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THE PUT-CALL SYMMETRY FOR AMERICAN OPTIONS IN THE HESTON STOCHASTIC VOLATILITY MODEL

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Abstract. For the American put-call option symmetry in the Heston (1993) model, we provide a new and simple proof that is easily accessible to the general finance readership. We also characterize the link between the free-boundary of the American call and the free boundary of the symmetric American put.

Keywords: American options; stochastic volatility; put-call symmetry; free-boundary; change of numeraire.

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1. Introduction

Several authors have studied American options within the Heston (1993) model (e.g. Broadie and Kaya (2006), Andersen (2008), and Vellekoop and Nieuwenhuis (2009)). We contribute by providing a proof of the pricing parity between the American call and its symmetric American put in the Heston (1993) model that is easily accessible to a general finance audience. In the European case, the put-call parity relates the prices of European call and put options on the same underlying asset, with the same maturity and the same strike via the law of one price. In the American case the put-call parity fails, but it is possible to derive a put-call symmetry relation. Such a relation is important for pricing purposes given the size of American derivatives

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markets. Moreover such a relation is useful for the analysis of optimal decision making for real option holders (see for instance Battauz et alii (2012) and (2014)). The American put-call symmetry relation equates the price of an American call to the price of an American put by swapping the initial underlying price with the strike price and the dividend yield with the interest rate. American put-call symmetry results[1] have been obtained by Carr and Chesney (1996) and McDonald and Schroder (1998) in the absence of stochastic volatility, and by Fajardo and Mordecki (2008) in a Levy process framework. We provide a change-of-numeraire-based proof of the American put-call symmetry in the Heston (1993) stochastic volatility model (see Geman et alii (1995) for a discussion on the change of numeraire technique and Battauz (2002) for applications to American options). We also characterize the link between the free-boundary of the American call and that of the symmetric American put. Meyer (2013) offers an alternative proof based on partial differential equations.

2. The American put-call symmetry in the Heston (1993) model

In the Heston (1993) model the stock price S is described by the following stochastic differential equation with respect to the risk-neutral measure Q

$$\frac{dS(s)}{S(s)} = (r-q)\,ds + \sqrt{v(s)}dW_1(s), \quad S(0) = S_0 \text{ for any } s \geq 0 \tag{1}$$

$$dv(s) = k(\bar{v} - v(s))\,ds + \xi\sqrt{v(s)}\left(\rho dW_1(s) + \sqrt{1-\rho^2}dW_2(s)\right) \quad v(0) = v_0, \tag{2}$$

where W_1 and W_2 are two independent standard Brownian motions under the risk neutral measure Q and the filtration $\mathscr{F}$; r is the riskless interest rate; q is the dividend yield of the stock; $\sqrt{v(s)}$ is the stochastic volatility of S at time s; $\bar{v}$ is the long variance; k is the speed of mean reversion of v towards $\bar{v}$; ξ is the *vol of vol*; ρ is the correlation between S and v. We assume that $2k\bar{v} > \xi^2$, to ensure that the volatility is always positive.

We denote by $B(t) = e^{rt}$ the riskless bond at date t.

Consider now an American call option on S. Its no-arbitrage price is

$$c(t) = \text{ess}\sup_{t\leq\tau\leq T} \mathbb{E}\left[e^{-r(\tau-t)}(S(\tau)-K)^+ \middle| \mathscr{F}_t\right] \tag{3}$$

[1]For a general treatment of the European put-call symmetry see Carr and Lee (2009).

for any $t \in [0,T]$, where $\mathbb{E}[\cdot]$ denotes the (conditional) risk neutral expectation, and τ denotes a stopping time with respect to the filtration $\mathscr{F}$. It can be shown that $c(t)$ is a deterministic function of t, $S(t)$ and current levels of volatility $\sqrt{v(t)}$. With a small abuse of notations we write

$$c(t) = c(t, S(t), v(t)).$$

The function c depends on the values of the fundamental parameters. We denote such dependence by writing

$$c(t) = c(t, S(t), v(t); r, q, \overline{v}, k, \xi, \rho, K).$$

The no-arbitrage price of the American put option on S is

$$p(t) = \operatorname*{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}\left[e^{-r(\tau - t)} (K - S(\tau))^+ \middle| \mathscr{F}_t \right] \tag{4}$$

for any $t \in [0,T]$. It can be shown that $p(t)$ is a deterministic function of t, $S(t)$ and current levels of volatility $\sqrt{v(t)}$. With a small abuse of notations we write

$$p(t) = p(t, S(t), v(t)) = p(t, S(t), v(t); r, q, \overline{v}, k, \xi, \rho, K).$$

As we already anticipated, in the American case it is possible to write $c(t)$ in terms of a *symmetric American put option,* whose definition in the stochastic volatility setting is provided here follows:

Definition 2.1. (The symmetric put option) The *symmetric American put option associated to the American call option* (3) is the American put option on a Heston (1993) underlying S_{put} driven by the following equations for $s \geq t$

$$\frac{dS_{put}(s)}{S_{put}(s)} = \mu_{put} ds + \sqrt{v_{put}(s)} dW_1(s), \tag{5}$$

$$dv_{put}(s) = k_{put}(\overline{v}_{put} - v_{put}(s)) ds + \xi_{put} \sqrt{v_{put}(s)} \left(\rho_{put} dW_1(s) + \sqrt{1 - \rho_{put}^2} dW_2(s) \right), \tag{6}$$

where the values for the fundamental parameters are: $S_{put}(t) = K$, $\mu_{put} = q - r$, $v_{put}(t) = v(t)$, $\overline{v}_{put} = \frac{k\overline{v}}{k - \xi\rho}$, $k_{put} = (k - \xi\rho)$, $\xi_{put} = \xi$, $\rho_{put} = -\rho$, $r_{put} = q$, and $K_{put} = S(t)$.

In the next theorem, we provide the fundamental symmetry result that relates the time$-t$ price of the American call option $c(t)$ to the time$-t$ price of the symmetric American put option.

Theorem 2.2. ***(American put-call symmetry)*** *Consider the American call option defined in* (3) *whose value at time* $t\in[0;T]$ *is denoted with* $c(t)=c(t,S(t),v(t);r,q,\overline{v},k,\xi,\rho,K)$.

Consider the symmetric *American put option defined in Definition 2.1, whose value at time* $t\in[0;T]$ *is denoted with*

$$p(t)=p\left(t,S_{put}(t),v_{put}(t);r_{put},q_{put},\overline{v}_{put},k_{put},\xi_{put},\rho_{put},K_{put}\right).$$

The value of the American call coincides with the value of the symmetric American put as defined in Definition 2.1. More precisely, for any $0\leq t\leq T$ *we have*

$$c(t,S(t),v(t);r,q,\overline{v},k,\xi,\rho,K)=p\left(t,S_{put}(t),v_{put}(t);r_{put},q_{put},\overline{v}_{put},k_{put},\xi_{put},\rho_{put},K_{put}\right). \tag{7}$$

Moreover, given $x=S(t)$, K *and* $v=v(t)$, *for any* $\widehat{x}_{put}$, $\widehat{K}_{put}$ *such that* $\frac{x}{K}=\frac{\widehat{K}_{put}}{\widehat{x}_{put}}$ *we have that*

$$c(t,x,v;r,q,\overline{v},k,\xi,\rho,K)=\sqrt{xK}\frac{p\left(t,\widehat{x}_{put},v_{put};r_{put},q_{put},\overline{v}_{put},k_{put},\xi_{put},\rho_{put},\widehat{K}_{put}\right)}{\sqrt{\widehat{x}_{put}\widehat{K}_{put}}}, \tag{8}$$

where $\widehat{x}_{put}$ *replaces* $S_{put}(t)$ *and* $\widehat{K}_{put}$ *replaces* K_{put} *in Definition 2.1.*

Proof. Define the numeraire (see Battauz (2002)) $N(t)=S(t)e^{-(r-q)t}$, which is a $Q-$martingale, since $\frac{dN(t)}{N(t)}=\sqrt{v(t)}dW_1(t)$. The numeraire N is associated to the equivalent martingale measure $\mathbb{Q}^N$ whose density with respect to Q is $L(T)=\frac{d\mathbb{Q}^N}{d\mathbb{Q}}=\frac{N(T)}{N(0)}$. Girsanov theorem ensures that

$$dW_1^N(t)=-\sqrt{v(t)}\,dt\;+\;dW_1(t),\quad dW_2^N(t)=\;dW_2(t) \tag{9}$$

are the differentials of two standard independent $\mathbb{Q}^N$ Brownian motions.

We apply the change of numeraire to $c(t)$ in (3).

To evaluate the American call option at any t, we consider a generic stopping time $t\leq\tau\leq T$ and compute

$$\mathbb{E}\left[e^{-r(\tau-t)}(S(\tau)-K)^+\middle|\mathscr{F}_t\right]=\frac{\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)}e^{-r(\tau-t)}(S(\tau)-K)^+\middle|\mathscr{F}_t\right]}{\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)}\middle|\mathscr{F}_t\right]}$$

$=\frac{\mathbb{E}^{\mathbb{Q}^N}\left[\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)}e^{-r(\tau-t)}(S(\tau)-K)^+\middle|\mathscr{F}_\tau\right]\middle|\mathscr{F}_t\right]}{\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)}\middle|\mathscr{F}_t\right]}$, where the first equation follows from Bayes theorem, and the second from the law of iterated conditional expectation. Since $e^{-r(\tau-t)}(S(\tau)-K)^+$ is $\mathscr{F}_\tau-$measurable and $\frac{1}{L(t)}$ is a $\mathbb{Q}^N-$martingale we get

$\mathbb{E}\left[e^{-r(\tau-t)}\left(S(\tau)-K\right)^{+}\middle|\mathscr{F}_t\right]=\frac{\mathbb{E}^{\mathbb{Q}^N}\left[e^{-r(\tau-t)}(S(\tau)-K)^{+}\frac{1}{L(\tau)}\middle|\mathscr{F}_t\right]}{\frac{1}{L(t)}}$

$=L(t)\,\mathbb{E}^{\mathbb{Q}^N}\left[e^{-r(\tau-t)}\left(S(\tau)-K\right)^{+}\frac{1}{L(\tau)}\middle|\mathscr{F}_t\right].$

Recalling the definition of L we obtain

$\mathbb{E}\left[e^{-r(\tau-t)}\left(S(\tau)-K\right)^{+}\middle|\mathscr{F}_t\right]=\frac{S(t)e^{-(r-q)t}}{S(0)}\,\mathbb{E}^{\mathbb{Q}^N}\left[e^{-r(\tau-t)}\left(S(\tau)-K\right)^{+}\cdot\quad\frac{S(0)}{S(\tau)e^{-(r-q)\tau}}\middle|\mathscr{F}_t\right]$

$=\frac{S(t)}{S(0)}\mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)}\left(S(0)-\frac{S(0)K}{S(\tau)}\right)^{+}\middle|\mathscr{F}_t\right]=\mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)}\left(S(t)-\frac{S(t)K}{S(\tau)}\right)^{+}\middle|\mathscr{F}_t\right].$

Passing to the essential supremum over all stopping times $t\le\tau\le T$ we get that

$$c(t)=\operatorname*{ess\,sup}_{t\le\tau\le T}\mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)}\left(S(t)-\frac{S(t)K}{S(\tau)}\right)^{+}\middle|\mathscr{F}_t\right]. \tag{10}$$

The argument of the $\mathscr{F}_t-$expectation under $\mathbb{Q}^N$ in Equation (10) is the payoff at $\tau\ge t$ of an American put option with maturity T, interest rate $r_{put}=q$, strike $K_{put}=S(t)=x$ on the asset $S_{put}(s)=\frac{xK}{S(s)}$. Applying Ito formula we derive the stochastic differential of S_{put} for any $s\ge t$: $dS_{put}(s)=xK\cdot d\left(\frac{1}{S(s)}\right)=\frac{xK}{S(s)}\cdot\left(-(r-q)\,ds-\sqrt{v(s)}dW_1(s)+v(s)\,ds\right)=S_{put}(s)\cdot\left(-(r-q)\,ds-\sqrt{v(s)}dW_1(s)+v(s)\,ds\right)$. From Equation (9) we substitute $dW_1(s)=\sqrt{v(s)}\,ds+dW_1^N(s)$ and get $\frac{dS_{put}(s)}{S_{put}(s)}=-(r-q)\,ds-\sqrt{v(s)}\cdot\left(\sqrt{v(s)}\,ds+dW_1^N(s)\right)+v(s)\,ds$ $=(q-r)\,ds-\sqrt{v(s)}dW_1^N(s)$. Therefore the underlying of the American put option is driven under the evaluation measure $\mathbb{Q}^N$ by the "Heston (1993) dynamics" of type (1)

$$\frac{dS_{put}(s)}{S_{put}(s)}=(q-r)\,ds-\sqrt{v(s)}dW_1^N(s),$$

with $r_{put}=q$ and $q_{put}=r$. We verify now that the volatility term follows a dynamics of the same type of Equation (2). By Girsanov theorem (9), $v(s)$ is driven by

$dv(s)=k(\overline{v}-v(s))\,ds+\xi\sqrt{v(s)}\left(\rho dW_1(s)+\sqrt{1-\rho^2}dW_2(s)\right)=(k-\xi\rho)\left(\frac{k\overline{v}}{k-\xi\rho}-v(s)\right)ds$ $+\xi\sqrt{v(s)}\left(\rho dW_1^N(s)+\sqrt{1-\rho^2}dW_2^N(s)\right)$. Since $d\widehat{W}_1^N(s)=-dW_1^N(s)$ defines a standard $\mathbb{Q}^N-$Brownian motion that is $\mathbb{Q}^N-$independent of W_2^N we have that

$$\frac{dS_{put}(s)}{S_{put}(s)}=(q-r)\,ds+\sqrt{v(s)}d\widehat{W}_1^N(s)\ \text{ and}$$

$$dv(s)=(k-\xi\rho)\left(\frac{k\overline{v}}{k-\xi\rho}-v(s)\right)ds+\xi\sqrt{v(s)}\left((-\rho)\,d\widehat{W}_1^N(s)+\sqrt{1-\rho^2}dW_2^N(s)\right).$$

Therefore under $\mathbb{Q}^N$ the underlying of the put option S_{put} follows an Heston (1993) dynamics with $S_{put}(t) = K$, $\overline{v}_{put} = \frac{k\overline{v}}{k-\xi\rho}$, $k_{put} = (k - \xi\rho)$, $\xi_{put} = \xi$, and $\rho_{put} = -\rho$, as in Definition 2.1. We conclude that Equation (10) can be rewritten as

$$c(t) = \operatorname{ess\,sup}_{t\leq\tau\leq T} \mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)} \left(K_{put} - S_{put}(\tau)\right)^+ \middle| \mathscr{F}_t \right]$$
$$= p\left(t, S_{put}(t), v_{put}(t);\; r_{put}, q_{put}, \overline{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, K_{put}\right), \text{ which is } (4).$$

To prove (8), take a $\beta > 0$ such that $\widehat{K}_{put} = \frac{x}{\beta}$, is an unconstrained strike for the put option, and let $\widehat{x}_{put} = \frac{S_{put}(t)}{\beta} = \frac{K}{\beta}$. The remaining parameters for the symmetric put are

$r_{put}, q_{put}, \overline{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, K_{put}$ as before: for simplicity we omit them. By formula (7) $c(t,x,...,K) = p(t,K,...,x) = \beta p\left(t, \frac{K}{\beta}, ..., \frac{x}{\beta}\right) = \beta \cdot p\left(t, \widehat{x}_{put}, ..., \widehat{K}_{put}\right)$, where the second equality follows from the homogeneity property of the put option. Since $\beta = \frac{x}{\widehat{K}_{put}} = \frac{K}{\widehat{x}_{put}}$, writing $\beta = \sqrt{\beta \cdot \beta} = \sqrt{\frac{x}{\widehat{K}_{put}} \cdot \frac{K}{\widehat{x}_{put}}}$, we arrive at (8).

In the constant volatility framework, the optimal exercise policy for an American call option is the first time the underlying asset exceeds the critical price. The critical price is time-varying, and its graph in the plane (t,S) separating the continuation region from the immediate exercise region is called the free boundary. In the Heston (1993) model, the free boundary is a surface in the space (t,S,v). The free boundary of the American call option is linked to the free boundary of the symmetric American put option via the following theorem.

Theorem 2.3. ***(The free boundary)*** *Consider the American call option defined in* (3) *whose value at time* $t \in [0;T]$ *is denoted with* $c(t) = c(t, S(t), v(t); r, q, \overline{v}, k, \xi, \rho, K) = c(t,x,v;...,K)$. *The free boundary for the American call option at* t *and* $v = v(t)$ *is*

$$fb(t,v) = \inf\left\{x \geq 0 : c(t,x,v;...,K) = (x-K)^+\right\}.$$

Let $\widehat{K}_{put} = 1$ *and consider the* symmetric *American put option where* $\widehat{x}_{put}$ *replaces* $S_{put}(t)$ *and* $\widehat{K}_{put} = 1$ *replaces* K_{put} *in Definition 2.1 as for* (8). *The free boundary of the symmetric American put option* $v_{put}(t, \widehat{x}_{put}, v_{put}; r_{put}, q_{put}, \overline{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, 1) = v_{put}(t, \widehat{x}_{put}, v_{put}; ..., 1)$ *is*

$$fb_{put}(t, v_{put}) = \sup\left\{\widehat{x}_{put} \geq 0 : v_{put}(t, \widehat{x}_{put}, v_{put}; ..., 1) = (1 - \widehat{x}_{put})^+\right\}.$$

Then

$$fb(t,v) = K \cdot fb_{put}(t, v_{put}).$$

Proof. The parameters x, K, and $\widehat{x}_{put}$ are constrained by the equality $\frac{x}{K} = \frac{1}{\widehat{x}_{put}}$. It follows that

$$fb(t,v) = \inf\left\{ \frac{K}{\widehat{x}_{put}} \geq 0 : \sqrt{xK}\frac{p\left(t, \widehat{x}_{put}, v_{put}; \ldots, 1\right)}{\sqrt{\widehat{x}_{put}\widehat{K}_{put}}} = \left(\frac{K}{\widehat{x}_{put}} - K\right)^{+} \right\}$$

$$= K \sup\left\{ \widehat{x}_{put} \geq 0 : \sqrt{xK}\frac{v_{put}\left(t, \widehat{x}_{put} v_{put}; \ldots, 1\right)}{\sqrt{\widehat{x}_{put}}} = \frac{K}{\widehat{x}_{put}}\left(1 - \widehat{x}_{put}\right)^{+} \right\}$$

$$= K \sup\left\{ \widehat{x}_{put} \geq 0 : \sqrt{\frac{K}{\widehat{x}_{put}}K}\frac{v_{put}\left(t, \widehat{x}_{put}, v_{put}; \ldots, 1\right)}{\sqrt{\widehat{x}_{put}}} = \frac{K}{\widehat{x}_{put}}\left(1 - \widehat{x}_{put}\right)^{+} \right\},$$ since $x = \frac{K}{\widehat{x}_{put}}$. Therefore

$$fb(t,v) = K \cdot \sup\left\{ \widehat{x}_{put} \geq 0 : v_{put}\left(t, \widehat{x}_{put}, v_{put}; \ldots, 1\right) = \left(1 - \widehat{x}_{put}\right)^{+} \right\} = K \cdot fb_{put}\left(t, \widehat{x}_{put}, v_{put}; \ldots, 1\right).$$

3. Conclusions

We employ a change-of-numeraire technique to provide a new and easy proof of the pricing parity between the American call and its symmetric American put in the Heston (1993) stochastic volatility model. We work out in detail the link between the free boundaries of the symmetric American options.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] L. Andersen, Simple and efficient simulation of the Heston stochastic volatility model, J. Comput. Finance 11 (2008), 1–42.

[2] A. Battauz, Change of numeraire and American options, Stochastic Anal. Appl. 20 (2002), 709-730.

[3] A. Battauz, M. De Donno, A. Sbuelz, Real options with a double continuation region, Quantitative Finance, 12 (2012), 465-475.

[4] A. Battauz, M. De Donno, A. Sbuelz, Real options and American derivatives: The double continuation region, Management Science, in press, (2014).

[5] M. Broadie, O. Kaya, Exact simulation of stochastic volatility and other affine jump diffusion processes, Operations Res. 54 (2006), 217–231.

[6] P. Carr, M. Chesney, American Put Call Symmetry, (1996).

[7] P. Carr, R. Lee, Put-call symmetry. Extensions and applications, Math. Finance, 19 (2009), 523–560.

[8] J. Fajardo, E. Mordecki, Duality and Symmetry with Time-Changed Levy Processes, Brazilian Rev. Econometrics (2008).

[9] H. Geman, N. El Karoui, J.C. Rochet, Changes of numeraire, changes of probability measure and option pricing, J. Appl. Probability 32 (1995), 443-458.

[10] S. L. Heston, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, The Review of Financial Studies, Volume 6 (1993), 327–343.

[11] R.L. McDonald and M. Schroder, A parity result for American options, J. Comput. Fiance 1 (1998), 5-13.

[12] G. H. Meyer, Pricing Options and Bonds with the Method of Lines, (2013).

[13] M. Vellekoop, H. Nieuwenhuis, A tree-based method to price American options in the Heston model, J. Comput. Finance 13 (2009), 1–21.