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Real options with a double continuation region

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If the average risk-adjusted growth rate of the project's present value V overcomes the discount rate but is dominated by the average risk-adjusted growth rate of the cost I of entering the project, a non-standard double continuation region can arise: The firm waits to invest in the project if V is insufficiently above I as well as if V is comfortably above I. Under a framework with diffusive uncertainty, we give exact characterization to the value of the option to invest, to the structure of the double continuation region, and to the subset of the primitives' values that support such a region.

Keywords: Asset pricing; American options; Capital investment theory; Optimal stopping; Free boundary

1. Introduction

The problem of optimal irreversible investment in a long-lived project has usually been studied under the assumption that the average risk-adjusted growth rate of the project's present value is dominated by the discount rate (for instance, see the classical textbook of Dixit and Pindyck 1993, pp. 138–141 and p. 211). This assumption is standard and serves the purpose of avoiding an explosive value for the perpetual option to invest. The assumption leads to the standard continuation region: The firm waits to invest if the project's present value V inadequately exceeds the unrecoverable cost I of entering it. Our first contribution is to show that such an assumption is conspicuously restrictive. In the presence of a stochastic cost I, the assumption rules out cases in which the value of the investment option does not explode. Our second contribution is to show that such uncharted cases are extremely interesting. They can give rise to a non-standard double continuation region: The firm waits to invest if V is insufficiently above I as well as if V is comfortably above I. To the best of our knowledge, this important result is novel in the literature on investment under uncertainty. Our third contribution is to provide a rigorous and explicit description of the firm's optimal decision in these unexplored cases. We use a setting with diffusive uncertainty to accurately characterize the analytical value of the option to invest, the structure of the double continuation region, and the subset of the primitives' values that support such a region. We offer numerical evidence that natural levels of business growth and risk, of prices of risk, and of discount rate can lead to the double continuation region.

The intuition behind our results is strikingly uncomplicated, as the following example shows. Under the valuation probability measure $\hat{\mathbf{P}}$, take the present value V to be a geometric Brownian motion (the default assumption in the real-options literature) with a percentage drift of 6% per annum and take the discount rate to be 5%. If the cost I is constant, the problem is unbounded. Given any initial level for V, the firm finds it optimal to postpone investment to the furthest future as an unlimited expected discounted gain can be reaped by doing so. However, if the cost I is a geometric Brownian motion (possibly correlated with V) with a percentage drift of 7% per annum,† the problem becomes bounded. The cost will eventually overwhelm the value if there is an indefinite postponement of investment. Optimal postponement becomes definite, as it applies only in a distinct region of the initial levels for V and I. The standard part of the

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†Although they do not explicitly appear, the correlations between V and I and their local volatilities contribute towards defining the $\hat{\mathbf{P}}$ drifts via the price(s) of risk and the associated risk adjustment.

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region is located where the initial V either does not exceed the initial I or insufficiently exceeds it. Notably, the non-standard part of the region is located where the initial V comfortably exceeds the initial I. Discounted values and costs have positive percentage drifts (1% and 2%, respectively). If the initial V tops the initial I largely enough, delaying the investment is optimal as, over the short/medium run, the discounted value exhibits a total average increase greater than the discounted cost's total average increase. This is best seen by fixing the initial value V_0 to 5 and the initial cost I_0 to 1 and by considering the suboptimal 'European-style' investment strategy of accessing the project exactly one year from the initial date. Jensen's inequality implies that such a suboptimal strategy outperforms the strategy of immediate investment:

$$\begin{aligned}&\mathbb{E}^{\hat{\mathbf{P}}}[e^{-0.05\cdot 1}(V_1 - I_1)^+ \mid \mathcal{F}_0]\\&\quad \geq e^{-0.05\cdot 1}(\mathbb{E}^{\hat{\mathbf{P}}}[V_1 \mid \mathcal{F}_0] - \mathbb{E}^{\hat{\mathbf{P}}}[I_1 \mid \mathcal{F}_0])^+\\&\quad = 5\cdot e^{0.01\cdot 1} - 1\cdot e^{0.02\cdot 1}\\&\quad = 4.03 > V_0 - I_0,\end{aligned}$$

where $\mathbb{E}^{\hat{\mathbf{P}}}$ denotes expectation† under the probability measure $\hat{\mathbf{P}}$. The straightforward implication is that the strategy of immediate investment is also suboptimal. The numerical example offers brazen evidence that plausible levels of the real-option primitives can lead to the non-standard optimal decision of continuing to hold a real option that is deep in-the-money.

Under the assumption that both V and I are diffusions, we start as in Aase (2005)‡ by describing the primitives (business growth, business risk, prices of risk, and discount rate) under the historical probability measure $\mathbf{P}$. We then tackle the optimal investment problem by transiting to a valuation probability measure $\hat{\mathbf{P}}$. If V and I are spanned by traded assets, the prices of risk are dictated by the market, the discount rate equals the risk-free rate, and $\hat{\mathbf{P}}$ corresponds to a martingale measure. Even under a martingale measure $\hat{\mathbf{P}}$, V and I can easily grow at an average rate different from the risk-free rate, since V and I are unlikely to match the values of traded self-financing portfolios at any point in time. In particular, V and I can grow on average more than the risk-free rate and at different speeds. Hence, the spanning condition is easily congruent with the uncharted cases we are focusing on in the present work.

In general, we do not restrict our analysis to any particular choice of the valuation measure. We allow for subjective prices of risk, which are used by the firm to risk-adjust future cashflows from the considered business. We also allow for a subjective discount rate, which is used by the firm to discount risk-adjusted future cashflows. We arrive at closed formulae via a change of numeraire that reduces our problem to the valuation of an American perpetual put option on the cost-to-value ratio with an endogenous 'deflating rate' that depends on the primitives' values and that can be either positive or negative. Importantly, our ability to arrive at an analytical solution of the investment problem does not depend on the change-of-numeraire technique. The investment problem can be fully solved even without slashing its two value-and-cost dimensions to the single cost-to-value dimension. However, passing from a problem written in V and in I to a reduced-form problem written in the ratio I/V greatly improves the comparability of our results with the existing literature. Change-of-numeraire techniques for American options have been used by Battauz (2002) and by Carr (1995) for the finite-maturity case. Carr (1995) explicitly applies his results to capital investment theory, formalizing early dimension-reducing practices reviewed by Dixit and Pindyck (1993) in the context of that theory.

The double continuation region is a novel result that contributes to the real-options literature. By starting from a fairly unrestricted structural problem under $\mathbf{P}$ and then moving to its reduced form, we heighten the economic flexibility of our approach. In turn, such flexibility empowers the uncovering of unsuspected but artless combinations of the structural parameters under which, even when the investment option is deep in-the-money, V and I can have a relative growth that makes the (bounded) present value of waiting to invest greater than the immediate-investment value.

Discounting is the perfect terrain for spotting the difference between the traditional assumptions on the real options parameters and the combination of structural parameters that enables the double continuation region. A non-negative 'deflating rate' is applied to the value-to-cost ratio by Dixit and Pindyck (1993, p. 211). In contrast, the sign of the endogenous 'deflating rate' that we apply to the cost-to-value ratio is determined by the relative size of the primitives. If such a sign is negative (the subjective discount rate is dominated by the value's average risk-adjusted growth rate) and if the cost's average risk-adjusted growth rate dominates the value's one, our novel result ensues: The firm waits to invest if V over I is low enough as well as if V over I is high enough. Hence, in the plane $[I, V]$, the early exercise region must be straddled by the double continuation region,§ as shown in figure 1.

The derivation of our results gives a technical contribution to the real-options literature by adapting the classical verification theorem (for the jump-diffusion case,

†Conditioning the expectation upon $\mathcal{F}_0$ is intended to stress the dependence of the resulting quantities on the initial values for V and I.
‡Aase (2005) highlights the key role played by risk adjustment for an American perpetual option pricing problem, and works out exact solutions when jump sizes cannot be negative. He investigates when his solution is an approximation also for negative jumps.
§Broadie and Detemple (1997) and Detemple (2006) have shown that a *single* continuation region surrounded by multiple immediate-exercise regions arises with a variety of finite-maturity American derivatives written on two or more traded risky underlying assets. Battauz *et al.* (2009) investigate the *double* continuation region in a number of finite-maturity American problems.

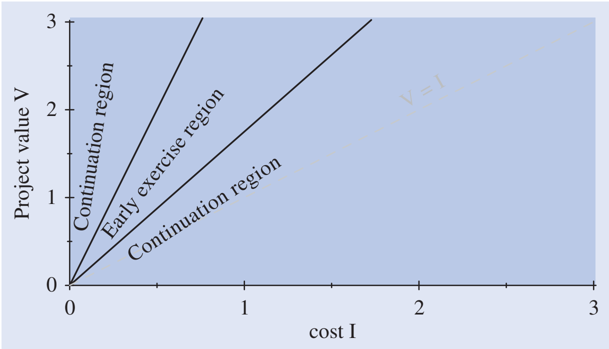

Figure 1. The double continuation region in the plane $[I, V]$.

see, for example, Mordecki 1999 and Øksendal and Sulem 2004) and by extending it to the situation of a non-integrable 'deflated' value function. Such a situation is originated by a negative 'deflating rate' and belongs to the non-standard cases we treat in the present work. Not surprisingly, it is associated with the possible emergence of the double continuation region.

The article is organized as follows. Section 2 introduces the investment problem and its primitives. Section 3 describes the endogenous 'deflating rate' in the reduced-form problem. Section 4 uses convexity, monotonicity, and value dominance to discuss the emergence of the double continuation region. In theorem 4.1 we provide the exact structure of the double continuation region. Section 5 collects numerical and graphical examples of the double continuation region. Section 6 concludes.

2. The investment problem and its primitives

Uncertainty is described by the historical probability space $(\Omega, \mathbf{P}, (\mathcal{F}_t)_t)$, and by two independent Brownian motions $W^{\mathbf{P}}$ and $\widetilde{W}^{\mathbf{P}}$. The two Brownian motions represent the diffusive risk that affects value and cost. The present value V of the project has $\mathbf{P}$ dynamics:

$$\mathrm{d}V_t = V_t\big(\mu_V\,\mathrm{d}t + \sigma_V\,\mathrm{d}W_t^{\mathbf{P}} + \widetilde{\sigma}_V\,\mathrm{d}\widetilde{W}_t^{\mathbf{P}}\big),$$

where μ_V, σ_V, and $\tilde{\sigma}_V$ are real positive constants. The investment cost I has $\mathbf{P}$ dynamics:

$$\mathrm{d}I_t = I_t(\mu_I\,\mathrm{d}t + \sigma_I\,\mathrm{d}W_t^{\mathbf{P}}),$$

where μ_I and σ_I are real positive constants. The current value of the perpetual option to invest is

$$\sup_{\tau\geq 0} \mathbb{E}^{\hat{\mathbf{P}}}[\mathrm{e}^{-\rho\tau}(V_\tau - I_\tau)^+ \mid \mathcal{F}_0], \tag{1}$$

where ρ is the constant subjective discount rate, and the expectation is taken under the valuation measure $\hat{\mathbf{P}}$. Problem (1) consists of finding the optimal exercise policy for a perpetual American call option on the asset value V with a non-constant strike I. The supremum in problem (1) runs among all stopping times τ. The optimal time to invest is the stopping time τ^* that attains the supremum in problem (1). The attained supremum is the value function and constitutes a Snell envelope, as it is the smallest supermartingale majorant of the discounted payoff from investing.

Given the subjective prices of W-type risk (θ) and of $\widetilde{W}$-type risk ($\widetilde{\theta}$), the $\hat{\mathbf{P}}$ dynamics of V and I can be derived by employing classical Cameron–Martin–Girsanov results for diffusion processes (see, for example, Protter (2004)):

$$\mathrm{d}V_t = V_t\big((\mu_V + \sigma_V\theta + \widetilde{\sigma}_V\widetilde{\theta})\,\mathrm{d}t + \sigma_V\,\mathrm{d}W_t^{\hat{\mathbf{P}}} + \widetilde{\sigma}_V\,\mathrm{d}\widetilde{W}_t^{\hat{\mathbf{P}}}\big)$$

and

$$\mathrm{d}I_t = I_t\big((\mu_I + \sigma_I\theta)\,\mathrm{d}t + \sigma_I\,\mathrm{d}W_t^{\hat{\mathbf{P}}}\big),$$

where $W^{\hat{\mathbf{P}}}$ and $\widetilde{W}^{\hat{\mathbf{P}}}$ are independent Brownian motions under $\hat{\mathbf{P}}$. If $\theta = \widetilde{\theta} = 0$, all the subjective prices of risk are null and the firm is risk-neutral ($\hat{\mathbf{P}} = \mathbf{P}$). If $\theta < 0$ and $\widetilde{\theta} < 0$, the firm is averse to W-type risk and to $\widetilde{W}$-type risk. If V and I are spanned by traded assets, the prices of risk $\{\theta, \widetilde{\theta}\}$ correspond to the market ones, the discount rate ρ equals the risk-free rate r, and $\hat{\mathbf{P}}$ becomes a martingale measure. However, it is well known that $V_t\mathrm{e}^{-rt}$ and $I_t\mathrm{e}^{-rt}$ are $\hat{\mathbf{P}}$-martingales only if V_t and I_t coincide with the values of traded self-financing portfolios at any date t, which is seldom the case for real asset values. It follows that, even under the spanning condition, the risk-adjusted percentage drifts of V and I typically differ from the discount rate:

$$\mu_V + \sigma_V\theta + \widetilde{\sigma}_V\widetilde{\theta} \neq \rho, \mu_I + \sigma_I\theta \neq \rho.$$

The choice of the risk-attitude parameters θ and $\widetilde{\theta}$ fixes the valuation probability measure $\hat{\mathbf{P}}$ for the assessment of the real option value in problem (1).

3. The endogenous 'deflating rate'

We reduce the option-value calculation in problem (1) to a one-dimensional problem by taking the process $V_t\mathrm{e}^{-k_V t}$ as numeraire, where

$$k_V = \mu_V + \sigma_V\theta + \widetilde{\sigma}_V\widetilde{\theta} - \rho. \tag{2}$$

The endogenous coefficient k_V is V's average growth rate under $\hat{\mathbf{P}}$, in excess of the subjective discount rate ρ. Analogously, we have that k_I,

$$k_I = \mu_I + \sigma_I\theta - \rho, \tag{3}$$

is I's average growth rate under $\hat{\mathbf{P}}$, in excess of ρ.

The change-of-numeraire technique† enables a useful dimension reduction of our investment problem. We denote by $\mathbf{P}^{\mathbf{A}}$ the auxiliary probability measure associated with the numeraire $V_t\mathrm{e}^{-k_V t}$ and re-express the investment option value in the following proposition.

†See, for example, Geman *et al.* (1995) and Battauz (2002) for a comprehensive application of change-of-numeraire techniques to American options.

Proposition 3.1: *Problem* (1) *admits the following representation*:

$$\sup_{\tau\geq 0}\mathbb{E}^{\hat{\mathbf{P}}}[\mathrm{e}^{-\rho\tau}(V_\tau - I_\tau)^+ \mid \mathcal{F}_0] = V_0 \cdot v(X_0), \tag{4}$$

with

$$v(X_0) = \sup_{\tau\geq 0}\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau}(1 - X_\tau)^+ \mid \mathcal{F}_0] \tag{5}$$

and

$$X_t = \frac{I_t}{V_t}.$$

Equations (4) and (5) relate the investment option value to the value,† calculated under the auxiliary probability measure $\mathbf{P}^{\mathrm{A}}$, of an American put option characterized by a unit strike price and written on an underlying item whose value is the cost-to-value ratio. The payoff from put option exercise is deflated by the endogenous 'deflating rate' $-k_V$.

The structure of the cost-to-value ratio under the probability measure $\mathbf{P}^{\mathrm{A}}$ can be derived by again employing the Cameron–Martin–Girsanov results:

$$X_t = X_0 \exp(at + \sigma_B B_t),$$

where B_t is a Brownian motion under $\mathbf{P}^{\mathrm{A}}$, and

$$\sigma_B^2 = (\sigma_I - \sigma_V)^2 + \tilde{\sigma}_V^2,$$
$$a = k_I - k_V - \frac{\sigma_B^2}{2}.$$

4. Option value properties and the structure of the continuation region

Given X's structure and

$$X_0 = x,$$

the American put value can be written as

$$v(x) = \sup_{u\geq 0}\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)u}(1 - X_0\exp(a\cdot u + \sigma_B B_u))^+ \mid \mathcal{F}_0], \tag{6}$$

where u runs among the stopping times. The function $v(x)$ inherits convexity and monotonicity from the exercise payoff, so that $v(x)$ is convex and monotonic decreasing in x. By construction, $v(x)$ dominates the exercise payoff:

$$0 \leq (1-x)^+ \leq v(x) \leq v(0),$$

for any non-negative value of x, where the last inequality stems from monotonicity. We now adapt the arguments of Lamberton and Lapeyre (1996) to show how convexity, monotonicity, and value dominance interact with the sign of the endogenous 'deflating rate' $-k_V$ in yielding the double continuation region.

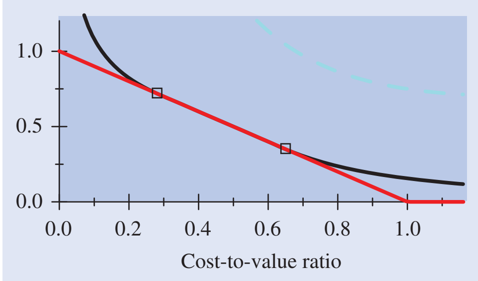

Figure 2. Put value with $-k_V<0$.

When the 'deflating rate' is non-negative, that is $-k_V \geq 0$, problem (1) is reduced to the valuation of a standard perpetual American put option with non-negative interest rates. A standard continuation region emerges: The firm waits to invest in the project if V is insufficiently above I.

When the 'deflating rate' is negative, i.e. $-k_V<0$, a non-standard double continuation region can emerge: The firm waits to invest in the project if V is insufficiently above I as well as V is comfortably above I. Indeed, the put value's supremum when the underlying is zero is infinite:

$$v(0) = \sup_{u\geq 0}\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)u}(1-0)^+ \mid \mathcal{F}_0] = \mathrm{e}^{-(-k_V)\infty} = +\infty$$
$$> (1-0)^+.$$

If the value $v(x)$ strictly dominates the exercise payoff $(1-x)^+$ for all $x>0$ (in figure 2, see the light-colored dashed line that starts from $+\infty$), early exercise is never optimal. However, the put value can also land on the exercise payoff for some x' belonging to the interval $(0,1)$. If it does so, it must remain grounded at the exercise payoff for a while and then take off from it for x sufficiently large (see the black solid line in figure 2). Hence, there exist two critical levels x_1 and x_2,

$$x_1 = \inf\{x \geq 0 : v(x) = 1-x\},$$
$$x_2 = \sup\{x \geq 0 : v(x) = 1-x\},$$
$$x_1 \leq x_2,$$

such that there is optimal early exercise only for $x_1 \leq x \leq x_2$. The non-standard double continuation region emerges. It is formed by all $x<x_1$ and $x>x_2$, straddling the early exercise region.

We remark that the arguments employed (convexity, monotonicity, and value dominance) do not depend on the dynamics of the underlying process X. This is suggestive that the robustness of the central result of our work (the surfacing of the double continuation region) under rather general types of uncertainty is quite plausible. The numerical and graphical output of section 5 exemplifies the emergence of the double continuation region.

†For an extensive review of valuation methods for American-style claims, see, for example, Broadie and Detemple (2004).

In theorem 4.1 we work out an exact characterization for the values of the option to invest, for the structure of the double continuation region, and for the subset of the primitives' values that support such a region. This is done via the explicit solutions to problem (1). With this aim, we adapt and extend the verification theorem as recounted in lemma 1 of Mordecki (1999). Lemma 1 of Mordecki (1999) states five standard optimality properties that, once satisfied by a function, render the function the value of a perpetual American option on the exponential of a jump-diffusion process. In our case, the assumed absence of jumps streamlines the structure of lemma 1. Mordecki (1999) starts from a reduced-form problem with zero 'deflating rate'. In theorem 4.1, we contribute by extending the streamlined lemma 1 to account for a negative 'deflating rate'. An extension of lemma 1 is required and not just an adaptation, as one of the five standard optimality properties (property (IV) in Mordecki's notation) is structurally violated. Such a property involves the integrability of the 'deflated' value function for any exercise policy and is clearly broken by the presence of a negative 'deflating rate', for which we have just seen that $v(0)=+\infty$.

As mentioned above, there are good reasons to believe that theorem 4.1's key economic finding (the materialization of the double continuation region) is resilient to a variety of stochastic settings, including environments with discontinuities. However, eschewing the presence of jumps enables a thrifty and agile structure for our theorems and proofs, which significantly enhances the visibility of our economic contribution to investment and decision making.

Theorem 4.1 : *Assume that*

$$-k_V < 0. \tag{7}$$

(1) *If*

$$k_I - k_V = a + \frac{\sigma_B^2}{2} \leq 0, \tag{8}$$

then

$$V_0 \cdot \sup_{\tau \geq 0} \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}\left[\mathrm{e}^{-(-k_V)\tau}\left(1 - \frac{I_\tau}{V_\tau}\right)^+ \middle| \mathcal{F}_0 \right] = +\infty,$$

and the corresponding optimal stopping time is

$$\tau^* = +\infty.$$

(2) *If*†

$$a - \sqrt{2\sigma_B^2 k_V} > 0, \tag{9}$$

then equation (*in the unknown p*)

$$\frac{1}{2}\sigma_B^2 p^2 + ap + k_V = 0 \tag{10}$$

has two negative solutions $\xi_2 < \xi_1$:

$$\xi_2 = \frac{-a - \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2},$$

$$\xi_1 = \frac{-a + \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2}.$$

We have

$$V_0 \cdot \sup_{\tau \geq 0} \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}\left[\mathrm{e}^{-(-k_V)\tau}\left(1 - \frac{I_\tau}{V_\tau}\right)^+ \middle| \mathcal{F}_0 \right]$$
$$= \begin{cases} \frac{I_0^{\xi_1}}{V_0^{\xi_1-1}} \frac{(1-\xi_1)^{\xi_1-1}}{(-\xi_1)^{\xi_1}}, & \text{if } \frac{I_0}{V_0} < \frac{\xi_1}{\xi_1-1}, \\ V_0 - I_0, & \text{if } \frac{\xi_1}{\xi_1-1} \leq \frac{I_0}{V_0} \leq \frac{\xi_2}{\xi_2-1}, \\ \frac{I_0^{\xi_2}}{V_0^{\xi_2-1}} \frac{(1-\xi_2)^{\xi_2-1}}{(-\xi_2)^{\xi_2}}, & \text{if } \frac{I_0}{V_0} > \frac{\xi_2}{\xi_2-1}. \end{cases} \tag{11}$$

and the corresponding optimal stopping time is

$$\tau^* = \inf\left\{ t \geq 0 : \frac{\xi_1}{\xi_1 - 1} \leq \frac{I_t}{V_t} \leq \frac{\xi_2}{\xi_2 - 1} \right\}.$$

(3) *If*

$$a - \sqrt{2\sigma_B^2 k_V} = 0, \tag{12}$$

then equation (10) *has one negative solution* ξ_0:

$$\xi_0 = -\frac{\sqrt{2k_V}}{\sigma_B}.$$

We have

$$V_0 \cdot \sup_{\tau \geq 0} \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}\left[\mathrm{e}^{-(-k_V)\tau}\left(1 - \frac{I_\tau}{V_\tau}\right)^+ \middle| \mathcal{F}_0 \right]$$
$$= \begin{cases} \frac{I_0^{\xi_0}}{V_0^{\xi_0-1}} \frac{(1-\xi_0)^{\xi_0-1}}{(-\xi_0)^{\xi_0}}, & \text{if } \frac{I_0}{V_0} \neq \frac{\xi_0}{\xi_0-1}, \\ V_0 - I_0, & \text{if } \frac{I_0}{V_0} = \frac{\xi_0}{\xi_0-1}, \end{cases} \tag{13}$$

and the corresponding optimal stopping time is

$$\tau^* = \inf\left\{ t \geq 0 : \frac{I_t}{V_t} = \frac{\xi_0}{\xi_0 - 1} \right\}.$$

The proof of theorem 4.1 is given in the appendix.

The condition of a negative 'deflating rate' in (7) violates the standard assumption in the literature on investment under uncertainty. It implies that, under the valuation measure $\hat{\mathbf{P}}$, V's percentage drift dominates the discount rate ρ. However, as points 2 and 3 of theorem 4.1 make clear, such a violation does not always lead to an infinite value of the investment opportunity, but instead opens the way to uncharted non-explosive cases of remarkable interest.

In point 1, condition (8) is conducive to the explosive solution. The percentage $\mathbf{P}^{\mathrm{A}}$-drift of the cost-to-value ratio process$\{I_t/V_t\}_{t\geq 0}$ is given by $a + (\sigma_B^2/2)$. Its non-positivity implies that, by Jensen's inequality,

†Notice that $a > 0$ implies $k_I - k_V > 0$.

delaying perpetually access to the payoff $e^{-(-k_V)t}[1-(I_t/V_t)]^+$ becomes optimal.

In points 2 and 3, conditions (9) and (12) grant that X has a sufficiently high rate of growth to counteract the effect of the negative 'deflating rate', resulting in a finite value for the put option.

In point 2, condition (9) keeps the problem bounded by ensuring that equation (10) admits two negative solutions† $\xi_1 > \xi_2$. In turn, this ensures that the process $\{e^{-(-k_V)t}v(I_t/V_t)\}_{t\geq 0}$ is a $\mathbf{P}^{\mathbb{A}}$-martingale in the continuation region and a $\mathbf{P}^{\mathbb{A}}$-supermartingale in the early exercise region.

In point 3, the immediate investment region collapses into a singleton, which represents the tangency point between the early exercise payoff and the put option value.

5. Examples of the double continuation region

In this section, we offer numerical evidence that credible levels of business growth and risk, of prices of risk, and of discount rate can lead to the double continuation region described in points 2 and 3 of theorem 4.1. Consider the values for the primitive parameters shown in table 1. These parameter values imply that the conditions in point 2 of theorem 4.1 are satisfied with

$$\xi_1 = -1.0877 > \xi_2 = -1.9123.$$

The American put value $v(I/V)$ (see equation (6)) is plotted versus the cost-to-value ratio I/V in the left-hand plot of figure 3. Optimal immediate investment occurs in the cost-to-value zone

$$\frac{\xi_1}{\xi_1 - 1} = 0.5210 \leq \frac{I}{V} \leq 0.6566 = \frac{\xi_2}{\xi_2 - 1}.$$

Fixing the cost I to 1 without loss of generality, the double continuation region is the union of the project-value zones

$$V < \frac{\xi_2 - 1}{\xi_2} = 1.5229, \quad 1.9194 = \frac{\xi_1 - 1}{\xi_1} < V.$$

The value $V \cdot v(1/V)$ of the corresponding option to invest (see formula (11)) is plotted versus the project's present value V in the right-hand plot of figure 3.

Coeteris paribus, we now take the discount rate to be $\rho = 4.46875\%$. The conditions in point 3 of theorem 4.1 are satisfied with

$$\xi_0 = -1.5.$$

The American put value $v(I/V)$ is plotted versus the cost-to-value ratio I/V in the left-hand plot of figure 4. Optimal immediate investment occurs only for a single cost-to-value ratio level:

$$\frac{I}{V} = \frac{\xi_0}{\xi_0 - 1}.$$

Fixing I to 1, the double continuation region swamps the whole real line but for a singleton:

$$V \neq \frac{\xi_0 - 1}{\xi_0} = 1.6667.$$

The value $V \cdot v(1/V)$ of the corresponding option to invest is plotted versus the project's present value V in the right-hand plot of figure 4.

Table 1. Values for the primitive parameters.

	Risk		
	W-type risk		$\widetilde{W}$-type risk
Growth			
$\mu_V = 25\%$	$\sigma_V = 20\%$		$\widetilde{\sigma}_V = 25\%$
$\mu_I = 30\%$	$\sigma_I = 20\%$		
Discount rate		Price of risk	
$\rho = 5\%$	$\theta = -30\%$		$\widetilde{\theta} = -30\%$

6. Conclusions

We have performed a thorough analysis of optimal irreversible investment in a long-lived project‡ when its present value V and its cost I are two possibly correlated diffusions. By removing the standard assumption that the average growth rate of the project's present value is always dominated by the discount rate under the valuation measure, we uncover the emergence of a non-standard double continuation region: The firm waits to invest if V is insufficiently above I as well as if V is comfortably above I.

Such a non-standard region emerges when the discounted value has an average growth rate that is positive but smaller than the discounted cost's one. The intuition behind our novel results is transparent. If the initial V tops the initial I largely enough, delaying the investment is optimal since, over the short/medium run, the discounted value exhibits a total average increase greater than the discounted cost's total average increase.

We start by describing the primitives (business growth, business risk, prices of risk, and discount rate) under the historical probability measure and then we tackle the optimal investment problem by transiting to a valuation probability measure. We argue that a double continuation region can occur even if V and I are spanned by traded assets, that is, even if the valuation measure is a martingale measure. We obtain closed formulae for the investment-option value by means of a change of

†The negative exponents ξ_1 and ξ_2 guarantee that the investment option value expressed in equation (11) remains increasing in V and decreasing in I also in the continuation region.
‡The project's predictable demise is considered in the real-options analysis of Magis and Sbuelz (2006).

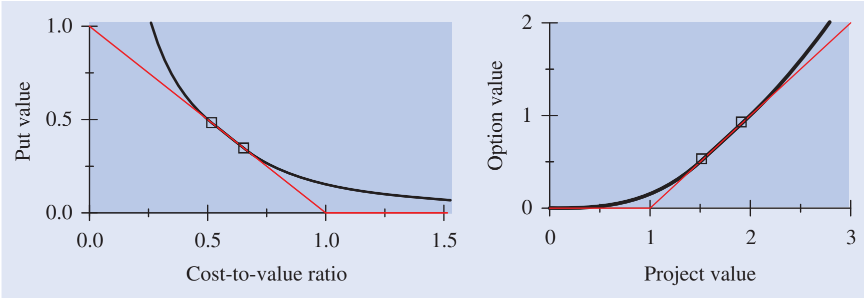

Figure 3. The value of the investment option with $I = 1$.

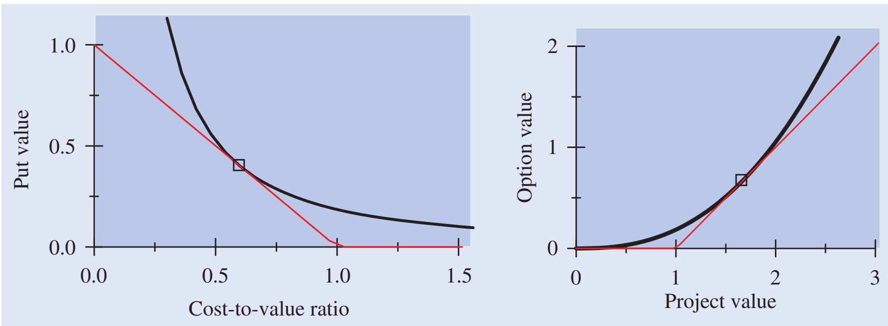

Figure 4. The value of the investment option with $I = 1$.

numeraire that reduces our problem to the assessment of an American perpetual option on the cost-to-value ratio.

The change of numeraire leads to a one-dimensional reduced-form problem whose comparability with the existing literature is strongly enhanced. However, our results can be directly accomplished under the valuation probability measure, without the use of dimension-cutting techniques.

By starting from the structural problem under the historical measure and then moving to its reduced form, we contribute to the traditional literature on real options, which usually starts its analysis directly in the reduced form and curbs economic flexibility via restrictions on the discounting procedure. Our novel result of the double continuation region shows that starting with flexible primitives under the historical measure greatly empowers the real-options analysis.

We give concrete examples of the double continuation region for plausible levels of business growth and risk, of prices of risk, and of discount rate. We offer reasons to believe that the emergence of the double continuation region is robust to varied types of uncertainty beyond the diffusive one.

Finally, the technical derivation of our results contributes to the optimal-stopping literature applied to real options by adapting the classical verification theorem for Snell envelopes and by extending it to the situation where integrability is violated. Such a situation arises exactly in the non-standard cases associated with the possible appearance of the double continuation region.

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References

Aase, K., The perpetual American put option for jump-diffusions with applications. Anderson Graduate School of Management Finance, Paper 15-05, 2005.

Battauz, A., Change of numeraire and American options. *Stochast. Anal. Applic.*, 2002, **20**(4), 709–730.

Battauz, A., De Donno, M. and Sbuelz, A., Real options and American derivatives: The double continuation region. Working Paper, Department of Mathematics, Quantitative Finance, and Econometrics, Catholic University of Milan, No. 42, 2009. Available online at: http://docenti.unicatt.it/eng/alessandro_sbuelz.

Broadie, M. and Detemple, J.B., The valuation of American options on multiple assets. *Math. Finance*, 1997, **7**, 241–285.

Broadie, M. and Detemple, J.B., Option pricing: Valuation models and applications (anniversary article). *Mgmt Sci.*, 2004, **50**(9), 1145–1177.

Carr, P., The valuation of American exchange options with application to real options. In *Real Options in Capital Investment: Models, Strategies and Applications*, edited by L. Trigeorgis, 1995 (Praeger: Westport).

Detemple, J.B., *American-Style Derivatives: Valuation and Computation*, 2006 (Chapman & Hall/CRC/Taylor & Francis Group: London).

Dixit, A.K. and Pindyck, R.S., *Investment Under Uncertainty*, 1993 (Princeton University Press: Princeton).

El Karoui, N., Les aspects probabilistes du controle stochastique (Ecole d'eté de probabilités de Saint-Flour IX). *Lecture Notes in Mathematics*, 1979, **876**, 73–238.

Geman, H., El Karoui, N. and Rochet, J.C., Changes of numeraire, changes of probability measure and option pricing. *J. Appl. Probab.*, 1995, **32**, 443–458.

Jacod, J. and Shiryaev, A.N., *Limit Theorems for Stochastic Processes*, 2002 (Springer: Berlin).

Lamberton, D. and Lapeyre, B., *Introduction to Stochastic Calculus Applied to Finance*, 1996 (Chapman and Hall: London).

Magis, P. and Sbuelz, A., Value of fighting irreversible demise by softening the irreversible cost. *Int. J. Theor. Appl. Finance*, 2006, **9**(4), 503–516.

Mordecki, E., Optimal stopping for a diffusion with jumps. *Finance Stochast.*, 1999, **3**(2), 227–238.

Mordecki, E., Optimal stopping and perpetual options for Lévy processes. *Finance Stochast.*, 2002, **6**(4), 473–493.

Øksendal, B. and Sulem, A., *Applied Stochastic Control of Jump Diffusions*, 2004 (Springer: Berlin).

Protter, P., *Stochastic Integration and Differential Equations, A New Approach*, 2nd ed., 2004 (Springer: Berlin).

Revuz, D. and Yor, M., *Continuous Martingales and Brownian Motion*, 3rd ed., 2001 (Springer: Berlin).

Shreve, S.E., Stochastic Calculus for Finance. *Continuous-Time Models*, Vol. II, 2004 (Springer: New York).

Appendix A: Proof of theorem 4.1

Proof: In proving point 1, we observe that the value function of equation (4) in proposition 3.1 is the value of an American put option with unit strike, with negative 'deflating rate' $-k_V$, and written on the underlying value X whose expectation is

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[X_t \mid \mathcal{F}_0] = X_0 \mathrm{e}^{(k_I - k_V)t}.$$

Since the put payoff is convex with respect to the underlying value and conditions (7) and (8),

$$-k_V < 0, \quad k_I - k_V < 0,$$

hold, Jensen's inequality implies that, for any non-stochastic τ', we have

$$\begin{aligned} v(x) &= \sup_{\tau \geq 0} \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau}(1 - X_\tau)^+ \mid \mathcal{F}_0] \\ &\geq \lim_{\tau' \to +\infty} \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau'}(1 - X_{\tau'})^+ \mid \mathcal{F}_0] \\ &\geq \lim_{\tau' \to +\infty} \mathrm{e}^{-(-k_V)\tau'}(1 - X_0 \mathrm{e}^{(k_I - k_V)\tau'})^+ \\ &= \mathrm{e}^{+\infty}(1 - 0) = +\infty. \end{aligned}$$

It follows that the optimal stopping time for equation (4) is $\tau^* = +\infty$ and that the value function is $v(x) = +\infty$. This completes the proof of point 1.

We turn to prove point 2. By proposition 3.1, the value function of problem (1) is given by the product of V_0 times the value function $v(x)$ in equation (6). Notice that this theorem can be proved without the use of change-of-numeraire techniques, by means of a direct application of the verification theorem for Snell envelopes to the value function on the two variables V_0 and I_0. The variable x represents the initial cost-to-value ratio:

$$x = X_0, \quad X_0 = \frac{I_0}{V_0}.$$

The value function $v(x)$ can also be written as

$$v(x) = \sup_{\tau \geq 0} \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau}(1 - X_\tau)^+ \mid \mathcal{F}_0]. \tag{A1}$$

We show that

$$v(x) = \begin{cases} A_1 \left(\frac{x}{x_1}\right)^{\xi_1}, & \text{if } 0 < x < x_1, \\ 1 - x, & \text{if } x_1 \leq x \leq x_2, \\ A_2 \left(\frac{x}{x_2}\right)^{\xi_2}, & \text{if } x > x_2, \end{cases} \tag{A2}$$

where $x_i = \xi_i/(\xi_i - 1)$ and $A_i = 1/(1 - \xi_i) = (1 - x_i)$ for $i = 1, 2$. ξ_1 and ξ_2 indicate the two negative solutions of equation (10):

$$\xi_1 = \frac{-a + \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2},$$

$$\xi_2 = \frac{-a - \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2}.$$

To prove that the function defined in (A2) is the value function of (A1) we have to show that

$$\begin{aligned} &\text{(a)} \quad v(x) = \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*}(1 - X_{\tau^*})^+ \mid \mathcal{F}_0], \\ &\text{(b)} \quad v(x) \geq \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau}(1 - X_\tau)^+ \mid \mathcal{F}_0], \end{aligned}$$

for any stopping time τ and for τ^* being the first instant at which X exits the continuation region (El Karoui 1979):

$$\tau^* = \inf\{t \geq 0 : x_1 \leq X_t \leq x_2\}.$$

Denoting by $\mathcal{L}$ the infinitesimal generator of the diffusion X and recalling X's $\mathbf{P}^{\mathbf{A}}$ dynamics,

$$\mathrm{d}X_t = X_t((k_I - k_V)\,\mathrm{d}t + \sigma_B\,\mathrm{d}B),$$

we have that

$$(\mathcal{L}v)(x) = \frac{1}{2}\sigma_B^2 x^2\, v''(x) + (k_I - k_V)x\, v'(x).$$

The function v defined in equation (A2) satisfies the following optimality properties:†

(I) $k_V v(x) + (\mathcal{L}v)(x) = 0$ for all $0 < x < x_1$ and for all $x > x_2$.
(II) $k_V v(x) + (\mathcal{L}v)(x) \leq 0$ for all $x_1 \leq x \leq x_2$.
(III) $0 \leq (1-x)^+ \leq v(x)$ for all x.
(IV) $v(X_{\tau^*}) = (1 - X_{\tau^*})^+$ almost surely.

We first verify property (I). For $0 < x < x_1$ or for $x > x_2$, the function v defined in equation (A2) has the form $v(x) = A_i(x/x_i)^p$ with $p = \xi_i$ for $i = 1, 2$. Hence, the condition $(k_V v + \mathcal{L}v)(x) = 0$ becomes

$$k_V A_i\left(\frac{x}{x_i}\right)^p + \frac{\sigma_B^2}{2} x^2 A_i p(p-1)\frac{1}{x_i^2}\left(\frac{x}{x_i}\right)^{p-2} + (k_I - k_V)x A_i p \frac{1}{x_i}\left(\frac{x}{x_i}\right)^{p-1} = 0,$$

which is equivalent to equation (10):

$$\frac{1}{2}\sigma_B^2 p^2 + ap + k_V = 0.$$

Under the assumptions in point 2, the equation has two negative solutions $\xi_2 < \xi_1$:

$$\xi_2 = \frac{-a - \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2},$$

$$\xi_1 = \frac{-a + \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2}.$$

Hence, v defined in equation (A2) satisfies property (I). Since we want the function v to be $\mathcal{C}^1$ everywhere, we impose *value matching* and *smooth pasting* that imply $x_i = \xi_i/(\xi_i - 1) < 1$ and $A_i = 1/(1 - \xi_i) = 1 - x_i$ for $i = 1, 2$. With these values, the function v is $\mathcal{C}^1$ everywhere, since $\lim_{x \to x_i} v(x) = 1 - x_i$ and $\lim_{x \to x_i^+} v'(x) = -1 = \lim_{x \to x_i^-} v'(x)$ for $i = 1, 2$.

We now verify property (II). Thanks to our assumptions, we have that, for any $x_1 \leq x \leq x_2$,

$$(k_V v + \mathcal{L}v)(x) = k_V(1-x) + \frac{1}{2}\sigma_B^2 x^2 \cdot 0 + (k_I - k_V)x \cdot (-1) = k_V - k_I x.$$

It is therefore sufficient to show that

$$k_V - k_I x_1 \leq 0,$$

to meet property (II). The above inequality can equivalently be written as

$$-(k_I - k_V)\xi_1 \geq k_V.$$

By replacing ξ_1 with its explicit value, we obtain

$$(k_I - k_V)\left(\frac{a - \sqrt{a^2 - 2\sigma_B^2 k_V}}{\sigma_B^2}\right) \geq k_V,$$

which is equivalent to

$$\sqrt{a^2 - 2\sigma_B^2 k_V} \leq a - \frac{\sigma_B^2 k_V}{k_I - k_V}. \tag{A3}$$

Assumption (9) guarantees that the right-hand side of (A3) is strictly positive. Indeed, by recalling that $k_I - k_V = a + \frac{1}{2}\sigma_B^2$, we have

$$\begin{aligned} a - \frac{\sigma_B^2 k_V}{k_I - k_V} &= a - \frac{\sigma_B^2 k_V}{a + \frac{1}{2}\sigma_B^2} \\ &= \frac{a^2 + \frac{1}{2}a\sigma_B^2 - \sigma_B^2 k_V}{a + \frac{1}{2}\sigma_B^2} \\ &> \frac{2\sigma_B^2 k_V + \frac{1}{2}a\sigma_B^2 - \sigma_B^2 k_V}{a + \frac{1}{2}\sigma_B^2} \\ &= \frac{\sigma_B^2 k_V + \frac{1}{2}a\sigma_B^2}{a + \frac{1}{2}\sigma_B^2} > 0. \end{aligned}$$

Simple algebra shows that (A3) holds if and only if

$$\frac{\sigma_B^4 k_V^2}{(k_I - k_V)^2} + 2\sigma_B^2 k_V - \frac{2a\sigma_B^2 k_V}{k_I - k_V} \geq 0. \tag{A4}$$

Since the left-hand term in (A4) can be rewritten as

$$\begin{aligned} &\frac{\sigma_B^4 k_V^2}{(k_I - k_V)^2} + 2\sigma_B^2 k_V\left(\frac{k_I - k_V - a}{k_I - k_V}\right) \\ &= \frac{\sigma_B^4 k_V^2}{(k_I - k_V)^2} + 2\sigma_B^2 k_V\left(\frac{\frac{1}{2}\sigma_B^2}{k_I - k_V}\right), \end{aligned}$$

we see that (A4) is equivalent to

$$\frac{\sigma_B^4 k_V^2}{(k_I - k_V)^2} + \frac{\sigma_B^4 k_V}{k_I - k_V} \geq 0.$$

This last inequality is clearly satisfied. Indeed, the right-hand side is the sum of strictly positive terms, as assumption (9) implies (see also footnote 1) that

$$k_I - k_V > 0.$$

We now verify property (III). To this aim, we focus on $x < x_1$ or $x_2 \leq x \leq 1$, since for $x_1 \leq x \leq x_2$ the function v coincides with $(1-x)^+$, and for $x \geq 1$ we have that $(1-x)^+ = 0 < v(x)$. On $x < x_1$ and on $x_2 \leq x \leq 1$ the function v is strictly convex, whereas $(1-x)^+$ is linear. Recalling that v and $(1-x)^+$ have the same derivative at x_i for $i = 1, 2$, we conclude that $v(x) \geq (1-x)^+$ on $x < x_1$ and on $x_2 \leq x \leq 1$.

Property (IV) will be verified at the end of the proof.

†These properties are four of the five optimality properties Mordecki (1999) quotes in his lemma 1. It is worth remarking that Mordecki (1999) considers solely a zero 'deflating rate'. We state the four properties in terms of the cost-to-value ratio X instead of $\ln X$ in order to save the convexity property of the put-payoff function. This allows us to apply the Meyer–Ito formula and minor modifications of its corollaries (Protter 2004). The missing property is a boundedness requirement, which is not satisfied in our case.

Thanks to properties (III) and (IV), conditions (a) and (b) are satisfied as soon as

$$\begin{aligned}&(\mathrm{a}') \quad v(x) = \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*} v(X_{\tau^*}) \mid \mathcal{F}_0],\\ &(\mathrm{b}') \quad v(x) \geq \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau} v(X_{\tau}) \mid \mathcal{F}_0],\\ &\qquad\qquad \text{for any stopping time } \tau.\end{aligned}$$

By applying the Meyer–Ito formula (see, for instance, Protter (2004, theorem IV.51)) to $\{\mathrm{e}^{-(-k_V)t} v(X_t)\}_{t\geq 0}$, we have

$$\mathrm{e}^{-(-k_V)t} v(X_t) - v(x) = \int_0^t \mathrm{e}^{-(-k_V)s}(k_V v + \mathcal{L}v)(X_s)\,\mathrm{d}s + M_t, \tag{A5}$$

where

$$M_t = \int_0^t \mathrm{e}^{-(-k_V)s} v'(X_s)\sigma_B X_s\,\mathrm{d}B_s.$$

Notice that we have to slightly modify the Meyer–Ito formula as done in the proof of lemma 1 of Mordecki (1999), since v is not $\mathcal{C}^2$. However, v is convex and its second derivative v'' exists continuous for any $x \neq x_1, x_2$. Also, v'' has finite left and right limits at x_1 and x_2. If τ is a stopping time, considering equation (A5) between 0 and τ yields

$$\begin{aligned}&\mathrm{e}^{-(-k_V)\tau} v(X(\tau)) - v(x)\\ &\quad= \int_0^\tau \mathrm{e}^{-(-k_V)s}(k_V v + \mathcal{L}v)(X_s)\,\mathrm{d}s + M_\tau \leq 0 + M_\tau,\end{aligned}$$

thanks to properties (I) and (II). Hence, (b′) is implied by

$$(\mathrm{b}'') \quad \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[M_\tau \mid \mathcal{F}_0] \leq 0, \quad \text{for any stopping time } \tau.$$

We now prove (b″). From (A5), we obtain

$$\begin{aligned}M_t = &-\int_0^t \mathrm{e}^{-(-k_V)s}(k_V v + \mathcal{L}v)(X_s)\,\mathrm{d}s\\ &+ \mathrm{e}^{-(-k_V)t} v(X_t) - v(x) \geq 0 - v(x),\end{aligned}$$

thanks to properties (I) and (II) and to the positivity of v. Hence, the local martingale M is bounded from below. It follows that M is a supermartingale bounded from below and, by theorem (1.39) of Jacod and Shiryaev (2002), we have $\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[M_\tau \mid \mathcal{F}_0] \leq M_0 = 0$ for any stopping time τ, which completes the proof of condition (b″).

It remains to prove condition (a′). To this aim, we compute explicitly

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*} v(X_{\tau^*}) \mid \mathcal{F}_0].$$

If $X_0 = x > x_2$, then $X_{\tau^*} = x_2$ and $v(X_{\tau^*}) = (1 - x_2)^+$. Hence,

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*} v(X_{\tau^*}) \mid \mathcal{F}_0] = \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{k_V\tau^*}(1 - x_2) \mid \mathcal{F}_0],$$

and the problem is reduced to the computation of the Laplace transform of the hitting time τ^* for the *positive* value $k_V > 0$. In this case the stopping time τ^* can be rewritten as

$$\begin{aligned}\tau^* &= \inf\{t \geq 0 : X_t = x_2\}\\ &= \inf\left\{t \geq 0 : at + \sigma_B B_t = \ln\frac{x_2}{x}\right\}\\ &= \inf\left\{t \geq 0 : -\frac{1}{\sigma_B}at - B_t = \frac{1}{\sigma_B}\ln\frac{x}{x_2}\right\}.\end{aligned}$$

We see that τ^* is the first instant at which the drifted Brownian motion $-(1/\sigma_B)at - B_t$ hits the barrier $(1/\sigma_B)\ln(x/x_2) > 0$. As highlighted, for example, by Shreve (2004), exploitation of the reflection principle for the standard Brownian motion $-B$ leads to the closed form of τ^*'s Laplace transform:

$$\begin{aligned}&\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{k_V\tau^*} \mid \mathcal{F}_0]\\ &\quad= \exp\left(\frac{1}{\sigma_B}\ln\frac{x}{x_2}\left(-\frac{1}{\sigma_B}a\right) - \frac{1}{\sigma_B}\ln\frac{x}{x_2}\sqrt{-2k_V + \left(-\frac{1}{\sigma_B}a\right)^2}\right).\end{aligned}$$

We observe that such a formula is usually employed for *negative* values of the parameter k_V, but it can be extended to the case of *positive* k_V provided that $-2k_V + [-(1/\sigma_B)a]^2 > 0$, which is exactly the assumption (9). Simple computations show that

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{k_V\tau^*} \mid \mathcal{F}_0] = \left(\frac{x}{x_2}\right)^{(1/\sigma_B^2)(-a-\sqrt{-2k_V\sigma_B^2+a^2})} = \left(\frac{x}{x_2}\right)^{\xi_2},$$

so that the equality

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*} v(X_{\tau^*}) \mid \mathcal{F}_0] = \left(\frac{x}{x_2}\right)^{\xi_2}(1 - x_2) = v(x)$$

is verified for any $x > x_2$.

If $X_0 = x < x_1$, then $X_{\tau^*} = x_1$ and $v(X_{\tau^*}) = (1 - x_1)^+$

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*} v(X_{\tau^*}) \mid \mathcal{F}_0] = \mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{k_V\tau^*}(1 - x_1) \mid \mathcal{F}_0].$$

In this case the stopping time τ^* can be rewritten as

$$\begin{aligned}\tau^* &= \inf\{t \geq 0 : X_t = x_1\}\\ &= \inf\left\{t \geq 0 : \frac{1}{\sigma_B}at + B_t = \frac{1}{\sigma_B}\ln\frac{x_1}{x}\right\}.\end{aligned}$$

We see that τ^* is the first instant at which the drifted Brownian motion $(1/\sigma_B)at + B_t$ hits the barrier $(1/\sigma_B)\ln(x_1/x) > 0$. By exploiting the reflection principle for the standard Brownian motion B, the Laplace transform of τ^* is now

$$\begin{aligned}&\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{k_V\tau^*} \mid \mathcal{F}_0]\\ &\quad= \exp\left(\frac{1}{\sigma_B}\ln\frac{x_1}{x}\left(\frac{1}{\sigma_B}a\right) - \frac{1}{\sigma_B}\ln\frac{x_1}{x}\sqrt{-2k_V + \left(\frac{1}{\sigma_B}a\right)^2}\right),\end{aligned}$$

for any k_V such that $-2k_V + [(1/\sigma_B)a\ \ ]^2 > 0$. Simple computations show that

$$\begin{aligned}\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{k_V\tau^*} \mid \mathcal{F}_0] &= \left(\frac{x_1}{x}\right)^{(1/\sigma_B^2)(a-\sqrt{-2k_V\sigma_B^2+a^2})}\\ &= \left(\frac{x_1}{x}\right)^{-(1/\sigma_B^2)(-a+\sqrt{-2k_V\sigma_B^2+a^2})} = \left(\frac{x_1}{x}\right)^{-\xi_1},\end{aligned}$$

so that the equality

$$\mathbb{E}^{\mathbf{P}^{\mathrm{A}}}[\mathrm{e}^{-(-k_V)\tau^*} v(X_{\tau^*}) \mid \mathcal{F}_0] = \left(\frac{x}{x_1}\right)^{\xi_1}(1 - x_1) = v(x)$$

is verified for any $x < x_1$.

Finally, we verify property (IV). If $\{\tau^* < +\infty\}$, then $v(X_{\tau^*}) = (1 - X_{\tau^*})^+$ from the definitions of the stopping time τ^* and of the function v.

We now focus on the event $\{\tau^* = +\infty\}$ and study the asymptotic behavior of X_t as t goes to infinity. Recall that $\mathbf{P}^{\mathrm{A}}$-almost surely, $\limsup_{t\to+\infty} B_t/\sqrt{2t\log_2 t} = 1$ and $\liminf_{t\to+\infty} B_t/\sqrt{2t\log_2 t} = -1$ (see, for instance, corollary 1.12, chapter II.1 of Revuz and Yor (2001)). Recalling the $\mathbf{P}^{\mathrm{A}}$ dynamics of the log cost-to-value ratio, we have

$$\ln X_t = \ln x + at + \sigma_B B_t.$$

Since $a > 0$, it follows that

$$\lim_{t\to+\infty} \ln X_t = +\infty, \quad \mathbf{P}^{\mathrm{A}}\text{-almost surely.}$$

Hence, the process X reaches any upper barrier almost surely as time goes by. In particular, if $X_0 < x_1$, then X reaches x_1 in a finite time. Therefore, the event $\{\tau^* = +\infty\}$ has a strictly positive $\mathbf{P}^{\mathrm{A}}$ probability of occurring only if $X_0 > x_2$, and can be rewritten as $\{\tau^* = +\infty\} = \{X_t > x_2 \text{ for all } t \geq 0\}$. But then

$$\begin{aligned} v(X_{\tau^*}) &= \lim_{t\to+\infty} A_2\left(\frac{X_t}{x_2}\right)^{\xi_2} \\ &= 0 \\ &= \lim_{t\to+\infty} (1 - X_t)^+ \\ &= (1 - X_{\tau^*})^+. \end{aligned}$$

The proof of point 3 follows by similar arguments. Note that, in this case, the function v defined in (A2) collapses to

$$v(x) = \frac{1}{1-\xi_0}\left(\frac{x}{x_0}\right)^{\xi_0},$$

where $x_0 = \xi_0/(\xi_0 - 1)$ and ξ_0 is the unique negative solution of equation (10). □