The geometry of the double continuation region

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Joint with

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1. The continuation region: single or double?
   - The usual situation
   - The American put
   - The American call
2. The problem is relevant
   - A capital investment option
   - The gold loan
3. American options with a negative 'interest rate'
   - The American perpetual put
   - The American put with finite maturity
4. Conclusions and extensions

The American put option

- Log-normal asset $X(t) = X(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)}$, with $\mu, \sigma > 0$, B standard Brownian motion
- strike K, interest rate ρ
- The American option value is

$$\operatorname*{ess\,sup}_{t \le \tau \le T} \mathbb{E}\left[e^{-\rho(\tau - t)} \left(K - X(\tau)\right)^+ \middle| \mathcal{F}_t \right] = v(t, X(t))$$

where v is

$$v(t,x) = \sup_{0 \le \Theta \le T-t} \mathbb{E}\left[e^{-\rho\Theta} \left(K - x \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma B(\Theta)} \right)^+ \right]$$

- When $T = +\infty$ then $v(t,x) = v_\infty(x)$

A point makes the difference

If $x = 0$ then $X(t) = 0$ for any $t \in [0; T]$. But then

- if $\rho \geq 0$,

$$\begin{aligned} v(t,0) &= \sup_{0 \leq \Theta \leq T-t} \mathbb{E}\left[e^{-\rho\Theta}(K-0)^{+}\right] = \\ &= \sup_{0 \leq \Theta \leq T-t} e^{-\rho\Theta} \cdot K = e^{-\rho \cdot 0} \cdot K = K, \end{aligned}$$

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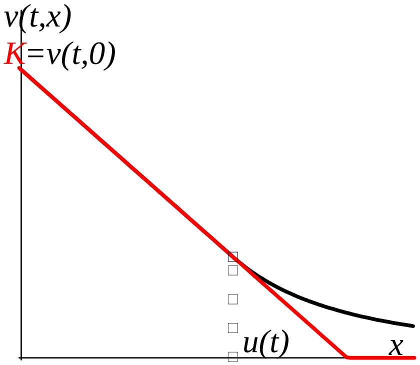
- if $\rho < 0$,

$$\begin{aligned} v(t,0) &= \sup_{0 \leq \Theta \leq T-t} \mathbb{E}\left[e^{-\rho\Theta}(K-0)^{+}\right] = \\ &= \sup_{0 \leq \Theta \leq T-t} e^{-\rho\Theta} \cdot K = e^{-\rho \cdot (T-t)} \cdot K > K. \end{aligned}$$

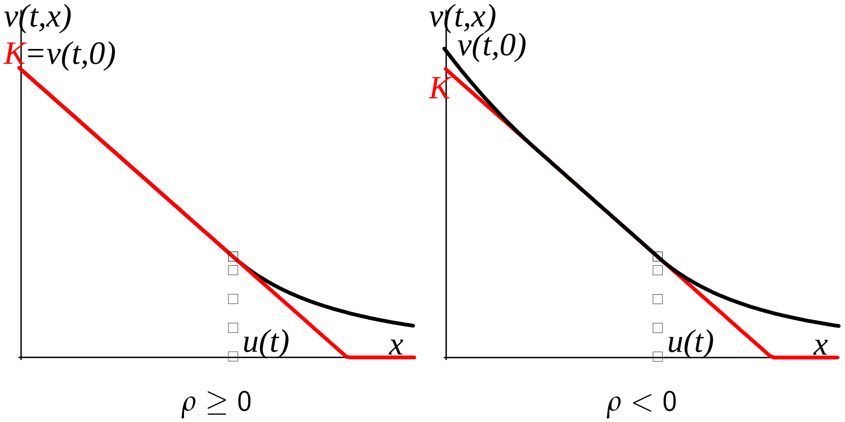
The put option: interplay with the interest rate's sign



$\rho \geq 0$

The value of the American put $v(t, \cdot)$ (black) and put payoff (red)

The put option: interplay with the interest rate's sign



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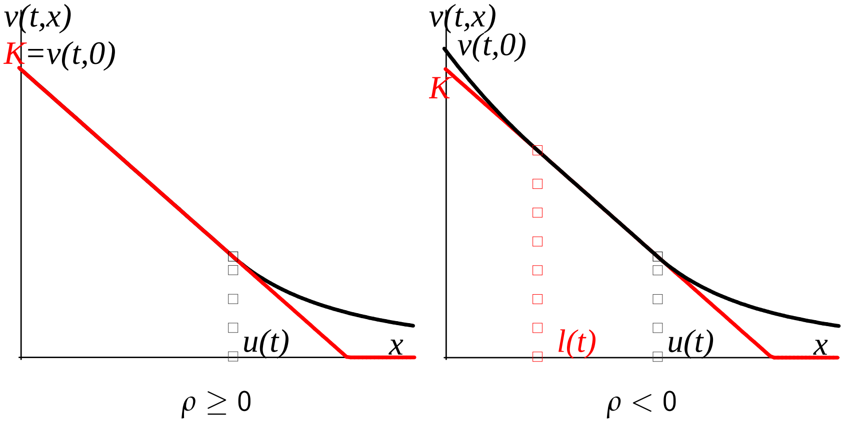


The value of the American put $v(t, \cdot)$ (black) and put payoff (red)

The put option: negative interest rate

There exist two critical prices

- $u(t) = \sup\{x \geq 0 : v(t,x) = (K-x)^+\} \wedge K$

and

- $l(t) = inf\{x \geq 0 : v(t,x) = (K-x)^+\}$ (the new one!)

such that

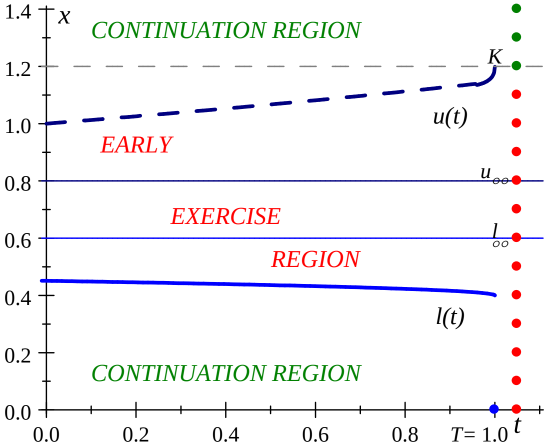
- exercise at t if $X(t) \in [l(t); u(t)]$
- continue at t if $\underbrace{X(t) < l(t)}$ or $X(t) > u(t)$

$\uparrow$

(new) branch of the continuation region

A double continuation region appears: Preview

$\rho = -4\%,\ K = 1.2,\ \sigma = 20\%,\ \mu = 8\%,\ T = 1.$



American put-call symmetry

- (Carr and Chesney, 1996):

$$v_{call}\left(t,x;K,\rho,\mu,\sigma\right)=v_{put}\left(t,\underbrace{K}_{x_{put}};\underbrace{x}_{K_{put}},\underbrace{\rho-\mu}_{\rho_{put}},\underbrace{-\mu}_{\mu_{put}},\underbrace{\sigma}_{\sigma_{put}}\right)$$

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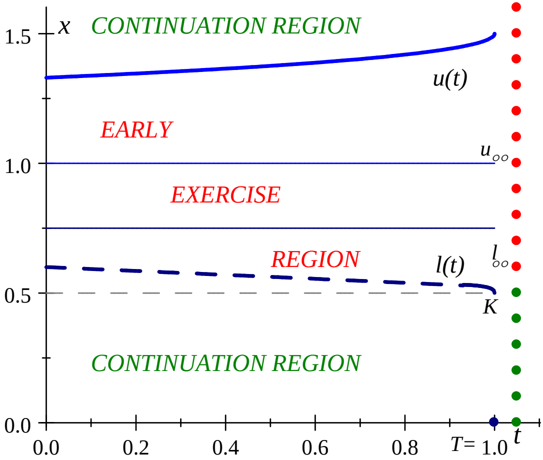
- $l_{put}(t)$ (resp. $u_{put}(t)$) lower (upper) critical price at t for put v_{put} with $K_{put} = 1$

$$l_{call}(t) = \frac{K}{u_{put}(t)} \text{ and } \overbrace{u_{call}(t)}^{new!} = \frac{K}{l_{put}(t)}$$

Translating the results for the call option

$\rho = -12\%,\ K = 0.5, \sigma = 20\%,\ \mu = -8\%,\ T = 1.$



A capital investment option I

A firm decides when to enter a project, whose *present value* V is

$$\mathrm{d}V_t = V_t \left(\widehat{\mu}_V \,\mathrm{d}t + \sigma_V \,\mathrm{d}W_t^{\hat{\mathbf{P}}}, + \widetilde{\sigma}_V \,\mathrm{d}\widetilde{W}_t^{\hat{\mathbf{P}}} \right),$$

at the *cost* I with

$$\mathrm{d}I_t = I_t \left(\widehat{\mu}_I \,\mathrm{d}t + \sigma_I \,\mathrm{d}W_t^{\hat{\mathbf{P}}} \right)$$

$\widehat{\mu}_V$, σ_V, $\tilde{\sigma}_V$, $\widehat{\mu}_I$ and σ_I, are real positive constants
$\widetilde{W}_t^{\hat{\mathbf{P}}}$, $W_t^{\hat{\mathbf{P}}}$ are independent Brownian motions on $(\Omega, \hat{\mathbf{P}}, (\mathcal{F}_t)_t)$.

The firm selects the *valuation measure* $\hat{\mathbf{P}}$ and the *discount rate* $\widehat{r}$.

A capital investment option II

The *value of the option to invest* at date t is

$$\operatorname{ess}\sup_{t \leq \tau \leq T} \mathbb{E}^{\hat{\mathsf{P}}} \left[e^{-\widehat{r}(\tau - t)} (V_\tau - I_\tau)^+ \middle| \mathcal{F}_t \right] \tag{1}$$

Solving the capital investment option

Let $\mathbf{P}^V$ the probability associated to the *numeraire* $V_T e^{\rho T}$ with

$$\rho = -\left(\widehat{\mu}_V - \widehat{r}\right) \ .$$

The investment option value is

$$\text{ess} \sup_{t \le \tau \le T} \mathbb{E}^{\hat{\mathbf{P}}} \left[e^{-\widehat{r}(\tau - t)} (V_\tau - I_\tau)^+ \middle| \mathcal{F}_t \right] = V_t \cdot v(t, X_t),$$

where

$$v(t, X_t) = \text{ess} \sup_{t \le \tau \le T} \mathbb{E}^{\mathbf{P}^V} \left[e^{-\rho(\tau - t)} \left(1 - X_\tau\right)^+ \middle| \mathcal{F}_t \right]$$

is **the value under $\mathbf{P}^V$ of an American put option on the cost-to-value ratio**

$$X_t = \frac{I_t}{V_t},$$

with 'interest rate' $\rho = -\left(\widehat{\mu}_V - \widehat{r}\right)$.

Solving the capital investment option

The cost-to-value ratio X is lognormal with

$$\begin{aligned} \mathbf{P}^V - \text{volatility} \qquad & \sigma^2 = (\sigma_I - \sigma_V)^2 + \widetilde{\sigma}_V^2, \\ \mathbf{P}^V - \text{drift} \qquad & \mu = \widehat{\mu}_I - \widehat{\mu}_V. \end{aligned}$$

- **Focus on very profitable investments** $\widehat{\mu}_V > \widehat{r}$ $\longrightarrow \rho = -(\widehat{\mu}_V - \widehat{r}) < 0$ **negative interest rate**

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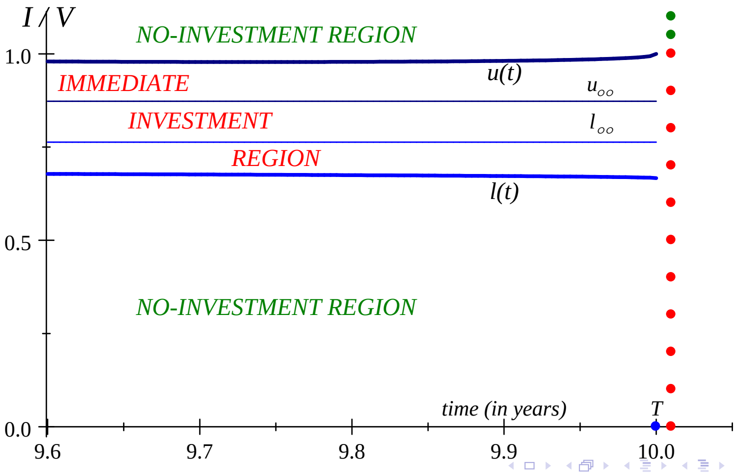
- If $\mu = \hat{\mu}_I - \hat{\mu}_V \leq 0 \xrightarrow{\text{Jensen's inequality}}$ invest at T only
- If $\mu = \hat{\mu}_I - \hat{\mu}_V > 0 \xrightarrow{\text{our result}}$ **double continuation region**

Preview of the results near maturity

$$\widehat{r} = 3\%,\ \widehat{\mu}_V = 5\%,\ \sigma_V = 7\%,\ \widetilde{\sigma}_V = 3\%,\ \widehat{\mu}_I = 6\%,\ \sigma_I = 10\%$$



The gold loan

- The borrower receives at time 0 the *loan amount* $q > 0$ using one unit of gold as collateral.
- The loan amount q grows at the rate γ, where $\gamma > r$ is the borrowing rate
- Prepayment option: The borrower has the right to redeem the gold at any $t \leq T$
- Gold is a tradable investment asset with storage and insurance costs $Gu > 0$ per unit of time:

$$\frac{dG(t)}{G(t)} = (r + u)\, dt + \sigma dW(t),$$

where r is the riskless interest rate, σ is the gold returns' volatility, and W is a B.M. under the risk-neutral $\mathbf{Q}$.

$\rightsquigarrow$ Differences with *stock loans* (Ekström and Wanntorp (2008))

The redemption option of the gold loan I

- Given a finite maturity T, the value of the redemption option at $t = 0$ is

$$C(G_0, 0) = \sup_{0 \le \tau \le T} \mathbb{E}^{\mathbf{Q}} \left[e^{-r\tau} \left(G(\tau) - q e^{\gamma \tau} \right)^+ \right]$$

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- C can be rewritten as

$$C(G_0, 0) = \sup_{0 \leq \tau \leq T} \mathbb{E}^{\mathbf{Q}} \left[e^{-(r-\gamma)\tau} \left(X(\tau) - q \right)^+ \right] \quad ,$$

that is **a call option on** $X(t) = G(t)\, e^{-\gamma t}$ **(the gold price deflated at the rate** γ**) and 'interest rate'**

$$\rho = r - \gamma < 0.$$

The redemption option of the gold loan II

The deflated gold price $X(t) = G(t)\, e^{-\gamma t}$ is lognormal with $\mathbb{Q}$-drift

$$\mu = r + u - \gamma$$

- if $\mu = r + u - \gamma > 0 \xrightarrow{\text{Jensen's inequality}}$ redemption at T only

The redemption option of the gold loan II

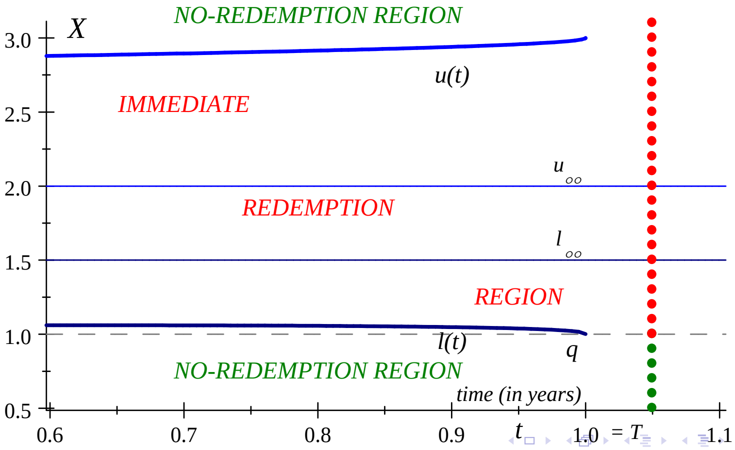
The deflated gold price $X(t) = G(t)\, e^{-\gamma t}$ is lognormal with $\mathbb{Q}$-drift

$$\mu = r + u - \gamma$$

- if $\mu = r + u - \gamma > 0 \stackrel{\text{Jensen's inequality}}{\longrightarrow}$ redemption at T only
- if $\mu = r + u - \gamma < 0 \stackrel{\text{our result}}{\longrightarrow}$ **double no-redemption region**

Preview of the results near maturity

$T = 1$, $r = 3\%$, $u = 1\%$, $\gamma = 6\%$, $\sigma = 10\%$, and $q = 1$.



The perpetual put with negative interest rates

Theorem Battauz, De Donno, Sbuelz [BDS] (2012, Quantitative Finance): *If* $\rho < 0$, *under condition*

$$\mu - \frac{\sigma^2}{2} > 0 \text{ and } \left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0 \qquad \text{(A1)}$$

the perpetual lower and upper free boundary are (resp.)

$$l_\infty = K\frac{\xi_l}{\xi_l - 1} \quad \text{and } u_\infty = K\frac{\xi_u}{\xi_u - 1}$$

where $\xi_u < \xi_l < 0$ *solve*

$$\frac{1}{2}\sigma^2\xi^2 + \left(\mu - \frac{\sigma^2}{2}\right)\xi - \rho = 0.$$

The perpetual put with negative interest rates (continued)

The value of the perpetual American put option is

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0; l_\infty) \\ K - x & \text{for } x \in [l_\infty; u_\infty] \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty; +\infty) \end{cases}$$

Proof:

Boundedness requirement violated: when $\rho < 0$ and $x = 0$ the optimal time to exercise is $\Theta^* = +\infty$, and the value of the American option is

$$v_\infty(0) = \mathbb{E}\left[e^{-\rho\Theta^*}(K-0)^+\right] = +\infty.$$

Hence direct verification, i.e.

$$(a) \quad v_\infty(x) = \mathbb{E}\left[e^{-\rho\tau^*}(K-X_{\tau^*})^+\right],$$

$$(b) \quad v_\infty(x) \geq \mathbb{E}\left[e^{-\rho\tau}(K-X_\tau)^+\right],$$

for any stopping time τ and for τ^* s.t.

$$\tau^* = \inf\left\{t \geq 0 : l_\infty \leq X_t \leq u_\infty\right\}.$$

The double continuation region appears when...

For $\rho < 0$, Condition $(A.1)$

$$\mu - \frac{\sigma^2}{2} > 0, \text{ and } \left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0$$

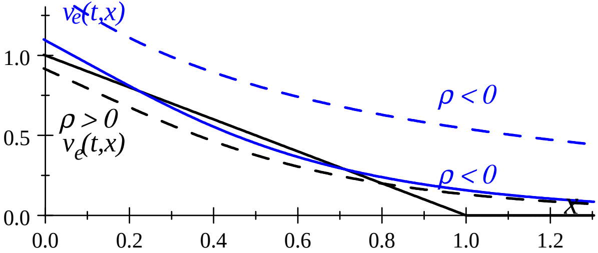
is a **sufficient** condition for early exercise.
A **necessary** condition for the optimal exercise of the finite-maturity American put option at $t \in [0; T)$ is

$$\mathcal{N}^{-1}\left(e^{\rho(T-t)}\right) - \mathcal{N}^{-1}\left(e^{(\rho-\mu)(T-t)}\right) \geq \sigma\sqrt{T-t}, \qquad (2)$$

where $\mathcal{N}^{-1}(\cdot)$ denotes the inverse of the standard normal cumulative distribution function.

The double continuation region appears when...

$(A.1)$ and (2) require μ high compared to $|\rho|$.
$(2) \Leftrightarrow \exists x > 0$ s.t. European put $v_e(t,x) \leq (K-x)^+$



$v_e(t,x)$ (solid) : $\rho = -1\%, \mu = 3\%, \sigma = 20\%$ (dashed) : $\rho = -4\%, \mu = 3\%, \sigma = 40\%$
$v_e(t,x)$ (dashed) : $\rho = 1\%, \mu = 3\%, \sigma = 20\%$

The (double) free-boundary geometry for finite maturity

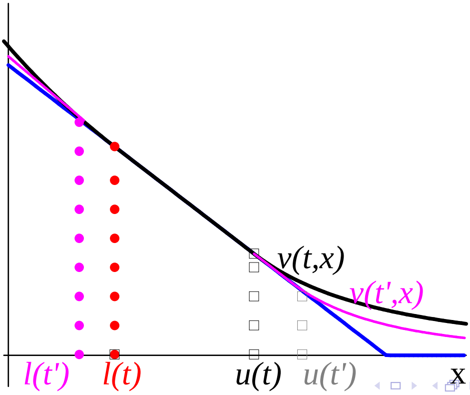
Theorem [BDS, forthcoming in Management Science]: *Under A1, for any* $t \in [0; T)$ *there exist*

$$0 < \frac{\rho K}{\rho - \mu} \leq l(t) < u(t) \leq K \tag{3}$$

such that $(K - x)^+ = v(t, x)$ *for any* $x \in [l(t); u(t)]$ *and* $(K - x)^+ < v(t, x)$ *for any* $x \notin [l(t); u(t)]$.
The lower free boundary $l : [0; T] \to [0; l_\infty)$ *is decreasing, continuous for any* $t \in [0; T)$, $l(T^-) = \frac{\rho K}{\rho - \mu} > l(T) = 0$.
The upper free boundary $u : [0; T] \to (u_\infty; K]$ *is increasing, continuous for any* $t \in [0; T]$, *and* $u(T) = u(T^-) = K$.

Proof: main steps I

1) **Monotonicity:** $v(t,x) \leq v(t',x)$ for any $t' > t \Rightarrow$
$\Rightarrow u(t)$ is increasing, $l(t)$ is decreasing



Proof: main steps II

2) *Right* **continuity** of l on $[0; T)$ follows from monotonicity of l, and continuity of v and $(K - \cdot)^+$.

3) *Left* **continuity** of l on $[0; T)$ follows from the V.I.:

$$\begin{cases} v(T, \cdot) = (K - \cdot)^+ \\ v(t, \cdot) \geq (K - \cdot)^+ \quad \text{for any } t \in [0; T] \\ \frac{\partial}{\partial t} v + \mathcal{L}v - \rho v \leq 0 \text{ on } (0; T) \times \Re^+ \\ \frac{\partial}{\partial t} v + \mathcal{L}v - \rho v = 0 \text{ on } \mathcal{CR} \end{cases}$$

where $(\mathcal{L}v)(t,x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} v(t,x) + \mu x \frac{\partial}{\partial x} v(t,x)$
and $\mathcal{CR} = \{(t,x) \in (0; T) \times \Re^+ : v(t,x) > (K - x)^+\}$.
4) $l(T^-) = \frac{\rho K}{\rho - \mu}$ follows from V.I.

Asymptotic behavior of the free boundaries at maturity

Theorem [BDS, forthcoming in Management Science]: *Under A1, for $t \longrightarrow T$ the upper free boundary*

$$u(t) - K \sim -K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi\,(T-t)\,\mu^2}}.$$

For $t \longrightarrow T$, the lower free boundary satisfies

$$l(t) - \frac{\rho K}{\rho-\mu} \sim \frac{\rho K}{\rho-\mu}\left(-y^*\sigma\sqrt{(T-t)}\right),$$

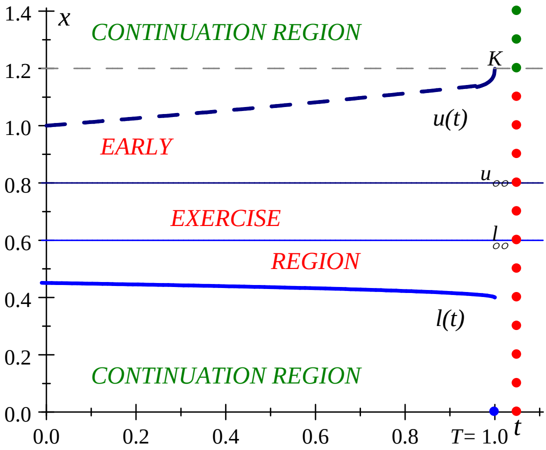
where $y^* \approx -0.638$ *s.t.* $\phi(y) = \sup_{0\le\Theta\le1} E\left[\int_0^\Theta (y+B(s))\,ds\right] = 0$ $\forall y \le y^*$ *and* $\phi(y) > 0$ $\forall y > y^*$.

Proof: main steps

- Evans, Kuske and Keller (2002): if $\rho \geq 0$, $\rho \geq \delta$ the critical price of the put tends to its left limit in a *parabolic-logarithmic* form as $t \longrightarrow T$ $\Rightarrow$ if $\rho < 0$ asymptotics of our *upper free boundary u*
- Lamberton and Villenevue (2003): if $\rho \geq 0$, $\rho < \delta$ the critical price of the put tends to its left limit in a *parabolic* form as $t \longrightarrow T$. The results relies on $\rho \geq 0$. We use an expansion result (Theorem 1 in LV2003) $\Rightarrow$ if $\rho < 0$ asymptotics of our *lower free boundary l*

A picture of the double free boundary for the put

$\rho = -4\%, \ K = 1.2, \ \sigma = 20\%, \ \mu = 8\%, \ T = 1.$



Conclusions: what is done

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- Examples of practical relevance that can be reduced to the valuation of American options with negative interest rates
- Worked out closed formulae in the perpetual case with lognormal underlying
- Described main features of the geometry of the free boundary in the finite maturity case (limits, continuity, asymptotics).

Extensions/open problems I:

- Convexity of upper free boundary u and concavity of lower free boundary l ?

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- Theorem 2.1 in Ekström (JMAA, 2004):
  For $\rho, \sigma > 0$, $\mu = \rho \Rightarrow$
  $u(t)\,\ddot{u}(t) \geq (\dot{u}(t))^2 > 0$ for all $t < T$

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- Theorem 2.1 in Ekström (JMAA, 2004):
  For $\rho, \sigma > 0$, $\mu = \rho \Rightarrow$
  $u(t)\,\ddot{u}(t) \geq (\dot{u}(t))^2 > 0$ for all $t < T$
- Still under investigation for $\rho < 0$, under condition $(A.1)$

$$\mu - \frac{\sigma^2}{2} > 0, \text{ and } \left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0$$

Extensions/open problems II: jump diffusion case

Extending formulae for l_∞ and u_∞ :

- The way we proved $v_\infty(x) \geq \mathbb{E}\left[e^{-\rho\tau}\left(K - X_\tau\right)^+\right]$ for any stopping time τ is feasible
- Direct computation to prove $v_\infty(x) = \mathbb{E}\left[e^{-\rho\tau^*}\left(K - X_{\tau^*}\right)^+\right] = \mathbb{E}\left[e^{-\rho\tau^*} v_\infty(X_{\tau^*})\right]$ is unfeasible.
- If $x < l_\infty$, then $\tau^* = \inf\left\{t \geq 0 : X_t = l_\infty\right\}$ and $v_\infty(x) = \mathbb{E}\left[e^{-\rho\tau^*}\right]\left(K - l_\infty\right)^+$ but usual techniques to compute $\mathbb{E}\left[e^{-\rho\tau^*}\right]$ require $-\rho \leq 0$.
- Alternative: prove $\mathbb{E}[M_{\tau^*}] = 0$, where M is the martingale part of the decomposition of $e^{-\rho t} v_\infty(X_t)$.
  But the Optional Sampling Theorem is an unfeasible tool (M and τ^* not unif. bdd).

Thanks for your attention!

Slides downloadable at my personal page at Bocconi

http://didattica.unibocconi.eu/docenti/

cv.php?rif=49395&cognome=BATTAUZ&nome=ANNA