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Research Article

Kim and Omberg Revisited: The Duality Approach

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We give an alternative duality-based proof to the solution of the expected utility maximization problem analyzed by Kim and Omberg. In so doing, we also provide an example of incomplete-market optimal investment problem for which the duality approach is conducive to an explicit solution.

1. Introduction

Kim and Omberg [1] study a problem of utility maximization from terminal wealth in a continuous-time market, allowing the Sharpe ratio of the risky asset to follow a mean-reverting process. As in the seminal paper [2], they address the problem using a stochastic control approach, despite the lack of a verification theorem supporting the uniqueness of their results. We provide a rigorous solution to Kim and Omberg's problem by using a probabilistic approach based on convex duality. Since the market is not complete, we refer to the very general results by Kramkov and Schachermayer [3, 4]. Other references on convex duality methods both in the complete and in the incomplete case can be found in [5–9]. Haugh et al. [6] exploit the dual formulation of the optimal portfolio problem to determine an upper bound on the unknown maximum expected utility in order to evaluate the quality of approximation of the optimal solution. As they observe, explicit solutions are rare in incomplete markets where the opportunity set is stochastic. We thus add to the literature by providing an example where the duality approach succeeds in characterizing explicitly the value function, the optimal solutions to both the primal and the dual problem, and the optimal strategy.

In Section 2, we describe the basic market model of Kim and Omberg. In Section 3, we apply the duality approach to the solution of the utility maximization problem from terminal wealth and find the optimal solutions to both the primal and the dual problems. Finally, in Section 4, we recover Kim and Omberg's results, characterizing the optimal value function and the optimal strategy.

2. The Market

The investor trades two assets, a risk-free asset and a risky asset in a frictionless continuous-time market, modeled through a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ satisfying the usual assumptions (in the sense of Definitions I.1.2 and I.1.3 in [10]). The price of the risky asset follows the dynamics

$$\frac{dP(t)}{P(t)} = \mu(t)\,dt + \Sigma(t)\,dZ(t), \tag{1}$$

where Z is a standard Brownian motion and μ and $\Sigma(t)$ are diffusion processes. The risk premium on the risky asset $X(t) = \Sigma(t)^{-1}(\mu(t) - r)$ follows a mean-reverting Ornstein-Uhlenbeck process. The coefficients $\mu(t) = \Sigma(t)X(t) + r$ and $\Sigma(t)$ are assumed to be sufficiently regular so that there exists a unique solution to the stochastic differential equation defining P. This is the case, e.g., when Σ is constant (and, consequently, $\mu(t) = \Sigma X(t) + r$). A list of sufficient conditions on the coefficients for the existence and uniqueness of the solution of the stochastic differential equation can be found in Section 1.2 of [11]. Weaker conditions can be found in [12] (see also the references therein). The dynamics of X is

$$dX(t) = -\lambda (X(t) - \overline{x})\, dt + \sigma dZ^X(t), \tag{2}$$

where $\overline{x}$, λ, and σ are positive constants and Z^X is a Brownian motion correlated with Z, with $d[Z, Z^X](t) = \rho dt$ and $0 \le |\rho| \le 1$. We can assume, without loss of generality, that $(\mathcal{F}_t)_{0\le t\le T}$ is the filtration generated by Z and Z^X.

Let $W = (W(t))_{0\le t\le T}$ be the value process of a self-financing portfolio, given the initial wealth w. The discounted value process is given by

$$\widetilde{W}(t) = e^{-rt} W(t) = w + \int_0^t H(s)\, d\widetilde{P}(s), \tag{3}$$

where $\widetilde{P}(t) = P(t)e^{-rt}$ and H is an adapted process representing the amount of shares of risky asset in the portfolio. We call the strategy H *admissible* if there exists some constant $C > 0$ such that $\int_0^t H(s)d\widetilde{P}(s) \ge -C$ almost surely, and, to rule out strategies generating arbitrage profit, we allow only for admissible strategies. In accordance with Kim and Omberg [1], we express the wealth in terms of the monetary investment in the risky asset $y(s)$; that is,

$$\widetilde{W}(t) = w + \int_0^t y(s) \frac{d\widetilde{P}(s)}{\widetilde{P}(s)}. \tag{4}$$

3. Utility Maximization with the Duality Approach

We consider the problem of an agent whose aim is to maximize his/her expected utility from terminal wealth. The investor has a HARA utility function, of the form

$$U(w) = \frac{\gamma}{\gamma - 1} w^{(\gamma-1)/\gamma} \tag{5}$$

for $w > 0$, $\gamma > 0$, $\gamma \ne 1$. We collect in the set $\mathcal{W}(w)$ all the nonnegative self-financing portfolios with initial value w and denote by $\widetilde{\mathcal{W}}(w)$ the set of the corresponding discounted portfolios, namely,

$$\widetilde{W} \in \widetilde{\mathcal{W}}(w) \Longleftrightarrow \widetilde{W}(t) = w + \int_0^t H(s)\, d\widetilde{P}(s) \ge 0. \tag{6}$$

The problem of utility maximization from terminal wealth can be written as

$$J(w) = \sup_{W\in\mathcal{W}(w)} E[U(W(T))] = \left(e^{rT}\right)^{\gamma/(\gamma-1)} u(w), \tag{7}$$

where we define

$$u(w) = \sup_{\widetilde{W}\in\widetilde{\mathcal{W}}(w)} E\left[U\left(\widetilde{W}(T)\right)\right]. \tag{8}$$

To find the function u, we apply the duality approach developed in [3, 4]. For convenience of the reader, we recall the results due to Kramkov and Schachermayer (in particular, [4, Theorems 1 and 2]) which will be exploited henceforth.

Let V denote the conjugate function of U (the functions U and V are conjugate if and only if $U(w) = \inf_{y>0}(V(y) + wy)$ and $V(y) = \sup_{w>0}(U(w) - wy)$); that is, $V(y) = (1/(\gamma - 1))y^{1-\gamma}$. We define the set $\mathcal{Y} = \{Y \ge 0 : Y_0 = 1$ and WY is a supermartingale for all $W \in \widetilde{\mathcal{W}}\}$ and consider the following optimization problem:

$$v(y) = \inf_{Y\in\mathcal{Y}} E[V(yY(T))]. \tag{9}$$

Let $\mathcal{D} = \{\eta \in L^1_+ : Q = \eta.\, P$ is an equivalent local martingale measure$\}$ and define

$$\widetilde{v}(y) = \inf_{\eta\in\mathcal{D}} E[V(y\eta)]. \tag{10}$$

Under the assumption that the set of equivalent local martingale measures is nonempty and that U satisfies Inada conditions ((2.4) in [3] or (3) in [4]), we have that if $\widetilde{v}(y) < \infty$ for all $y > 0$ (see Note 3 in [4]), then

(1) $u(w) < \infty$ for all $w > 0$; there exists some y_0 such that $v(y) < +\infty$ for $y > y_0$, and u and v are conjugate;

(2) the optimal solution $\widetilde{W}^* \in \mathcal{W}(w)$ to (8) exists and is unique. If η^* is the optimal solution to (10), with $y = u'(w)$ (or equivalently $w = -v'(y)$), we have the dual relation $\widetilde{W}^* = I(y\eta^*)$, where $I = -V'$;

(3) $v(y) = \widetilde{v}(y)$.

When $\rho = \pm 1$, the market is complete and the martingale measure is unique; therefore the set $\mathcal{D}$ is a singleton (provided that it is nonempty). In this case, the above results reduce to Theorem 2.0 in [3]. The complete case can be outlined following [13], where a similar analysis is carried on, under the assumption that the price process follows CEV dynamics. In the present paper, we limit our analysis to the more complex case $\rho \ne \pm 1$, where the set of local equivalent martingale measures has infinitely many elements, among which we look for the optimal solution of the dual problem. Observe that the utility function U defined as (5) satisfies Inada conditions. So we only need to prove that the set of equivalent martingale measures is nonempty, in order to apply the duality approach. This is done in the next lemma.

Lemma 1. *The set $\mathcal{D}$ is not empty.*

Proof. It is sufficient to prove that there exists at least an equivalent local martingale measure. Let

$$\eta^X = \mathcal{E}\left(-\int_0^T X(s)\, dZ(s)\right) = \exp\left(-\int_0^T X(s)\, dZ(s) - \frac{1}{2}\int_0^T X^2(s)\, ds\right), \tag{11}$$

where we denote by $\mathcal{E}$ the stochastic exponential; see, e.g., [10, section II.8a.]. The random variable η^X is the Radon-Nikodym density of an equivalent local martingale measure, provided that $E[\eta^X] = 1$ (see [14]). There are several ways to show this result. One can, e.g., prove Novikov's condition on sufficiently small intervals and exploit Corollary 3.5.14 in [15] or one can follow an argument similar to that used in [16], recalling that an Ornstein-Uhlenbeck process satisfies $\int_0^T X_s^2 ds < +\infty$ almost surely. We choose to exploit the relation between the martingale property and the solution of stochastic differential equations and, in particular, the recent results in [17] that in turn extend the method of [18] (an alternative approach has been recently developed in [19]).

We first introduce the process

$$Z' = \frac{Z_X}{\sqrt{1-\rho^2}} - \frac{\rho Z}{\sqrt{1-\rho^2}} \tag{12}$$

which is a Brownian motion independent of Z. Note that the filtration $(\mathcal{F}_t)_{0\le t\le T}$ coincides with the filtration generated by Z and Z'. Using the same notation as in [17], we set $B = [Z \;\; Z']^T$:

$$\begin{aligned} a_s(x) &= -\lambda(x-\overline{x}), \\ b_s(x) &= \left(\sigma\rho \;\; \sigma\sqrt{1-\rho^2}\right), \\ \sigma_s(x) &= (-x \;\; 0). \end{aligned} \tag{13}$$

It follows that $\|\sigma_s(x)\|^2 = x^2$, $L_s(x) = -2\lambda(x-\overline{x})x + \sigma^2$, and $\mathcal{L}_s(x) = -2[(\lambda+\sigma\rho)x^2 + \overline{x}x] + \sigma^2$.

Then, one can find $r > X(0) > 0$ such that

$$\|\sigma_s(x)\|^2 + L_s(x) + \mathcal{L}_s(x) \le r\left(1+x^2\right) \tag{14}$$

and the assumptions of Theorem 8.1 in [17] are satisfied. As a consequence $E[\eta^X] = 1$. □

To solve explicitly our problem, we proceed as follows. We first assume that there exists an optimal solution $\eta^* \in \mathcal{D}$ for the dual problem (10). From η^*, we derive the solution of the primal problem $\widetilde{W}^*$, and, exploiting the fact that it must be the final value of a self-financing portfolio, we are able to explicitly characterize the candidate solutions $\widetilde{W}^*$ and η^*. Finally, we show that the characterized η^* is really the density of an equivalent martingale measure and the optimal solution of problem (10), namely, that our initial assumption is satisfied and $J(w) = (e^{rT})^{\gamma/(\gamma-1)}E[U(\widetilde{W}^*(T))]$ is the maximal expected utility given the initial wealth w.

As announced before, we make the following assumption, which will be proved to be true in Lemma 4.

Assumption 1. There exists $\eta^* \in \mathcal{D}$ such that $\widetilde{v}(\eta^*) = E[V(y\eta^*)]$.

Our first results characterize both solutions of the primal and the dual problem, given that they exist.

Lemma 2. *Under Assumption 1, the optimal solution $\widetilde{W}^*(T) \in \widetilde{\mathcal{W}}(w)$ to (8) exists and is given by*

$$\widetilde{W}^*(T) = w\frac{(\eta^*)^{-\gamma}}{E\left[(\eta^*)^{1-\gamma}\right]}. \tag{15}$$

Proof. From the condition $w = -v'(y)$, we get $y = (w/E[(\eta^*)^{1-\gamma}])^{-1/\gamma}$. Hence

$$\begin{aligned} \widetilde{W}^*(T)(w) = I(y\eta) &= \left(\left(\frac{w}{E\left[(\eta^*)^{1-\gamma}\right]}\right)^{-1/\gamma}\eta^*\right)^{-\gamma} \\ &= w\frac{(\eta^*)^{-\gamma}}{E\left[(\eta^*)^{1-\gamma}\right]}. \end{aligned} \tag{16}$$

The claim is then proved. □

The discounted optimal wealth (15) is the final value of a self-financing discounted portfolio, which, under the optimal martingale measure $Q^* = \eta^*.P$, admits the representation

$$\begin{aligned} \widetilde{W}^*(t) &= w + \int_0^t y^*(s)\frac{d\widetilde{P}(s)}{\widetilde{P}(s)} \\ &= w + \int_0^t \Sigma(s)y^*(s)\,dZ^*(s), \end{aligned} \tag{17}$$

where $Z^*(t) = Z(t) + \int_0^t X(s)ds$ is a Q^*-Brownian motion. We denote $\eta^*(t) = E[\eta^* \mid \mathcal{F}_t]$ and $M(t) = E[(\eta^*)^{1-\gamma} \mid \mathcal{F}_t]$. Recalling that $\widetilde{W}^*(t) = E^{Q^*}[\widetilde{W}^*(T) \mid \mathcal{F}_t]$ and the expression for $\widetilde{W}^*(T)$ in (15) we obtain that

$$\begin{aligned} \xi(t) = \widetilde{W}^*(t)\frac{E\left[(\eta^*)^{1-\gamma}\right]}{w} &= E^{Q^*}\left[(\eta^*)^{-\gamma} \mid \mathcal{F}_t\right] \\ &= \frac{E\left[(\eta^*)^{1-\gamma} \mid \mathcal{F}_t\right]}{E\left[\eta^* \mid \mathcal{F}_t\right]} = \frac{M(t)}{\eta^*(t)}. \end{aligned} \tag{18}$$

Denote

$$\eta^\theta = \mathcal{E}\left(-\int_0^T \theta(t)\,dZ'(t)\right), \tag{19}$$

where Z' is defined in (12), and let Θ be the set of processes θ such that η^θ is a Radon-Nikodym density. The set Θ is nonempty since it contains $\theta \equiv 0$. Every equivalent local martingale measure has a Radon-Nikodym density of the form $\eta^X\eta^\theta$, with $\theta \in \Theta$. Assumption 1 guarantees that there exists $\theta^* \in \Theta$ such that $\eta^* = \eta^X\eta^{\theta^*}$. Then

$$d\eta^*(t) = -\eta^*(t)\left(X(t)\,dZ(t) + \theta^*(t)\,dZ'(t)\right). \tag{20}$$

To find the stochastic differential of $M(t)$, we denote

$$\begin{aligned} F(T,x) &= E\left[(\eta^*)^{1-\gamma}\right] \\ &= E\left[e^{(\gamma-1)\int_0^T X(s)dZ(s) + (\gamma-1)\int_0^T \theta^*(s)dZ'(s) + ((\gamma-1)/2)\int_0^T (X^2(s)+(\theta^*(s))^2)ds}\right], \end{aligned} \tag{21}$$

where $x = X(0)$ and X satisfies the stochastic differential equation (2). Then

$$M(t) = \exp\left[(\gamma-1)\int_0^t X(s)\,dZ(s) + (\gamma-1)\int_0^t \theta^*(s)\,dZ'(s) + \left(\frac{(\gamma-1)}{2}\right)\int_0^t \left(X^2(s)+(\theta^*(s))^2\right)ds\right] F(T-t, X(t)), \tag{22}$$

provided that the process on the right-hand side is a martingale. Letting $G(T,x) = \ln F(T,x)$ (so that $F(T,x) = e^{G(T,x)}$) and denoting by G_t, G_x, and G_{xx}, respectively, the first partial derivatives of G with respect to t, x, and the second partial derivative of G with respect to x, Ito's formula yields (for sake of notation, we omit the dependence of G_t, G_x, and G_{xx} on $(T-t, X(t))$)

$$\begin{aligned}\frac{dM(t)}{M(t)} = {} & \left[\frac{\gamma(\gamma-1)}{2}\left(X(t)^2+(\theta(t)^*)^2\right) - G_t - \lambda G_x(X(t)-\overline{x}) + \frac{\sigma^2}{2}\left(G_x^2+G_{xx}\right) + (\gamma-1)\sigma G_x\left(\rho X(t)+\sqrt{1-\rho^2}\theta^*(t)\right)\right]dt \\ & + \left[(\gamma-1)X(t)+\sigma\rho G_x\right]dZ(t) + \left[(\gamma-1)\theta^*(t) + \sigma\sqrt{1-\rho^2}G_x\right]dZ'(t).\end{aligned} \tag{23}$$

Applying again Ito's formula to $\xi(t) = M(t)/\eta^*(t)$ and recalling that ξ is a Q^*-martingale (hence the drift part in its Ito decomposition is null), we find

$$\begin{aligned}\frac{d\widetilde{W}^*(t)}{\widetilde{W}^*(t)} &= \frac{d\xi(t)}{\xi(t)} \\ &= \left[\gamma X(t)+\sigma\rho G_x\right]dZ^*(t) + \left[\gamma\theta^*(t)+\sigma\sqrt{1-\rho^2}G_x\right]dZ'^*(t),\end{aligned} \tag{24}$$

where Z^* and Z'^* are Q^* independent Brownian motions. The process $\widetilde{W}^*(t)$ coincides with the value of a self-financing discounted portfolio (note that, in this case, the process $\widetilde{W}^*(t)$ is an exponential martingale under Q^*, and hence it is strictly positive, namely, $\widetilde{W}^* \in \mathscr{W}(w)$) if and only if it has form (17), namely,

$$\gamma\theta^*(t) + \sigma\sqrt{1-\rho^2}G_x(T-t, X_t) = 0 \tag{25}$$

or, equivalently,

$$\theta^*(t) = -\frac{\sigma\sqrt{1-\rho^2}}{\gamma}G_x(T-t, X_t). \tag{26}$$

Lemma 3. *The function $F(T,x) = E[(\eta^*)^{1-\gamma}]$ is given by*

$$F(T,x) = e^{\widehat{A}(T)+x\widehat{B}(T)+x^2\widehat{C}(T)/2}, \tag{27}$$

where $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ are the solutions of the differential equations

$$\begin{aligned}\widehat{A}' &= \frac{\widehat{c}\widehat{B}^2}{2} + \lambda\overline{x}\widehat{B} + \frac{\sigma^2\widehat{C}}{2}, \\ \widehat{B}' &= \widehat{c}\widehat{B}\widehat{C} + \widehat{b}\widehat{B} + \lambda\overline{x}\widehat{C}, \\ \widehat{C}' &= \widehat{c}\widehat{C}^2 + 2\widehat{b}\widehat{C} + \widehat{a}\end{aligned} \tag{28}$$

with the initial conditions $\widehat{A}(0) = \widehat{B}(0) = \widehat{C}(0) = 0$, and

$$\begin{aligned}\widehat{a} &= \gamma(\gamma-1), \\ \widehat{b} &= (\gamma-1)\rho\sigma - \lambda, \\ \widehat{c} &= \frac{\sigma^2}{\gamma}\left(1+(\gamma-1)\rho^2\right)\end{aligned} \tag{29}$$

provided that T belongs to $(0,\widehat{T})$, which is the largest interval on which (28) admits bounded solutions.

Proof. We exploit the fact that the process $M(t)$ defined in (22) is a martingale, and hence the drift part in its Ito decomposition must be 0; that is,

$$\begin{aligned}&\frac{\gamma(\gamma-1)}{2}\left(X(t)^2 + \frac{\sigma^2}{\gamma^2}\left(1-\rho^2\right)G_x^2\right) - G_t - \lambda G_x(X(t)-\overline{x}) + \frac{\sigma^2}{2}\left(G_x^2+G_{xx}\right) \\ &\quad + (\gamma-1)\sigma G_x\left(\rho X(t) - \frac{\sigma}{\gamma}\left(1-\rho^2\right)G_x\right) = 0.\end{aligned} \tag{30}$$

Since the above equation must hold when applied at any pair $(\tau, X(t))$ with $\tau = T-t$, the martingale condition amounts to require that G satisfies the following partial differential equation:

$$\begin{aligned}G_t &= \frac{\sigma^2}{2}G_{xx} + \left(\frac{\sigma^2}{2} - \frac{\sigma^2(\gamma-1)\left(1-\rho^2\right)}{2\gamma}\right)G_x^2 + \left(\left((\gamma-1)\rho\sigma-\lambda\right)x + \lambda\overline{x}\right)G_x + \frac{\gamma(\gamma-1)}{2}x^2, \\ G(0,x) &= 0,\end{aligned} \tag{31}$$

for all $\tau \in [0,T]$, and for all x. If we guess a solution of the form

$$G(\tau,x) = \widehat{A}(\tau) + x\widehat{B}(\tau) + \frac{x^2\widehat{C}(\tau)}{2}, \tag{32}$$

we obtain the partial differential equations (28) for $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ with the initial conditions $\widehat{A}(0) = 0$, $\widehat{B}(0) = 0$, and $\widehat{C}(0) = 0$. □

We show now that η^* is the density of an equivalent local martingale measure and the optimal solution of the dual problem (10).

Lemma 4. *(1) $E[\eta^*] = 1$ and hence $\eta^* \in \mathcal{D}$.*

(2) The density η^* is the optimal solution to problem (10), and, as a consequence,

$$v(y) = \tilde{v}(y) = E[V(y\eta^*)] = \frac{1}{\gamma-1} y^{1-\gamma} E\left[(\eta^*)^{1-\gamma}\right]. \quad (33)$$

Proof. (1) We can follow the same argument as in the proof of Lemma 1 and apply the results in [17], recalling that $\widehat{B}$ and $\widehat{C}$ defined in (28) are bounded on $[0,T]$ when $T < \widehat{T}$ (see Appendix in [1]).

(2) We already know that the density of every equivalent martingale measure has the form $\eta^X\eta^\theta$ with $\theta \in \Theta$. Let $\theta' = \theta - \theta^*$, and observe that $\theta' \in \Theta$. Denote $\theta'^* = \theta'\theta^*$. Then

$$\begin{aligned}\eta^X\eta^\theta &= \exp\left(-\int_0^T X(s)\,dZ(s) - \int_0^T \theta(s)\,dZ'(s) - \frac{1}{2}\int_0^T \left(X^2(s) + \theta^2(s)\right) ds\right) \\ &= \exp\left(-\int_0^T X(s)\,dZ(s) - \int_0^T \left(\theta^*(s) + \theta'(s)\right) dZ'(s) - \frac{1}{2}\int_0^T \left(X(s)^2 + \left(\theta^*(s)+\theta'(s)\right)^2\right) ds\right) = \eta^* \\ &\quad \cdot \exp\left(-\int_0^T \theta'(s)\,dZ'(s) - \frac{1}{2}\int_0^T \left(2\theta'^*(s) + \theta'^2(s)\right) ds\right)\end{aligned} \quad (34)$$

with $\theta' \in \Theta$. Then

$$\begin{aligned}\tilde{v}(y) &= y^{1-\gamma} \inf_{\theta\in\Theta} E\left[\frac{1}{\gamma-1}\left(\eta^X\eta^\theta\right)^{1-\gamma}\right] = y^{1-\gamma} \\ &\quad \cdot \inf_{\theta'\in\Theta} E\left[\frac{1}{\gamma-1}(\eta^*)^{1-\gamma} \cdot e^{(\gamma-1)(\int_0^T \theta'(s)dZ'(s) + (1/2)\int_0^T (2\theta'^*(s)+\theta'^2(s))ds)}\right] = y^{1-\gamma} \\ &\quad \cdot \inf_{\theta'\in\Theta} E\left[(\eta^*)^{1-\gamma}\right] E\left[\frac{(\eta^*)^{1-\gamma}}{E\left[(\eta^*)^{1-\gamma}\right]}\frac{1}{\gamma-1} \cdot e^{(\gamma-1)(\int_0^T \theta'(s)dZ'(s) + (1/2)\int_0^T (2\theta'^*(s)+\theta'^2(s))ds)}\right] \\ &= y^{1-\gamma} E\left[(\eta^*)^{1-\gamma}\right] \inf_{\theta'\in\Theta} \widetilde{E}\left[\frac{1}{\gamma-1} \cdot e^{(\gamma-1)(\int_0^T \theta'(s)dZ'(s) + (1/2)\int_0^T (2\theta'^*(s)+\theta'^2(s))ds)}\right],\end{aligned} \quad (35)$$

where $\widetilde{E}$ is the expectation with respect to the probability measure $\widetilde{Q} = (\eta^*)^{1-\gamma}/E[(\eta^*)^{1-\gamma}]. P$. Note that the process

$$\begin{aligned}\widetilde{Z}'(t) &= Z'(t) - \int_0^t \left((\gamma-1)\theta^*(s) + \sigma\sqrt{1-\rho^2}G_x\right) ds \\ &= Z'(t) - \int_0^t \left((\gamma-1)\theta^*(s) - \gamma\theta^*(s)\right) ds \\ &= Z'(t) + \int_0^t \theta^*(s)\,ds\end{aligned} \quad (36)$$

is a $\widetilde{Q}$-Brownian motion and

$$\begin{aligned}&\int_0^T \theta'(s)\,dZ'(s) + \frac{1}{2}\int_0^T \left(2\theta'^*(s) + \theta'^2(s)\right) ds \\ &\quad = \int_0^T \theta'(s)\,d\widetilde{Z}'(s) + \frac{1}{2}\int_0^T \theta'^2(s)\,ds.\end{aligned} \quad (37)$$

Since the function $V(y) = (1/(\gamma-1))y^{1-\gamma}$ is convex, Jensen's inequality yields

$$\begin{aligned}&\widetilde{E}\left[\frac{1}{\gamma-1}e^{(\gamma-1)(\int_0^T \theta(s)d\widetilde{Z}'(s) + (1/2)\int_0^T \theta^2(s)ds)}\right] \\ &\quad \geq \frac{1}{\gamma-1}\widetilde{E}\left[e^{-\int_0^T \theta(s)d\widetilde{Z}'(s) - (1/2)\int_0^T \theta^2(s)ds}\right]^{1-\gamma} = \frac{1}{\gamma-1},\end{aligned} \quad (38)$$

where the last equality comes from $\widetilde{Z}'$ being a $\widetilde{Q}$-Brownian motion and $\theta' \in \Theta$. The equality holds, and the infimum is attained, when $\theta' \equiv 0$, that is, $\theta = \theta^*$. This shows that the density η^* is the optimal solution to the dual problem (10). □

4. Value Function and Optimal Strategy

We exploit the results of the previous section to obtain the explicit expressions of the value function and of the optimal strategy, depending on the current wealth level w, the current Sharpe ratio x, and the time to maturity T. Of course, our results coincide with Kim and Omberg's results ([1, formulas (16)–(21)]).

Theorem 5. *(1) The maximum expected utility is*

$$J(w) = U\left(we^{rT}\right) e^{A(T) + xB(T) + x^2C(T)/2}, \quad (39)$$

where $A = \widehat{A}/\gamma$, $B = \widehat{B}/\gamma$, and $C = \widehat{C}/\gamma$.

(2) The optimal monetary investment in the risky asset is

$$y^*(t) = \frac{e^{-rt} W(t) \gamma}{\Sigma(t)} \left[X(t) + \sigma\rho B(T-t) + \sigma\rho C(T-t) X(t) \right]. \quad (40)$$

Proof. (1) We replace the optimal wealth (15) in (8). The value function is then

$$\begin{aligned} J(w) &= \left(e^{rT}\right)^{\gamma/(\gamma-1)} u(w) \\ &= \left(e^{rT}\right)^{\gamma/(\gamma-1)} E\left[U\left(\widetilde{W}^*(T)(w)\right)\right] \\ &= U\left(we^{rT}\right) E\left[\left(\frac{(\eta^*)^{-\gamma}}{E\left[(\eta^*)^{1-\gamma}\right]}\right)^{(\gamma-1)/\gamma}\right] \\ &= U\left(we^{rT}\right) E\left[(\eta^*)^{1-\gamma}\right]^{1/\gamma} \\ &= U\left(we^{rT}\right) F(T,x)^{1/\gamma}. \end{aligned} \quad (41)$$

(2) Comparing (17) and (24), $d\widetilde{W}^*(t)/\widetilde{W}^*(t) = d\xi(t)/\xi(t) = [\gamma X(t) + \sigma\rho G_x] dZ^*(t)$, we obtain

$$y^*(t) \Sigma(t) = \widetilde{W}(t) \left(\gamma X(t) + \sigma\rho G_x\right). \quad (42)$$

Replacing $G_x = \widehat{B} + \widehat{C}X = \gamma(B + CX)$, the claim is proved. □

Remark 6. In our results we worked out the solution of the problem of utility maximization from terminal wealth starting at the initial date $t = 0$. The analysis extends to any date $t \in [0,T]$ by replacing unconditional expectations with time t-conditional expectations, the initial investment horizon T with the current investment horizon $T-t$, and the initial values of W and X with the current ones.

5. Conclusions

We work out a duality-based proof to the solution of the optimal incomplete-market portfolio problem studied by Kim and Omberg [1]. Our analysis offers an example where the duality approach succeeds in finding explicit results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] T. S. Kim and E. Omberg, "Dynamic nonmyopic portfolio behavior," *Review of Financial Studies*, vol. 9, no. 1, pp. 141–161, 1996.

[2] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuoustime case," *The Review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[3] D. Kramkov and W. Schachermayer, "The asymptotic elasticity of utility functions and optimal investment in incomplete markets," *The Annals of Applied Probability*, vol. 9, no. 3, pp. 904–950, 1999.

[4] D. Kramkov and W. Schachermayer, "Necessary and sufficient conditions in the problem of optimal investment in incomplete markets," *The Annals of Applied Probability*, vol. 13, no. 4, pp. 1504–1516, 2003.

[5] J. Cvitanic and I. Karatzas, "Convex duality in constrained portfolio optimization," *The Annals of Applied Probability*, vol. 2, no. 4, pp. 767–818, 1992.

[6] M. B. Haugh, L. Kogan, and J. Wang, "Evaluating portfolio policies: a duality approach," *Operations Research*, vol. 54, no. 3, pp. 405–418, 2006.

[7] I. Karatzas, J. P. Lehoczky, S. P. Sethi, and S. E. Shreve, "Explicit solution of a general consumption/investment problem," *Mathematics of Operations Research*, vol. 11, no. 2, pp. 261–294, 1986.

[8] I. Karatzas, J. P. Lehoczky, S. E. Shreve, and G.-L. Xu, "Martingale and duality methods for utility maximization in an incomplete market," *SIAM Journal on Control and Optimization*, vol. 29, no. 3, pp. 702–730, 1991.

[9] W. Schachermayer, "Optimal investment in incomplete markets when wealth may become negative," *The Annals of Applied Probability*, vol. 11, no. 3, pp. 694–734, 2001.

[10] J. Jacod and A. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, Germany, 2003.

[11] A. S. Cherny and H.-J. Engelbert, *Singular Stochastic Differential Equations*, vol. 1858 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2005.

[12] G. Kulinich and S. Kushnirenko, "Strong uniqueness of solutions of stochastic differential equations with jumps and non-Lipschitz random coefficients," *Modern Stochastics: Theory and Applications*, vol. 1, no. 1, pp. 65–72, 2014.

[13] A. Battauz, M. De Donno, and A. Sbuelz, "Reaching Nirvana with a defaultable asset?" Working Paper, 2014.

[14] I. V. Girsanov, "On transforming a certain class of stochastic processes by absolutely continuous substitution of measures," *Theory of Probability and Its Applications*, vol. 5, no. 3, pp. 285–301, 1960.

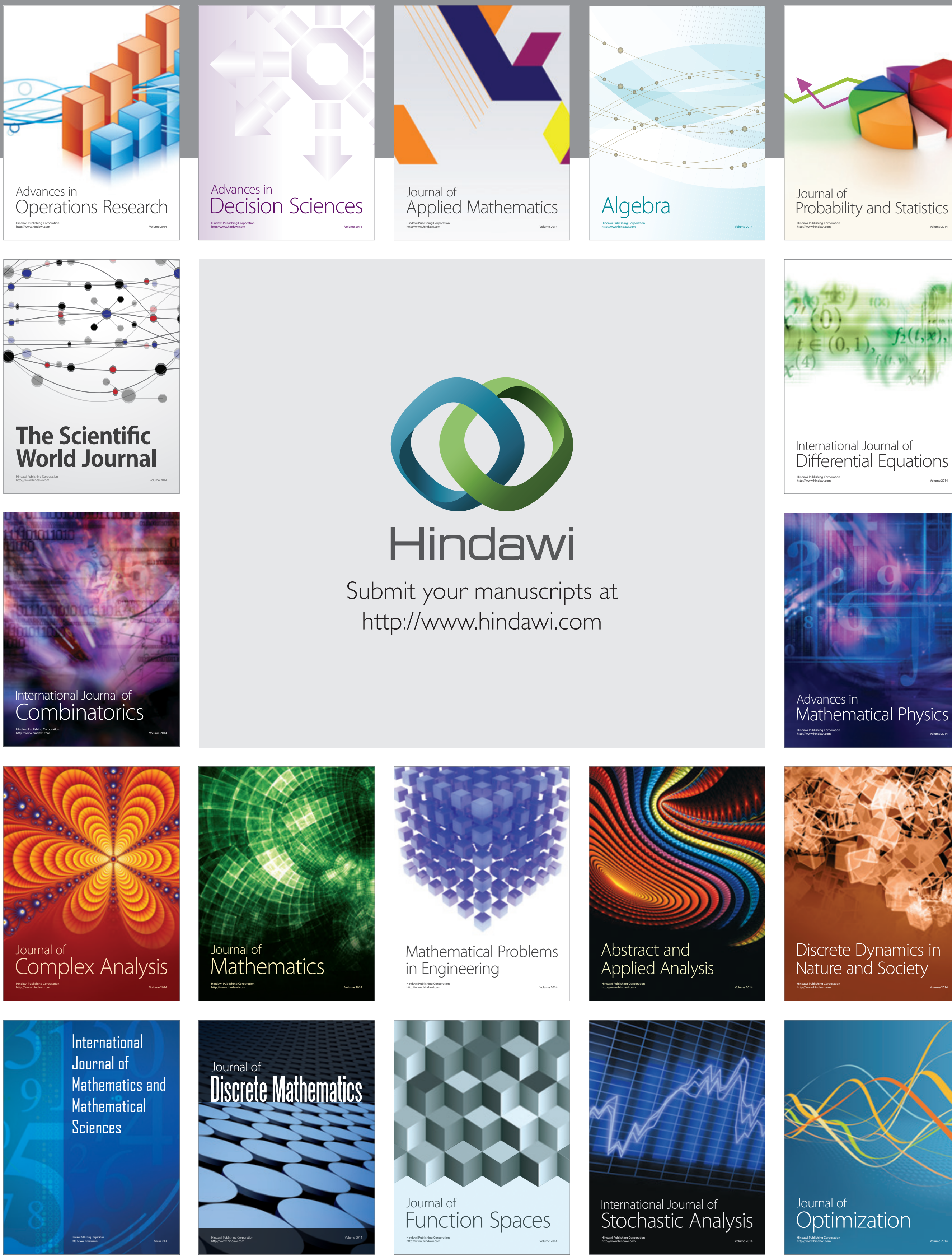
[15] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, Berlin, Germany, 1988.

[16] F. Delbaen and H. Shirakawa, "A note on option pricing for the constant elasticity of variance model," *Asia-Pacific Financial Markets*, vol. 9, no. 2, pp. 85–99, 2002.

[17] F. Klebaner and R. Liptser, "When a stochastic exponential is a true martingale. Extension of the Beneš method," *Theory of Probability and Its Applications*, vol. 58, no. 1, pp. 38–62, 2014.

[18] V. E. Beneš, "Existence of optimal stochastic control laws," *SIAM Journal on Control and Optimization*, vol. 9, pp. 446–472, 1971.

[19] J. Ruf, "The martingale property in the context of stochastic differential equations," *Electronic Communications in Probability*, vol. 20, no. 34, pp. 1–10, 2015.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