U(c^*)$. Assume then that (17) holds true for $0 \leq s \leq t-1$ and consider $H(t, \gamma^*(t-1), W^*(t))$. If we look at (5), we immediately see that the pair $(c^*(s), \theta^*(s))_{s=t}^{T}$ satisfies the budget constraint, that is $\varphi^* = u_t(\gamma^*(t)) + \sum_{s=t+1}^{T} E_t[u_s(\gamma^*(s))]$ belongs to $\Phi(t, \gamma^*(t-1), W^*(t))$ (where $\Phi(t, \gamma(t-1), W)$ is defined in the proposition above). Hence we deduce:

$$H(t, \gamma^*(t-1), W^*(t)) \geq u_t(\gamma^*(t)) + \sum_{s=t+1}^{T} E_t[u_s(\gamma^*(s))]$$

Assume by contradiction that the strict inequality holds with positive probability; this means that there exists $\varphi \in \Phi(t, \gamma^*(t-1), W^*(t))$ such that $P(\varphi > \varphi^*) > 0$. Since $\Phi(t, \gamma^*(t-1), W^*(t))$ is directed upwards (Proposition 5 (i)), there exists $\tilde{\varphi}$ such that $\tilde{\varphi} \geq \max(\varphi, \varphi^*)$, namely there exists a pair $(\tilde{c}(s), \tilde{\theta}(s))_{s=t}^{T}$ which satisfies the budget constraint and

$$\sum_{s=t}^{T} E_t[u_s(\gamma^*(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))] \geq \sum_{s=t}^{T} E_t[u_s(\gamma^*(s))] \tag{18}$$

and

$$P\left(\sum_{s=t}^{T} E_t[u_s(\gamma^*(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))] > \sum_{s=t}^{T} E_t[u_s(\gamma^*(s))]\right) > 0.$$

If we take the conditional expectation with respect to $\mathcal{F}_{t-1}$ of both sides in (18), we obtain that:

$$\sum_{s=t}^{T} E_{t-1}[u_s(\gamma^*(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))] \geq \sum_{s=t}^{T} E_{t-1}[u_s(\gamma^*(s))]$$

and

$$P\left(\sum_{s=t}^{T} E_{t-1}[u_s(\gamma^*(t-1), \tilde{c}(t), \ldots, \tilde{c}(s))] > \sum_{s=t}^{T} E_{t-1}[u_s(\gamma^*(s))]\right) > 0.$$

Adding to both sides the quantity $u_{t-1}(\gamma^*(t-1))$ and recalling the inductive hypothesis, we obtain:

$$u_{t-1}(\gamma^*(t-1))+\sum_{s=t}^{T} E_{t-1}\left[u_s(\gamma^*(t-1),\tilde{c}(t),\ldots,\tilde{c}(s))\right] \geq H(t-1,\gamma^*(t-2),W^*(t-1)) \quad (19)$$

and

$$P\left(u_{t-1}(\gamma^*(t-1))+\sum_{s=t}^{T} E_{t-1}\left[u_s(\gamma^*(t-1),\tilde{c}(t),\ldots,\tilde{c}(s))\right] > H(t-1,\gamma^*(t-2),W^*(t-1))\right) > 0. \quad (20)$$

Since $\left(c^*(t-1),(\tilde{c}(s))_{s=t}^T\right),\left(\theta^*(t-1),(\tilde{\theta}(s))_{s=t}^T\right)$ satisfies the budget constraints of (5) at time $t-1$, (19) and (20) gives a contradiction. Hence the claim is proved.

2. To show that inequality (7) holds, we denote by (c^*,θ^*) the optimal consumption-portfolio choice of the agent whose optimal wealth is W^*, and observe that given any self-financing dynamic investment strategy $\theta^{SF}\in\Theta$ we have that

$$H(t,\gamma^*(t-1),W^*(t)+\varepsilon V_{\theta^{SF}}(t)) \geq \operatorname*{ess\,sup}_{(c,\theta)\in\mathcal{C}\times\Theta} u_t(\gamma^*(t-1),c(t))+\sum_{s=t+1}^{T} E_t\left[u_s(\gamma^*(t-1),c(t),\ldots,c(s))\right]$$

$$\text{such that}\begin{cases} c(t)+V_\theta(t)\leq W^*(t)+\varepsilon V_{\theta^{SF}}(t)\\ c(s)\leq x_\theta(s), \qquad s=t+1,\ldots,T\\ c(t)=c^*(t)\\ \theta(t)=\theta^*(t)+\varepsilon\,\theta^{SF}(t)\end{cases} \quad (21)$$

where the inequality is a consequence of the fact that the supremum that defines the random variable $H(t,\gamma^*(t-1),W^*(t)+\varepsilon V_{\theta^{SF}}(t))$ in (5) is taken on a larger set than the feasible set in (21). Substitute now the last two constraints in the first two and into the objective function in (21). The first constraint is always satisfied since it reduces to $c^*(t)+V_{\theta^*}(t)+\varepsilon V_{\theta^{SF}}(t)\leq W^*(t)+\varepsilon V_{\theta^{SF}}(t)$, that is $c^*(t)+V_{\theta^*}(t)\leq W^*(t)$ which holds since (c^*,θ^*) is optimal and hence budget-feasible. Recalling the definition of cash-flow process in (1), the second constraint for $s=t+1$ becomes $c(t+1)\leq x_\theta(t+1)=W^*(t+1)+\varepsilon V_{\theta^{SF}}(t+1)-V_\theta(t+1)$, that is $c(t+1)+V_\theta(t+1)\leq W^*(t+1)+\varepsilon V_{\theta^{SF}}(t+1)$. Therefore problem (21) can be equivalently rewritten as follows:

$$
\begin{aligned}
H(t, \gamma^*(t-1), W^*(t) + \varepsilon V_{\theta^{SF}}(t)) \geq u_t(\gamma^*(t)) +& \\
+ \underset{(c,\theta)\in\mathcal{C}\times\Theta}{\text{ess sup}} \sum_{s=t+1}^{T} E_t \left[u_s(\gamma^*(t-1), c(t), \dots, c(s)) \right]& \\
s.t. \begin{cases} c(t+1) + V_\theta(t+1) \leq W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1) \\ c(s) \leq x_\theta(s), \qquad s = t+2, \dots, T \end{cases}&
\end{aligned}
$$

By Proposition 5 (iii), the essential supremum in the above inequality coincides with $E_t \left[H(t+1, \gamma^*(t), W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1)) \right]$ and this yields (7). ■

Proof of Proposition 2.

Denote by $\mathcal{D}(t, W^*(t))$ the Fréchet-differential of H at the optimal wealth. We exploit Equation (6) together with inequality (7) to show first that $\left\{ \mathcal{D}(t, W^*(t)) \left(V_{\theta^{SF}}(t) \right) \right\}_{0 \leq t \leq T}$ is a martingale for every self-financing admissible strategy θ^{SF}, that is,

$$
\mathcal{D}\left(t, W^*(t)\right) \left(V_{\theta^{SF}}(t) \right) = E_t \left[\mathcal{D}\left(t+1, W^*(t+1)\right) \left(V_{\theta^{SF}}(t+1) \right) \right] \tag{22}
$$

for all $t = 0, \dots, T-1$. For sake of notation, we omit the dependence of the value function at time t on the stream of optimal past consumptions $\gamma^*(t-1)$. Taking ε sufficiently small so that $H\left(t+1, W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1)\right) - H\left(t+1, W^*(t+1)\right)$ is integrable and subtracting (6) from (7) we obtain

$$
\begin{aligned}
&H\left(t, W^*(t) + \varepsilon V_{\theta^{SF}}(t)\right) - H\left(t, W^*(t)\right) \geq \\
&\geq E_t \left[H\left(t+1, W^*(t+1) + \varepsilon V_{\theta^{SF}}(t+1)\right) - H\left(t+1, W^*(t+1)\right) \right]
\end{aligned} \tag{23}
$$

Let (ε_n) be a decreasing sequence in $\Re$ such that $\varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$. Consider inequality (23) for $\varepsilon = \varepsilon_n$ and divide both sides for ε_n. For sake of notation, we denote

$$
\begin{aligned}
W_n^{SF}(t) &= W^*(t) + \varepsilon_n V_{\theta^{SF}}(t) & \qquad W_n^{SF}(t+1) &= W^*(t+1) + \varepsilon_n V_{\theta^{SF}}(t+1) \\
\mathcal{D}_t^* &= \mathcal{D}\left(t, W^*(t)\right) & \qquad \mathcal{D}_{t+1}^* &= \mathcal{D}\left(t+1, W^*(t+1)\right)
\end{aligned}
$$

then:

$$
\frac{H\left(t, W_n^{SF}(t)\right) - H\left(t, W^*(t)\right)}{\varepsilon_n} \geq \frac{E_t \left[H\left(t+1, W_n^{SF}(t+1)\right) - H\left(t+1, W^*(t+1)\right) \right]}{\varepsilon_n} \tag{24}
$$

Let $\varepsilon_n \to 0^+$. By definition of Fréchet-differentiability,

$$
\lim_{\varepsilon_n \to 0} E \left[\left| \frac{H\left(t, W_n^{SF}(t)\right) - H\left(t, W^*(t)\right)}{\varepsilon_n} - \mathcal{D}_t^* \left(V_{\theta^{SF}}(t) \right) \right| \right] = 0
$$

Hence, there exists a subsequence (which, for sake of simplicity, we still denote by n) such that $\frac{H\left(t,W_n^{SF}(t)\right)-H(t,W^*(t))}{\varepsilon_n}$ converges to $\mathcal{D}_t^*\left(V_{\theta^{SF}}(t)\right)$ P-almost surely.

Moreover, we also have that

$$E\left[E_t\left[\left|\frac{H\left(t+1,W_n^{SF}(t+1)\right)-H\left(t+1,W^*(t+1)\right)}{\varepsilon_n}-\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right|\right]\right]$$
$$=E\left[\left|\frac{H\left(t+1,W_n^{SF}(t+1)\right)-H\left(t+1,W^*(t+1)\right)}{\varepsilon_n}-\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right|\right]$$

which converges to 0 thanks to the Fréchet-differentiability of $H(t+1,W^*(t+1))$. Hence, possibly passing to a subsequence, we have that

$$E_t\left[\left|\frac{H\left(t+1,W_n^{SF}(t+1)\right)-H\left(t+1,W^*(t+1)\right)}{\varepsilon_n}-\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right|\right]$$

converges to 0 P-a.s. Applying Jensen's inequality for conditional expectation to the convex function $x\mapsto|x|$, we obtain that

$$\left|E_t\left[\frac{H\left(t+1,W_n^{SF}(t+1)\right)-H\left(t+1,W^*(t+1)\right)}{\varepsilon_n}-\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right]\right|$$
$$\leq E_t\left[\left|\frac{H\left(t+1,W_n^{SF}(t+1)\right)-H\left(t+1,W^*(t+1)\right)}{\varepsilon_n}-\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right|\right]$$

which implies that $\frac{E_t\left[H\left(t+1,W_n^{SF}(t+1)\right)-H(t+1,W^*(t+1))\right]}{\varepsilon_n}$ converges P-a.s. to the random variable $E_t\left[\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right]$. From (24) we deduce therefore the inequality

$$\mathcal{D}_t^*\left(V_{\theta^{SF}}(t)\right)\geq E_t\left[\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\right].$$

If we replace the positive sequence ε_n with $(-\varepsilon_n)$, inequality (24) is reversed, so we obtain (22).

We show now that (22) implies (10). We first observe that for any $A\in\mathcal{F}_t$

$$\mathcal{D}_t^*\left(Y\right)\cdot\mathbb{I}_A=\mathcal{D}_t^*\left(Y\ \mathbb{I}_A\right)$$

for all $Y\in L^p(\Omega,\mathcal{F}_t,P)$ and for all $t=1,\ldots,T$. Hence, recalling the definition of marginal utility of wealth, we have for any $A\in\mathcal{F}_t$ and for all $t=0,\ldots,T-1$

$$\begin{aligned}
E\left[H_W^*\left(t\right)\cdot\left(V_{\theta^{SF}}(t)\mathbb{I}_A\right)\right] &= E\left[\mathcal{D}_t^*\left(V_{\theta^{SF}}(t)\ \mathbb{I}_A\right)\right]= \\
&= E\left[\mathcal{D}_t^*\left(V_{\theta^{SF}}(t)\right)\cdot\mathbb{I}_A\right]= \\
&= E\left[\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\right)\cdot\mathbb{I}_A\right]= \\
&= E\left[\mathcal{D}_{t+1}^*\left(V_{\theta^{SF}}(t+1)\mathbb{I}_A\right)\right]= \\
&= E\left[H_W^*\left(t+1\right)\cdot V_{\theta^{SF}}(t+1)\mathbb{I}_A\right],
\end{aligned}$$

where the third equality is implied by the martingality of $\left\{\mathcal{D}_t^*\left(V_{\theta^{SF}}(t)\right)\right\}_{t=0}^T$. The chain of equalities shows that $\left\{H_W(t, W^*(t)) \cdot V_{\theta^{SF}}(t)\right\}_{t=0}^T$ is a martingale. ■

Proof of Theorem 2.

We first observe that if the market is not viable, then both Ψ and $\mathcal{A}^*$ are empty, hence the claim is trivially satisfied.

(If part) Assume that there exists an optimizing agent in $\mathcal{A}^*$ and let $\psi : X \to \Re$ be defined by equation (11).

Note that

$$\psi(x) = \frac{1}{H_W^*(0)} \sum_{t=1}^T E\left[\mathcal{D}(t, W^*(t))(x(t))\right];$$

this means that ψ is a continuous linear functional on X by definition. Therefore, we need only to prove that such ψ is strictly positive on X^+ and $\psi(m) = V_\theta(0)$ for all $m \in M$ and for all $\theta \in \Theta$ such that $m(t) = x_\theta(t)$, $t = 1, \ldots, T$.

To prove that ψ is strictly positive on X^+ is equivalent to show that $P\left(H_W^*(t) > 0\right) = 1$ for all $t \leq T$. This fact is established by backward induction, exploiting Proposition 2.

We first recall that by assumption $P\left(H_W^*(T) > 0\right) = 1$. Fix then $t < T$ and suppose that $P\left(H_W^*(t+1) > 0\right) = 1$. Given any $\theta^{BN} \in \Theta^{BN}$ (which is non-empty thanks to Condition 1), equation (10) can be rewritten as

$$H_W^*(t) = E_t\left[H_W^*(t+1) \frac{V_{\theta^{BN}}(t+1)}{V_{\theta^{BN}}(t)}\right] \tag{25}$$

Since $H_W^*(t+1)$, $V_{\theta^{BN}}(t+1)$, and $V_{\theta^{BN}}(t)$ are almost surely strictly positive, equation (25) implies $P[H_W^*(t) > 0] = 1$.

To conclude the *if part*, we prove that (10) implies

$$\frac{1}{H_W^*(0)} E\left[\sum_{t=1}^T H_W^*(t)\, x_\theta(t)\right] = V_\theta(0), \quad \textit{for all } \theta \in \Theta \tag{26}$$

which in turns implies that ψ as defined in (11) satisfies $\psi(m) = V_\theta(0)$ for all $m \in M$ and for all $\theta \in \Theta$ such that $m(t) = x_\theta(t)$, $t = 1, \ldots, T$. To do so, given any $\theta \in \Theta$ and $\theta^{BN} \in \Theta^{BN}$ define the admissible dynamic investment strategy $\widehat{\theta}$ as follows:

$$\widehat{\theta}(t) = \theta(t) + \left[\sum_{s=0}^t \frac{x_\theta(s)}{V_{\theta^{BN}}(s)}\right] \theta^{BN}(t), \quad t = 0, \ldots, T-1. \tag{27}$$

Observe that the strategy $\widehat{\theta}$ is self-financing since it consists in 'buying' the strategy θ at time $t = 0$, re-investing at every intermediate date the cash-flow $x_\theta(t)$ in θ^{BN}, hence it satisfies (10) :

$$H_W^*(t)\, V_{\widehat{\theta}}(t) = E_t\left[H_W^*(t+1)\, V_{\widehat{\theta}}(t+1)\right], \quad t = 0, \ldots, T-1$$

Substituting (27) into this last equation for $t = 0, 1, \ldots, T-2$ we obtain therefore

$$H_W^*(t)\left(V_\theta(t) + \left[\sum_{s=0}^{t} \frac{x_\theta(s)}{V_{\theta^{BN}}(s)}\right] V_{\theta^{BN}}(t)\right) =$$

$$= E_t\left[H_W^*(t+1)\left(V_\theta(t+1) + \left[\sum_{s=0}^{t+1} \frac{x_\theta(s)}{V_{\theta^{BN}}(s)}\right] V_{\theta^{BN}}(t+1)\right)\right]$$

Recalling that $\sum_{s=0}^{t} \frac{x_\theta(s)}{V_{\theta^{BN}}(s)}$ is $\mathcal{F}_t-$measurable, and that θ^{BN} satisfies (10), our last equation reduces to

$$H_W^*(t)\, V_\theta(t) = E_t\left[H_W^*(t+1)\left(V_\theta(t+1) + x_\theta(t+1)\right)\right], \quad t \leq T-2. \tag{28}$$

while for $t = T-1$ we have that

$$H_W^*(T-1)\, V_\theta(T-1) = E_{T-1}\left[H_W^*(T)\, x_\theta(T)\right]$$

Exploiting the law of iterated expectation in a backward iterative fashion, these two last equations lead to

$$H_W^*(0)\, V_\theta(0) = E\left[\sum_{t=1}^{T} H_W^*(t)\, x_\theta(t)\right]$$

which shows that (26) holds for all $\theta \in \Theta$.

(Only if part). Given any linear pricing functional $\psi \in \Psi$ with Riesz representation

$$\psi(x) = \sum_{t=1}^{T} E[\rho(t)x(t)]$$

we recall (see Duffie, 2001) that the sequence $\rho(0) = 1$, $\{\rho(t)\}_{t=1}^{T}$ satisfies for all $\theta \in \Theta$

$$\rho(t)V_\theta(t) = E_t[\rho(t+1)\left(V_\theta(t+1) + x_\theta(t+1)\right)], \quad t = 0, \ldots, T-2 \tag{29}$$

and

$$\rho(T-1)V_\theta(T-1) = E_{T-1}[\rho(T)x_\theta(T)].$$

Consider then an agent with $e_0 = 0$ and preferences given by

$$U(c) = c(0) + \sum_{t=1}^{T} E[\rho(t)c(t)]$$

As shown in the proof of Theorem 1, for such an agent $U(c) \leq 0$ for any budget-feasible consumption sequence, while $U(c) = 0$ for any consumption sequence that satisfies his budget set with equality. It is readily seen that for this agent we have, independently of the past consumptions

$$H(t, W(t)) = \begin{cases} W(0), & t = 0 \\ \rho(t)W(t), & t = 1, \dots, T \end{cases}$$

and $W^*(t) = 0$ for all t. The marginal utility of wealth are

$$H_W(t, W^*(t)) = \begin{cases} 1, & t = 0 \\ \rho(t), & t = 1, \dots, T \end{cases}$$

well defined and satisfy (8), hence our proof is complete. ■

Proof of Corollary 1.

If $H_W^*(t)$ is the marginal utility of optimal intertemporal wealth for some optimizing agent in $\mathcal{A}^*$, then $P(H_W^*(t) > 0) = 1$ and it satisfies equation (28), for every admissible strategy θ as we showed in the proof of *if part* of Theorem 2. Let θ be the strategy consisting in buying one unit of the asset S_j at time t and selling it at time $t+1$. For such θ, equation (28) implies

$$S_j(t) = \frac{1}{H_W^*(t)} E_t \left[H_W^*(t+1) \left(S_j(t+1) + d_j(t+1) \right) \right].$$

Since j is arbitrary, this shows that $H_W^*(t)$ is a state price density. The converse implication can be proved similarly to the *only if* part in Theorem 2, assuming that (29) is satisfied with π in place of ρ. ■

Proof of Proposition 3.

As in the Proof of Proposition 2, for sake of notation, we omit the dependence of the function $H(t, \cdot)$ and $h(t, \cdot)$ on the stream of optimal past consumptions $\gamma^*(t-1)$.

Let (ε_n) a sequence of real numbers going to 0 as n tends to ∞ and $Y \in L^p(\Omega, \mathcal{F}_t, P)$. Since H is Fréchet differentiable at the optimum,

$$\frac{H(t, W^*(t) + \varepsilon_n Y) - H(t, W^*(t))}{\varepsilon_n} = \frac{h(t, W^*(t) + \varepsilon_n Y, z(t)) - h(t, W^*(t), z(t))}{\varepsilon_n}$$

converges in L^1 to $\mathcal{D}(t, W^*(t))(Y)$, as $n \to \infty$. So we can subtract a subsequence such that the limit is $P-$a.s.

At the same time, due to the differentiability of h, we have that for all ω

$$\lim_{n\to\infty}\frac{h(t,W^*(t)(\omega)+\varepsilon_n Y(\omega),z(t)(\omega))-h(t,W^*(t)(\omega),z(t)(\omega))}{\varepsilon_n}=h_w^*(t,z(t)(\omega))\cdot Y(\omega).$$

Because the limit must be unique, we can claim that

$$\mathcal{D}(t,W^*(t))(Y)=h_w^*(t,z(t))\cdot Y \tag{30}$$

and, in particular, for $Y=1$,

$$\mathcal{D}(t,W^*(t))(1)=h_w^*(t,z(t))$$

This implies that $h_w^*(t,z(t))\in L^q(\Omega,\mathcal{F}_t,P)$. Moreover, from equation (30), we deduce that

$$E\left[\mathcal{D}(t,W^*(t))(Y)\right]=E\left[h_w^*(t,z(t))\cdot Y\right]$$

for all $Y\in L^p(\Omega,\mathcal{F}_t,P)$; this, together with equation (9) and the uniqueness of Riesz representation implies that $h_w^*(t,z(t))=H_W^*(t)$. ■

Proof of Proposition 4.

For sake of notation, in the proof of this proposition, we omit the dependence on the state variables $z(t)$. Let (ε_n) a sequence of positive real numbers decreasing to 0 as n tends to ∞ and $Y\in L^p(\Omega,\mathcal{F}_T,P)$ and denote by $\mathcal{H}_n$ the ratio:

$$\begin{aligned}\mathcal{H}_n &= \frac{\tilde{u}_T(\gamma^*(T-1),W^*(T)+\varepsilon_n Y)-\tilde{u}_T(\gamma^*(T-1),W^*(T))}{\varepsilon_n}\\ &= \frac{h(T,\gamma^*(T-1),W^*(T)+\varepsilon_n Y)-h(T,\gamma^*(T-1),W^*(T))}{\varepsilon_n}.\end{aligned}$$

We shall prove that for all $t\le T$ the sequence $E_t\left[\mathcal{H}_n\right]$ converges in L^1 and almost surely to $E_t\left[h_w^*(T)Y\right]$. For $t=T$, the differentiability of $\tilde{u}_T(\gamma,\cdot)=h(T,\gamma,\cdot)$ implies that $\mathcal{H}_n$ converges P-a.s. to $h_w^*(T)\,Y$.

To obtain L^1-convergence, we write:

$$\mathcal{H}_n=\mathcal{H}_n\mathbb{I}_{\{Y<0\}}+\mathcal{H}_n\mathbb{I}_{\{Y>0\}}.$$

Since $\varepsilon_n\ge\varepsilon_{n+1}>0$ (hence $\varepsilon_n Y\mathbb{I}_{\{Y<0\}}\le\varepsilon_{n+1}Y\mathbb{I}_{\{Y<0\}}$) and $h(T,\gamma^*(T-1),\cdot)=\tilde{u}_T(\gamma^*(T-1),\cdot)$ is increasing, the following inequalities hold:

$$\mathcal{H}_n\mathbb{I}_{\{Y<0\}}\le\frac{h(T,\gamma^*(T-1),W^*(T)+\varepsilon_{n+1}Y)-h(T,\gamma^*(T-1),W^*(T))}{\varepsilon_n}\mathbb{I}_{\{Y<0\}}\le\mathcal{H}_{n+1}\mathbb{I}_{\{Y<0\}},$$

namely the sequence $\mathcal{H}_n \mathbb{I}_{\{Y<0\}}$ is an increasing sequence. On the other hand, since $\varepsilon_n Y \mathbb{I}_{\{Y>0\}} > \varepsilon_{n+1} Y \mathbb{I}_{\{Y>0\}}$, concavity implies that

$$\begin{aligned} & \frac{\tilde{u}_T(\gamma^*(T-1), W^*(T) + \varepsilon_n Y) - \tilde{u}_T(\gamma^*(T-1), W^*(T))}{\varepsilon_n Y} \mathbb{I}_{\{Y>0\}} \\ \leq \quad & \frac{\tilde{u}_T(\gamma^*(T-1), W^*(T) + \varepsilon_{n+1} Y) - \tilde{u}_T(\gamma^*(T-1), W^*(T))}{\varepsilon_{n+1} Y} \mathbb{I}_{\{Y>0\}} \end{aligned}$$

which means that also $\mathcal{H}_n \mathbb{I}_{\{Y>0\}}$ is increasing. So, $\mathcal{H}_n$ is an increasing sequence which converges almost surely to the integrable random variable $h^*_w(T)\,Y$ and $\sup_n E[\mathcal{H}_n] \leq E[h^*_w(T)\,Y]$. Then, the monotone convergence theorem implies that $\mathcal{H}_n$ converges in L^1 to $h^*_w(T)\,Y$. Moreover, the monotone convergence theorem for conditional expectation implies that $E_t\,[\mathcal{H}_n]$ is an increasing sequence which converges almost surely and in L^1 to $E_t\,[h^*_w(T)\,Y]$.

In a similar way, one can prove that the sequence

$$E_t \left[\frac{h(T, \gamma^*(T-1), W^*(T) + \varepsilon_n Y) - h(T, \gamma^*(T-1), W^*(T))}{\varepsilon_n} \right]$$

converges almost surely and in L^1 to $E_t\,[h^*_w(T)\,Y]$ also when (ε_n) is a sequence of negative real numbers increasing to 0, for all $t \leq T$ and $Y \in L^p(\Omega, \mathcal{F}_T, P)$.

Let now $V_{\theta^{SF}}$ be a self-financing portfolio and ε sufficiently small so that

$$H(t, \gamma^*(T-1), W^*(t) + \varepsilon\, V_{\theta^{SF}}(t)) = h(t, \gamma^*(T-1), W^*(t) + \varepsilon\, V_{\theta^{SF}}(t))$$

is integrable. If we apply recursively (6) and (7), we obtain:

$$h(t, \gamma^*(T-1), W^*(t)) = E_t \left[\sum_{s=t}^{T-1} u_s(\gamma^*(s)) \right] + E_t\left[h(T, \gamma^*(T-1), W^*(T))\right]$$

$$h(t, W^*(t) + \varepsilon V_{\theta^{SF}}(t)) \geq E_t \left[\sum_{s=t}^{T-1} u_s(\gamma^*(s)) \right] + E_t\left[h(T, \gamma^*(T-1), W^*(T) + \varepsilon\, V_{\theta^{SF}}(T))\right]$$

Subtracting the first equation from the second inequality, we get

$$\begin{aligned} & h(t, \gamma^*(t-1), W^*(t) + \varepsilon V_{\theta^{SF}}(t)) - h(t, \gamma^*(t-1), W^*(t)) \\ \geq \quad & E_t\left[h(T, \gamma^*(T-1), W^*(T) + \varepsilon\, V_{\theta^{SF}}(T)) - h(T, \gamma^*(T-1), W^*(T))\right]. \end{aligned} \tag{31}$$

Take a decreasing sequence (ε_n) in $\Re$ such that $\varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$, then write inequality (31), with $\varepsilon = \varepsilon_n$ and divide both sides of the inequality for ε_n.

$$\begin{aligned} & \frac{h(t, \gamma^*(t-1), W^*(t) + \varepsilon V_{\theta^{SF}}(t)) - h(t, \gamma^*(t-1), W^*(t))}{\varepsilon_n} \geq \\ \geq & E_t \left[\frac{h(T, \gamma^*(T-1) W^*(T) + \varepsilon\, V_{\theta^{SF}}(T)) - h(T, \gamma^*(T-1) W^*(T))}{\varepsilon_n} \right] \end{aligned}$$

Let $\varepsilon_n \to 0^+$: since h is differentiable, the left-hand side of the above inequality converges to $h_w^*(t)V_{\theta^{SF}}(t)$ while the right-hand side converges almost surely to $E_t\left[h_w^*(T)V_{\theta^{SF}}(T)\right]$. This implies that:

$$h_w^*(t)V_{\theta^{SF}}(t) \geq E_t\left[h_w^*(T)V_{\theta^{SF}}(T)\ \right]$$

Replacing the sequence (ε_n) with $(-\varepsilon_n)$, we obtain the reverse inequality. It follows that

$$h_w^*(t)V_{\theta^{SF}}(t) = E_t\left[h_w^*(T)V_{\theta^{SF}}(T)\ \right] \tag{32}$$

so we can conclude that $h_w^*(t)V_{\theta^{SF}}(t)$ is integrable and $\left(h_w^*(t)V_{\theta^{SF}}(t)\right)_{t=0}^T$ is a martingale for any self-financing portfolio $V_{\theta^{SF}}$.

Finally, given any $\theta^{BN} \in \Theta^{BN}$ (which is non-empty thanks to Condition 1), equation (32) can be rewritten as

$$h_w^*(t) = E_t\left[h_w^*(T)\frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)}\right].$$

Jensen's inequality for conditional expectations applied to the convex function $x \to |x|^q$ implies that

$$|h_w^*(t)|^q = \left|E_t\left[h_w^*(T)\frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)}\right]\right|^q \leq E_t\left[\left|h_w^*(T)\frac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)}\right|^q\right].$$

Taking the expectation of both sides, and exploiting the fact that $h_w^*(T) \in L^q(\Omega, \mathcal{F}_T, P)$ and $\dfrac{V_{\theta^{BN}}(T)}{V_{\theta^{BN}}(t)} \in L^\infty(\Omega, \mathcal{F}_T, P)$, we obtain that $h_w^*(t) \in L^q(\Omega, \mathcal{F}_t, P)$. ■

Detailed Computations of Example 4.3. For sake of simplicity, we denote dependence on time with subscripts, for instance $c_t = c(t)$. Given any initial endowment e_0, the optimal consumption-portfolio problem for our agent is

$$\sup_{(c,\theta)} U(c) = \sup_{(c,\theta)} E\left[u_0(c_0) + u_1(c_1) + u_2(c_2)\right]$$

$$s.t. \begin{cases} c_0 + \theta_0 \leq e_0 \\ c_1 + \theta_1 S_1 \leq \theta_0 S_1 \\ c_2 \leq \theta_1 S_2 \end{cases}$$

We first compute the maximum remaining utilities of wealth. For $t = T = 2$, we have

$$H(2, W_2) = u_2(W_2) = 3\beta W_2 z_2$$

namely, $H(2,W_2) = h(2,W_2,z_2)$, where $h(2,w,z) = 3\beta wz$.For $t=1$, the maximum remaining utility of wealth, given the level W_1 is given by:

$$H(1,W_1) \equiv \underset{(c,\theta)\in\widehat{\mathcal{C}}\times\Theta}{\text{ess sup}}\, u_1(c_1) + \ E_1\left[u_2(c(2))\right]$$
$$s.t. \begin{cases} c_1 + \theta_1 S_1 \leq W_1 \\ c_2 \leq \theta_1 S_2 \end{cases}$$

Clearly the time 2 constraint is binding at the optimum, hence we can write:

$$H(1,W_1) \equiv \underset{c_1,\theta_1}{\text{ess sup}} u_1(c_1) + \ E_1\left[u_2(\theta_1 S_2)\right]$$
$$s.t.\ c_1 + \theta_1 S_1 \leq W_1.$$

where $u_1(c_1) +\ E_1\left[u_2(\theta_1 S_2)\right] = v(c_1)\cdot z_1 + \overline{v}(c_1)\left(1-z_1\right) + E_1\left[3\beta\theta_1 S_2 z_2\right]$.Since $\theta_1$has the form $\theta_1 = \vartheta \cdot z_1 + \overline{\vartheta}\cdot(1-z_1)$ (where $\vartheta, \overline{\vartheta} \in \mathbb{R}$), we can explicitly compute $E_1\left[u_2(\theta_1 S_2)\right]$

$$\begin{aligned}
E_1\left[3\beta\theta_1 S_2 z_2\right] &= E_1\left[3\beta\left(\vartheta\cdot z_1 + \overline{\vartheta}\cdot(1-z_1)\right)R(z_2)^2\right] \\
&= 3\beta R\left(\vartheta E_1\left[z_1(z_2)^2\right] + \overline{\vartheta}E_1\left[(1-z_1)((z_2)^2\right]\right) \\
&= 3\beta R\left(2\vartheta z_1\int_0^{1/2} x^2dx + 2\overline{\vartheta}\left(1-z_1\right)\int_{1/2}^1 x^2dx\right) \\
&= \frac{\beta R}{4}\left(\vartheta z_1 + 7\overline{\vartheta}(1-z_1)\right) \\
&= 2\delta\left(\vartheta z_1 + 7\overline{\vartheta}(1-z_1)\right)
\end{aligned}$$

where we have defined $\delta = \frac{\beta R}{8}$.Note that we can also write

$$E_1\left[u_2(\theta_1 S_2)\right] = 2\delta\left(\frac{\vartheta S_1}{s}\cdot z_1 + \frac{7\overline{\vartheta}S_1}{\overline{s}}(1-z_1)\right).$$

Then, our problem becomes

$$H(1,W_1) \equiv \underset{c_1,\theta_1}{\text{ess sup}} F_1(c_1,\theta_1,z_1)$$
$$s.t.\ c_1 + \theta_1 S_1 \leq W_1.$$

where

$$\begin{aligned}
F(c_1,\theta_1,z_1) &= v(c_1)\cdot z_1 + \overline{v}(c_1)\cdot(1-z_1) + 2\delta\left(\frac{\vartheta S_1}{s}\cdot z_1 + \frac{7\overline{\vartheta}S_1}{\overline{s}}(1-z_1)\right) \\
&= \left(v(c_1) + 2\delta\frac{\vartheta S_1}{s}\right)\cdot z_1 + \left(\overline{v}(c_1) + 2\delta\frac{7\overline{\vartheta}S_1}{\overline{s}}\right)\cdot(1-z_1)
\end{aligned}$$

The consumption at time 1 takes the form $c_1 = \gamma \cdot z_1 + \overline{\gamma} \cdot (1 - z_1)$ where $\gamma, \overline{\gamma} \in \mathbb{R}$. We analyze separately the two addends in $F(c_1, \theta_1, z_1)$ In particular, if $\gamma \leq 1$, then

$$\begin{aligned}
\left(v(c_1) + 2\delta\frac{\vartheta S_1}{s}\right) \cdot z_1 &= \left(c_1 + 2\delta\frac{\vartheta S_1}{s}\right) \cdot z_1 \\
&= \left[\left(1 - \frac{2\delta}{s}\right) c_1 + \frac{2\delta}{s} \left(c_1 + \vartheta S_1\right)\right] \cdot z_1 \\
&\leq \left[1 - \frac{2\delta}{s} + \frac{2\delta}{s} W_1\right] \cdot z_1
\end{aligned}$$

where the last inequality follows from the assumption that $\frac{2\delta}{s} = \frac{\beta R}{4s} < 1$. On the other hand, for $\gamma > 1$,

$$\begin{aligned}
\left(v(c_1) + 2\delta\frac{\vartheta S_1}{s}\right) \cdot z_1 &= \left(1 - c_1 + 2\delta\frac{\vartheta S_1}{s}\right) \cdot z_1 \\
&= \left[1 - \left(1 + \frac{2\delta}{s}\right) c_1 + \frac{2\delta}{s} \left(c_1 + \vartheta S_1\right)\right] \cdot z_1 \\
&\leq \left[-\frac{2\delta}{s} + \frac{2\delta}{s} W_1\right] \cdot z_1 \\
&< \left[1 - \frac{2\delta}{s} + \frac{2\delta}{s} W_1\right] \cdot z_1.
\end{aligned}$$

As for the second part of $F(c_1, \theta_1, z_1)$,if $\overline{\gamma} \leq 1$,then

$$\begin{aligned}
\left(\overline{v}(c_1) + 2\delta\frac{7\overline{\vartheta} S_1}{\overline{s}}\right) \cdot (1 - z_1) &= \left((c_1)^3 + \delta\frac{14\overline{\vartheta} S_1}{\overline{s}}\right) \cdot (1 - z_1) \\
&= \left[(c_1)^3 - \frac{14\delta}{\overline{s}} c_1 + \frac{14\delta}{\overline{s}} \left(c_1 + \vartheta S_1\right)\right] \cdot (1 - z_1) \\
&\leq \left[1 - \frac{14\delta}{\overline{s}} + \frac{14\delta}{\overline{s}} W_1\right] \cdot (1 - z_1)
\end{aligned}$$

where the last inequality follows from the assumption that $\frac{14\delta}{\overline{s}} = \frac{14\beta R}{8\overline{s}} < \frac{1}{2}$ (hence the function $(c_1)^3 - \frac{14\delta}{\overline{s}} c_1$ takes its maximum value in $c_1 = 1$). Finally, in the case $\overline{\gamma} > 1$,

$$\begin{aligned}
\left(\overline{v}(c_1) + 2\delta\frac{7\overline{\vartheta} S_1}{\overline{s}}\right) \cdot (1 - z_1) &= \left(1 - (c_1)^2 + \delta\frac{14\overline{\vartheta} S_1}{\overline{s}}\right) \cdot (1 - z_1) \\
&= \left[1 - \left((c_1)^2 + \frac{14\delta}{\overline{s}} c_1\right) + \frac{14\delta}{\overline{s}} \left(c_1 + \vartheta S_1\right)\right] \cdot (1 - z_1) \\
&\leq \left[-\frac{14\delta}{\overline{s}} + \frac{14\delta}{\overline{s}} W_1\right] \cdot (1 - z_1) \\
&< \left[1 - \frac{14\delta}{\overline{s}} + \frac{14\delta}{\overline{s}} W_1\right] \cdot (1 - z_1)
\end{aligned}$$

So, we can conclude that, given the wealth level W_1, the maximum remaining utility reaches its optimum at the consumption $c_1^* = 1$, with the strategy $\theta_1^* = \frac{W_1 - c_1^*}{S_1}$ and its value is

$$\begin{aligned} H(1, W_1) &\equiv \left[1 - \frac{2\delta}{s} + \frac{2\delta}{s} W_1\right] \cdot z_1 + \left[1 - \frac{14\delta}{\overline{s}} + \frac{14\delta}{\overline{s}} W_1\right] \cdot (1 - z_1) \\ &= 1 + 2\delta \left[\left(\frac{1}{s} - \frac{7}{\overline{s}}\right) z_1 + \frac{7}{\overline{s}}\right] (W_1 - 1) \end{aligned}$$

that is, $H(1, W_1) = h(1, W_1, z_1)$ where

$$h(1, w, z) = 1 + 2\delta \left[\left(\frac{1}{s} - \frac{7}{\overline{s}}\right) z + \frac{7}{\overline{s}}\right] (w - 1).$$

Finally, we compute the maximum remaining wealth at time 0, given the wealth level $W(0) = e_0$:

$$\begin{aligned} H(0, e_0) = \sup_{(c,\theta)} & E\left[u_0(c_0) + u_1(c_1) + u_2(c_2)\right] \\ s.t. & \begin{cases} c_0 + \theta_0 \le e_0 \\ c_1 + \theta_1 S_1 \le \theta_0 S_1 \\ c_2 \le \theta_1 S_2 \end{cases} \end{aligned}$$

At the times $t = 0$ and $t = T = 2$, the constraints are binding at the optimum since the period-utilities are strictly increasing, so the problem becomes

$$\begin{aligned} H(0, e_0) = \sup_{(c_1, \theta_0, \theta_1)} & E\left[u_0(e_0 - \theta_0) + u_1(c_1) + u_2(\theta_1 S_2)\right] \\ & s.t. c_1 + \theta_1 S_1 \le \theta_0 S_1 \end{aligned}$$

where $E\left[u_0(e_0 - \theta_0)\right] = \alpha(e_0 - \theta_0)$ where $\alpha = \beta R$ and

$$\begin{aligned} E\left[u_2(\theta_1 S_2)\right] &= E\left[3\beta\theta_1 S_2 z_2\right] = E\left[3\beta\left(\vartheta \cdot z_1 + \overline{\vartheta} \cdot (1 - z_1)\right) R(z_2)^2\right] \\ &= 3\beta R\left(\vartheta E\left[z_1(z_2)^2\right] + \overline{\vartheta} E\left[(1 - z_1)((z_2)^2\right]\right) \\ &= 3\beta R\left(\vartheta \int_0^{1/2} x^2 dx + \overline{\vartheta} \int_{1/2}^1 x^2 dx\right) \\ &= \frac{\beta R}{8}\left(\vartheta + 7\overline{\vartheta}\right) \\ &= \delta\left(\vartheta + 7\overline{\vartheta}\right) \end{aligned}$$

where, as above, $\theta_1 = \vartheta \cdot z_1 + \overline{\vartheta} \cdot (1 - z_1)$ with $\vartheta, \overline{\vartheta} \in \mathbb{R}$. Therefore, with the usual notation $c_1 = \gamma \cdot z_1 + \overline{\gamma} \cdot (1 - z_1)$, $(\gamma, \overline{\gamma} \in \mathbb{R})$ the problem becomes

$$\begin{aligned} H(0, e_0) \equiv \sup_{\gamma, \overline{\gamma}, \theta_0, \vartheta, \overline{\vartheta}} & G(\gamma, \overline{\gamma}, \theta_0, \vartheta, \overline{\vartheta}) \\ s.t. & \begin{cases} \gamma + \vartheta s \le \theta_0 s \\ \overline{\gamma} + \vartheta \overline{s} \le \theta_0 \overline{s} \end{cases} \end{aligned}$$

with $G(\gamma,\overline{\gamma},\theta_0,\vartheta,\overline{\vartheta}) = \alpha(e_0-\theta_0)+E\left[u_1(c_1)\right]+\delta\left(\vartheta+7\overline{\vartheta}\right)$. Since $\delta>0$, the function $G(\gamma,\overline{\gamma},\theta_0,\vartheta,\overline{\vartheta})$ reaches its maximum in $\vartheta=\theta_0-\frac{\gamma}{s}, \overline{\vartheta}=\theta_0-\frac{\overline{\gamma}}{\overline{s}}$, so in fact we have to maximize over $\theta_0,\gamma,\overline{\gamma}$ the function

$$\begin{aligned}
\widetilde{G}(\gamma,\overline{\gamma},\theta_0) &= \alpha(e_0-\theta_0)+E\left[u_1(c_1)\right]+\left(\theta_0-\frac{\gamma}{s}+7\theta_0-7\frac{\overline{\gamma}}{\overline{s}}\right)\\
&= \alpha e_0+\theta_0(8\delta-\alpha)+E\left[u_1(c_1)\right]-\frac{\gamma}{s}-7\frac{\overline{\gamma}}{\overline{s}}\\
&= \alpha e_0+E\left[u_1(c_1)\right]-\frac{\gamma}{s}-7\frac{\overline{\gamma}}{\overline{s}}
\end{aligned}$$

(we recall $\alpha=\beta R=8\delta$) where

$$E\left[u_1(c_1)\right]=\begin{cases}\frac{\gamma+\overline{\gamma}^3}{2} & \text{if } \gamma,\overline{\gamma}\le 1\\ \frac{\gamma+1-\overline{\gamma}^2}{2} & \text{if } \gamma\le 1,\overline{\gamma}>1\\ \frac{1-\gamma+\overline{\gamma}^3}{2} & \text{if } \gamma>1,\overline{\gamma}\le 1\\ \frac{2-\gamma-\overline{\gamma}^2}{2} & \text{if } \gamma>1,\overline{\gamma}>1\end{cases}$$

Then, for $\gamma,\overline{\gamma}\le 1$,

$$\begin{aligned}
\widetilde{G}(\gamma,\overline{\gamma},\theta_0) &= \alpha e_0+\frac{\gamma+\overline{\gamma}^3}{2}-\frac{\gamma}{s}-7\frac{\overline{\gamma}}{\overline{s}}\\
&= \alpha e_0+\gamma\left(\frac{1}{2}-\frac{\delta}{s}\right)+\overline{\gamma}\left(\frac{\overline{\gamma}^2}{2}-\frac{7\delta}{\overline{s}}\right)\\
&\le \alpha e_0+1-\frac{\delta}{s}-\frac{7\delta}{\overline{s}}
\end{aligned}$$

recalling that $\frac{\delta}{s}<\frac{1}{2}$, $\frac{7\delta}{\overline{s}}<\frac{1}{4}$.In the case $\gamma\le 1,\overline{\gamma}>1$, we have

$$\begin{aligned}
\widetilde{G}(\gamma,\overline{\gamma},\theta_0) &= \alpha e_0+\frac{\gamma+1-\overline{\gamma}^2}{2}-\frac{\gamma}{s}-7\frac{\overline{\gamma}}{\overline{s}}\\
&= \alpha e_0+\gamma\left(\frac{1}{2}-\frac{\delta}{s}\right)+\frac{1}{2}-\overline{\gamma}\left(\frac{\overline{\gamma}}{2}+\frac{7\delta}{\overline{s}}\right)\\
&< \alpha e_0+\frac{1}{2}-\frac{\delta}{s}+\frac{1}{2}-\frac{1}{2}-\frac{7\delta}{\overline{s}}\\
&< \alpha e_0+1-\frac{\delta}{s}-\frac{7\delta}{\overline{s}}
\end{aligned}$$

In the case $\gamma>1,\overline{\gamma}\le 1$, we have

$$\begin{aligned}
\widetilde{G}(\gamma,\overline{\gamma},\theta_0) &= \alpha e_0+\frac{1-\gamma+\overline{\gamma}^3}{2}-\frac{\gamma}{s}-7\frac{\overline{\gamma}}{\overline{s}}\\
&= \alpha e_0+\frac{1}{2}-\gamma\left(\frac{1}{2}+\frac{\delta}{s}\right)+\overline{\gamma}\left(\frac{\overline{\gamma}^2}{2}-\frac{7\delta}{\overline{s}}\right)\\
&< \alpha e_0+\frac{1}{2}-\frac{1}{2}-\frac{\delta}{s}+\frac{1}{2}-\frac{7\delta}{\overline{s}}\\
&< \alpha e_0+1-\frac{\delta}{s}-\frac{7\delta}{\overline{s}}
\end{aligned}$$

In the case $\gamma > 1, \overline{\gamma} > 1$, we have

$$\begin{aligned}
\widetilde{G}(\gamma, \overline{\gamma}, \theta_0) &= \alpha e_0 + \frac{2 - \gamma - \overline{\gamma}^2}{2} - \frac{\gamma}{s} - 7\frac{\overline{\gamma}}{\overline{s}} \\
&= \alpha e_0 + 1 - \gamma \left(\frac{1}{2} + \frac{\delta}{s} \right) - \overline{\gamma} \left(\frac{\overline{\gamma}}{2} + \frac{7\delta}{\overline{s}} \right) \\
&< \alpha e_0 + 1 - \frac{1}{2} - \frac{\delta}{s} - \frac{1}{2} - \frac{7\delta}{\overline{s}} \\
&< \alpha e_0 + 1 - \frac{\delta}{s} - \frac{7\delta}{\overline{s}}
\end{aligned}$$

Therefore the maximum is reached at $\gamma = \overline{\gamma} = 1$ and the maximum remaining utility at time 0 is:

$$\begin{aligned}
H(0, e_0) &= \alpha e_0 + 1 - \frac{\delta}{s} - \frac{7\delta}{\overline{s}} \\
&= \beta R . e_0 + 1 - \frac{\beta R}{8s} - \frac{7\beta R}{8\overline{s}}
\end{aligned}$$

The value functions are all differentiable with respect to the level of wealth and the marginal utilities of optimal wealth are given by:

$$H_W(0, e_0) = \beta R$$

$$H_W(1, W_1) = \frac{\beta R}{4} \left[\left(\frac{1}{s} - \frac{7}{\overline{s}} \right) z_1 + \frac{7}{\overline{s}} \right]$$

$$H_W(2, W_2) = 3\beta z_2$$

Hence a linear pricing rule can be defined by:

$$\begin{aligned}
\psi(x) &= \frac{1}{\beta R} E \left[\frac{\beta R}{4} \left[\left(\frac{1}{s} - \frac{7}{\overline{s}} \right) z_1 + \frac{7}{\overline{s}} \right] x_1 + 3\beta z_2 x_2 \right] \\
&= \frac{1}{R} E \left[\frac{R}{4} \left[\left(\frac{1}{s} - \frac{7}{\overline{s}} \right) z_1 + \frac{7}{\overline{s}} \right] x_1 + 3 z_2 x_2 \right]
\end{aligned}$$

attained.

Proof of Theorem 4.

Let $Q \in \mathcal{Q}_N$ and let ψ be defined as in (15). Thanks to condition (12), ψ is well-defined on X, it is continuous and linear. Since Q is an equivalent martingale measure we also have that

$$V_\theta(0) = E^Q \left[\sum_{t=1}^{T} x(t) \frac{N(0)}{N(t)} \right]$$

for any $\theta \in \Theta$. Therefore ψ as in (15) defines a linear pricing functional in Ψ.

Conversely, let $\psi \in \Psi$: then ψ admits the representation (3) for some $\rho \in \prod_{t=1}^{T} L^q_{++}(\Omega, \mathcal{F}_t, P)$. Let $L(t) = \frac{\rho(t)N(t)}{N(0)}$ for $t = 0, \ldots, T$ with $\rho(0) = 1$: it is evident that $L(t) > 0$ for all t and $L(0) = 1$. Moreover, since ψ is a linear pricing rule, L is a P-martingale. To see this, let $t \in \{0, 1, \ldots, T-1\}$ and take an arbitrary $B \in \mathcal{F}_t$. Let $\theta^N \in \Theta^N$ be the strategy which generates the numeraire N and consider the strategy that consists in buying one unit of the strategy θ^N at time t if B occurs and then liquidating the positions in $t+1$, namely $\theta(s) = 0$ for $s \neq t$, $\theta(t) = \theta^N(t)\mathbb{I}_B$. The cash-flow generated by such a strategy is

$$x_\theta(s) = 0 \text{ for } s \neq t,\, t+1$$

$$x_\theta(t) = -\mathbb{I}_B N(t) \qquad\qquad x_\theta(t+1) = \mathbb{I}_B N(t+1).$$

Hence we have that $0 = \psi(x_\theta) = E^P\left[\sum_{t=1}^{T} \rho(t)x_\theta(t)\right] = N(0)E^P\left[\mathbb{I}_B\left(-L_t + L_{t+1}\right)\right]$, and therefore L is a P-martingale. Define the probability measure via $\frac{dQ}{dP} = L(T)$: it is easy to see that Q is the probability measure defined as in (14). Hence, to prove that Q is an equivalent martingale measure, we only need to verify property (13). To this aim, let $0 \leq t < T$, take an arbitrary $B \in \mathcal{F}_t$ and a generic $j \in \{1, \ldots, J\}$. Then buy 1 of the security S_j at time t if the event B occurs, and liquidate at $t+1$ the position. More formally, consider the dynamic investment strategy defined by:

$$\theta(s) = 0 \text{ for all } s \neq t,$$

and at time t by

$$\theta_j(t) = \mathbb{I}_B \qquad\qquad \theta_h(t) = 0 \text{ for } h \neq j$$

The cash-flow produced by this strategy is

$$x_\theta(s) = 0 \text{ for } s \neq t,\, t+1$$

$$x_\theta(t) = -\mathbb{I}_B S_j(t) \qquad\qquad x_\theta(t+1) = \mathbb{I}_B\left(S_j(t+1) + d_j(t+1)\right)$$

Hence, by definition of linear pricing functional and by Bayes' rule

$$\begin{aligned} 0 &= \psi(x_\theta) = E^P\left[\mathbb{I}_B\left(-S_j(t)\rho(t) + \left(S_j(t+1) + d_j(t+1)\right)\rho(t+1)\right)\right] = \\ &= N(0)E^Q\left[\mathbb{I}_B\left(-\frac{S_j(t)}{N(t)} + \frac{S_j(t+1) + d_j(t+1)}{N(t+1)}\right)\right], \end{aligned}$$

that is property (13) holds true. ■

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