

American Options and Stochastic Interest Rates

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Abstract

We study finite-maturity American equity options in a stochastic interest rate framework of Vasicek type (Vasicek (1977)). We allow for a non-zero correlation between the innovations driving the equity price and the interest rate. Importantly, we also allow for the interest rate to assume negative values, which is the case for some investment grade government bonds in Europe in recent years. In this setting we focus on American equity call and put options and characterize analytically their two-dimensional free boundary, i.e. the underlying equity and the interest rate values that trigger the optimal exercise of the option before maturity. We show that non-standard double continuation regions may appear, extending the findings obtained by Battauz et al. (2015) in a constant interest rate framework. Moreover, we contribute by developing a bivariate discretization of the equity price and interest rate processes that converges in distribution as the time step shrinks. The discretization, described by a recombining quadrinomial tree, allows us to compute American equity options' prices and their related free boundaries. In particular, we document the existence of non-standard optimal exercise policies for American call options on a non-dividend-paying equity. We also verify the existence of a non-standard double continuation region for American equity options, and provide a detailed analysis of the associated free boundaries with respect to the time and the current interest rate variables.

Keywords: Finance, Stochastic Processes, American Options, Stochastic Interest Rates.

1 Introduction

In an arbitrage-free financial market the role of the short-term interest rate is twofold: on one hand it represents the rate at which the equity price appreciates; on the other hand it drives the locally risk-free asset and the related discount rate. Therefore, neglecting the variability of short-term interest rates may induce significant mispricing on both interest rates and equity derivatives. This issue is particularly relevant when derivatives are path-dependent and therefore sensitive to the entire path of the short-term interest rate, and not just to its expected value at maturity. American equity call and put options, due to the optionality of their exercise policy, fall within this category. In fact, the holder of an American option has to timely chose when to cash in by exercising the option, balancing the effects from the discount rate and from the expected rate of return of the underlying asset. When both of these effects depend on a stochastic process, the valuation of the option becomes tricky. Our paper develops an extensive analysis of American call and put options written on equity with constant volatility in a stochastic interest rate framework of Vasicek type (see Vasicek (1977)). We employ the Vasicek mean-reverting model for the interest rate, because it allows the interest rate to assume mildly negative values, as the ones documented in recent years in the Eurozone¹. The feasibility of negative interest rates within the Vasicek model, once a source of major criticism, has very recently become the reason of renewed interest in the model itself because of the aforementioned market circumstances. We also allow for a non-zero constant correlation between the Brownian innovations of the interest rate and the equity price processes. A positive (resp. negative) correlation between the interest rate and the equity price corresponds to a negative (resp. positive) correlation between the bond price and the equity price. After 2000 the market observed persistent negative stock-bond correlation as shown by Connolly et al. (2005). Perego and Vermeulen (2016) find that the correlation between equities and bonds is now consistently negative also in the Eurozone but for Southern Europe. Thus, in line with the recent empirical evidence, in our numerical examples we consider

¹It is widely accepted to proxy the risk-free rate in Europe by the recently negative rates of German bonds.

a positive correlation between the interest rate and the equity price.

The literature on American equity options has so far focused on alternative stochastic interest rates models, such as the CIR one, based on the seminal work of Cox et al. (1985) (See Medvedev and Scaillet (2010) and Boyarchenko and Levendorskiĭ (2013)). Our paper is, to our knowledge, the first that addresses the evaluation of American equity options in a stochastic interest rate framework of Vasicek type, allowing for the possibility of negative interest rates (see Detemple (2014) for an exhaustive review of the state of the art.²)

We contribute to the literature by offering an intuitive and effective lattice method to compute the price, the optimal exercise policies and the related free boundaries of American equity options in the presence of market risk and interest rate risk. In the spirit of Cox et al. (1979), building on Nelson and Ramaswamy (1990), who provide a tree approximation for an univariate process, we construct a discrete joint approximation for the both the equity price and the interest rate processes. Hahn and Dyer (2008) develop a similar discretization for a correlated two-dimensional mean reverting process representing the price of two correlated commodities and they use it to evaluate the value of an oil and gas switching option. Our paper is different, as the mean reverting stochastic interest rate process enters the risk-neutral drift of our equity price, that has constant volatility and correlates with the interest rate. In this framework, we provide a throughout investigation of American equity call and put options and their free boundaries. Our findings contribute to the literature on American options with stochastic interest rates, that usually restricts on non-negative interest rates. In particular, we unveil two novel significant features of the free boundary that appear when the stochastic interest rate may take mildly negative values.

²Fabozzi et al. (2016) provide a new recent quasi-analytic method to price and hedge American options on a lognormal asset with constant interest rate. Jin et al. (2013) introduce a computationally effective pricing algorithm for American options in a multifactor setting. First attempts to evaluate the impact of stochastic interest rates on American derivatives date back to Amin and Jr. (1995) and Ho et al. (1997). Nevertheless, both of them proxy American with Bermudan options with few exercise dates. Although this allows them to obtain closed form solutions for both the price of the options and their optimal exercise policy, the approximation of a continuum of exercise dates by just a couple of possible exercise dates leads to a heavy mispricing of the options and provides no accurate insight on the free boundary.

First, we show that for American put (resp. call) options the early exercise region is not always downward (resp. upward) connected. The early exercise region is downward (resp. upward) connected if optimal exercise at t of the put (resp. call) option for some underlying equity price implies optimal exercise at t for all lower (resp. greater) values of the underlying equity price. In a stochastic interest rate framework Detemple (2014) retrieves the free boundary by a discretization of an integral equation for the early exercise premium decomposition. For American call options he argues that the exercise region is connected in the upward direction. Our results show that this property holds true if interest rates are always non-negative, but may fail if the interest rates' positivity assumption is not satisfied. In this case, we document the existence of a non standard double continuation region first described by Battauz et al. (2015) in a constant interest rate framework. In particular, a non-standard additional continuation region appears where the option is most deeply in the money and the underlying pays a negative dividend.³ Under these circumstances a mildly negative interest rate may lead to optimal postponement of the deeply in the money option as the holder is confident the option will still be in the money later and prefers to delay the cash-in. Second, we show that early exercise may be optimal for an American call option even if the underlying equity does not pay any dividend. This happens when a mildly negative initial interest rate causes the underlying equity's drift to be negative as well, pushing the underlying equity towards the out of the money region. In this case, immediate exercise turns out to be optimal as soon as the option is sufficiently in the money. The critical equity price that triggers optimal early exercise is increasing with respect to the interest rate value, as the higher the interest rate, the higher the underlying equity drift, the lower the risk of ending up in the out of the money region for the call option, and thus the higher has to be the immediate payoff to be optimally exercised before maturity.

The remaining of the paper is organized as follows: in Section 2 we introduce the financial market and

³A negative dividend can be interpreted as a storage cost for commodities (e.g. gold or silver) or as the result of the interplay of domestic and foreign interest rates when evaluating options on foreign equities (see Battauz et al. (2018), (2015)).

develop its lattice-based discretization, that we call *quadrinomial tree*. In Section 3 we deal with American put and call equity options in our stochastic interest rate environment, characterizing their optimal exercise policies and the main analytical features of their free boundaries. We also provide numerical pricing results for the discretized market via our quadrinomial tree, showing the pricing differences from the standard constant interest rate case, and providing a graphical characterization of the free boundaries that confirm their analytical features in the continuous-time setting. Section 4 concludes. All proofs are in the Appendix.

2 The market and the Quadrinomial Tree

2.1 The Assets in the Market

Consider a stylized financial market in a continuous time framework with investment horizon $T > 0$. A risky security $S(t)$ is traded. Following the seminal work of Vasicek (1977), we assume a mean-reverting stochastic process for the prevailing short term interest rate on the market $r(t)$. We allow for a non zero correlation between the innovations of S and r . A market player can invest in the short-term interest rate, which is locally risk-free, through the money market account $B(t)$.

The dynamics of the risky equity price, of the short-term interest rate and of the money market account under the risk-neutral measure \mathbb{Q} are:

$$\begin{cases} \frac{dS(t)}{S(t)} = (r(t) - q)dt + \sigma_S dW_S^{\mathbb{Q}}(t) \\ dr(t) = \kappa(\theta - r(t))dt + \sigma_r dW_r^{\mathbb{Q}}(t) \\ dB(t) = r(t)B(t)dt \end{cases} \quad (1)$$

with $\langle dW_S^{\mathbb{Q}}(t), dW_r^{\mathbb{Q}}(t) \rangle = \rho dt$ and given some initial conditions $S(0) = S_0$, $r(0) = r_0$ and $B(0) = 1$. The parameter q is the constant annual dividend rate of the equity, $\sigma_S > 0$ the volatility of the equity price, κ the speed of mean-reversion of the short-term interest rate, θ its long-run mean, $\sigma_r > 0$ the volatility of the short-term interest rate and $\rho \in [-1, 1]$ the correlation between the Brownian shocks on S and r .

The stochastic differential equations (SDEs) of System (1) can be rewritten equivalently in the following

vectorial specification:

$$\begin{cases} \frac{dS(t)}{S(t)} = \mu_S dt + \nu_S \cdot dW^{\mathbb{Q}}(t) \\ dr(t) = \mu_r dt + \nu_r \cdot dW^{\mathbb{Q}}(t) \end{cases} \quad (2)$$

where $\mu_S = (r(t) - q)$, $\mu_r = \kappa(\theta - r(t))$, $\nu_S = [\sigma_S \ 0]$, $\nu_r = [\sigma_r \rho \ \sigma_r \sqrt{1 - \rho^2}]$, $W^{\mathbb{Q}}(t) = [W_1^{\mathbb{Q}}(t) \ W_2^{\mathbb{Q}}(t)]'$ is a standard two-dimensional Brownian motion and \cdot is the matrix product.

The explicit solution to the System (1) is

$$\begin{cases} S(t) = S_0 \exp \left[\int_0^t r(s) ds - \left(q + \frac{\sigma_S^2}{2} \right) t + \sigma_S W_S(t) \right] \\ r(t) = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma_r \int_0^t e^{-\kappa(t-s)} dW_r(s) \\ B(t) = \exp \left[\int_0^t r(s) ds \right] \end{cases} \quad (3)$$

The zero-coupon bond with maturity T pays 1 at its holder at T and its price at $t \in (0, T)$ is labelled with $p(t, T)$. By no arbitrage valuation, we have

$$p(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{B(T)} \middle| \mathcal{F}_t \right] = e^{A(t, T) - B(t, T)r(t)}, \quad (4)$$

where the deterministic functions $A(t, T)$ and $B(t, T)$ are defined in Section 3.2.1 of Brigo and Mercurio (2007).

In this fairly general pricing framework, the price of European put and call options on S can be derived in closed formulae. Indeed, exploiting a change of numéraire as described in Geman et al. (1995), it is possible to obtain the following:

Proposition 1 (*Value of the European put/call equity option*) *In the financial market specified in (1), the price at $t \in [0, T]$ of an European put option on S with strike K is equal to*

$$\pi_E^{put}(t, S(t), r(t)) = Kp(t, T)N(-\tilde{d}_2) - S(t)e^{-q(T-t)}N(-\tilde{d}_1) \quad (5)$$

with⁴:

$$\begin{aligned}\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma_{t,T}^2}} \left(\ln \frac{S(t)}{Kp(t,T)} + \frac{1}{2} \Sigma_{t,T}^2 - q(T-t) \right), \\ \tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma_{t,T}^2}, \\ \Sigma_{t,T}^2 &= \sigma_S^2(T-t) + 2\sigma_S\sigma_r\rho \left(\frac{-1 + e^{-\kappa(T-t)} + \kappa(T-t)}{k^2} \right) + \\ &\quad -\sigma_r^2 \left(\frac{3 + e^{-2\kappa(T-t)} - 4e^{-\kappa(T-t)} - 2\kappa(T-t)}{2k^3} \right).\end{aligned}$$

The price at $t \in [0, T]$ of an European call option on S with strike K is equal to

$$\pi_E^{call}(t, S(t), r(t)) = S(t)e^{-q(T-t)}N(\tilde{d}_1) - Kp(t, T)N(\tilde{d}_2). \quad (6)$$

Proof. See Appendix B. ■

Within this market model, Bernard et al. (2008) price European barrier options by properly approximating the hitting time of the equity price of the exogenously given barrier. Unfortunately the same approach does not work when the derivative is of American style, as the barrier has to be endogenously determined. For this reason we develop in the next session the lattice discretization of our market that will allow us to compute the American free boundaries.

2.2 The Quadrinomial Tree

In their seminal work, Cox et al. (1979) show how to discretize a lognormal risky security and how to easily exploit such a binomial discretization in order to evaluate derivatives written on the primary asset. Embedding this geometric Brownian motion case into a more general class of diffusion processes, Nelson and Ramaswamy (1990) propose a one-dimensional scheme to properly define a binomial process that approximates a one-dimensional diffusion process. They do so by matching the diffusion's instantaneous drift and its variance and imposing a recombining structure to their discretized process.

⁴Notice that the current value of the interest rate $r(t)$ enters the current price of the zero-coupon bond $p(t, T)$ in \tilde{d}_1 and \tilde{d}_2 .

The discretization via a tree/lattice structure of correlated processes, possibly of different kind, is more challenging. Gamba and Trigeorgis (2007) model two or more correlated geometric Brownian motion representing the price processes of different risky assets exploiting a log-transformation of the processes first and then an orthogonal decomposition of the shocks. In this way they are able to efficiently price derivatives on five correlated assets. Moving away from lognormality, Hahn and Dyer (2008) construct a quadrinomial lattice to approximate two mean-reverting processes in order to model two correlated one-factor commodity prices and evaluate derivatives on them.

We propose here a quadrinomial tree to jointly model a mean-reverting process for the short term interest rate as suggested first by Vasicek (1977) and the process for the risky equity's price with constant volatility and the drift that embeds the stochastic interest rate as in Equation (1). Non constant short-term interest rates are surely more suitable from an option pricing perspective and an Ornstein–Uhlenbeck process enables us to investigate some interesting features of options when the discount rate becomes slightly negative, as documented in the Euro zone in recent years.

We first show how to build the discretization of the processes $(S(t), r(t))$ described by (1) and then we address the convergence issue of the discretization itself.

We apply Itô's Lemma to $Y(t) := \ln(S(t))$ and we get

$$\begin{cases} dY(t) = \mu_Y dt + \sigma_S dW_S(t) \\ dr(t) = \mu_r dt + \sigma_r dW_r(t) \end{cases} \quad (7)$$

where $\mu_Y := \left(r(t) - q - \frac{\sigma_S^2}{2} \right)$ and $\mu_r := \kappa(\theta - r(t))$. Again, the vectorial version of (2) is:

$$d \begin{bmatrix} Y(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \left(r(t) - q - \frac{\sigma_S^2}{2} \right) \\ \kappa(\theta - r(t)) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 \\ \sigma_r \rho & \sigma_r \sqrt{1 - \rho^2} \end{bmatrix} \cdot dW(t). \quad (8)$$

We refer here to the general technique of Section 11.3 of Stroock and Varadhan (1997) exploiting the very convenient notation introduced in Section 3.2.1 of Prigent (2003). For the ease of the reader we recall here

their template. Consider the following bivariate SDE:

$$dX(t) = \mu(x, t)dt + \sigma(x, t) \cdot dW(t) \quad (9)$$

where $X(t)_{t \geq 0} = (Y(t), r(t))_{t \geq 0}$, $W(t)$ is a standard two-dimensional Brownian motion, $\mu(x, t) : (\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, $\sigma(x, t) : (\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}^+ \rightarrow \mathbb{R}^{2 \times 2}$ and an initial condition $X(0) = (x_0, r_0)$ is given.

Consider a discrete uniform partition of the time interval $[0, T]$ like $\{i \frac{T}{n}, i = 1, \dots, n\}$ and define $\Delta t := \frac{T}{n}$. For each n consider a bivariate stochastic process $\{X_n\}$ on $[0, T]$ which is constant between the nodes of the partition. At each node, both of the components of X_n jump up (or down) a certain distance with a certain probability. The sizes of the jumps and the probabilities are allowed to be time-dependent and state-contingent. Since after any jump each component of X_n can assume two new different values, there will be globally four possible outcomes after each jump. For sake of clarity, fix n and consider the generic i -th step of the bivariate discrete process $X_i = (Y_i, r_i)$. In the following step, the process can assume only one of the following four values:

$$(Y_{i+1}, r_{i+1}) = \begin{cases} (Y_i + \Delta Y^+, r_i + \Delta r^+) & \text{with probability } q_{uu} \\ (Y_i + \Delta Y^+, r_i + \Delta r^-) & \text{with probability } q_{ud} \\ (Y_i + \Delta Y^-, r_i + \Delta r^+) & \text{with probability } q_{du} \\ (Y_i + \Delta Y^-, r_i + \Delta r^-) & \text{with probability } q_{dd} \end{cases} \quad (10)$$

where $\Delta Y^\pm, \Delta r^\pm$ are the jumping increments and the four transition probabilities are both time-dependent and state-contingent. Figure (1) provides a graphical intuition for the bivariate binomial discretization over one step.

Globally, there are 8 parameters to pin down at each step: $\Delta Y^\pm, \Delta r^\pm, q_{uu}, q_{ud}, q_{du}$ and q_{dd} .

In order to obtain a discretization that converges in distribution to the solution of (9), we need to match the first two moments of $Y(t)$ and $r(t)$ as well as their cross variation. In this way we impose five conditions on the eight parameters we need to determine. One more constraint has to be imposed on the four transition probabilities that need to sum up to one. Finally, we may want to impose a recombining

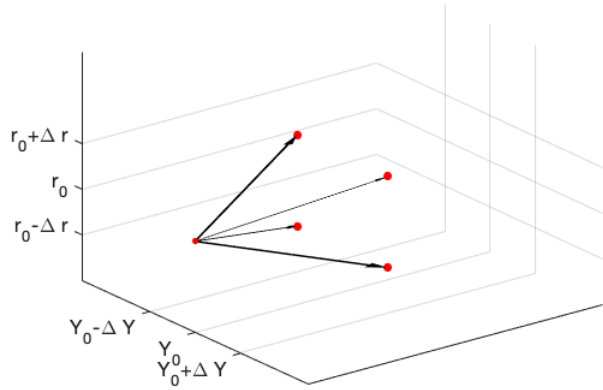


Figure 1: One step of the bivariate binomial discretization.

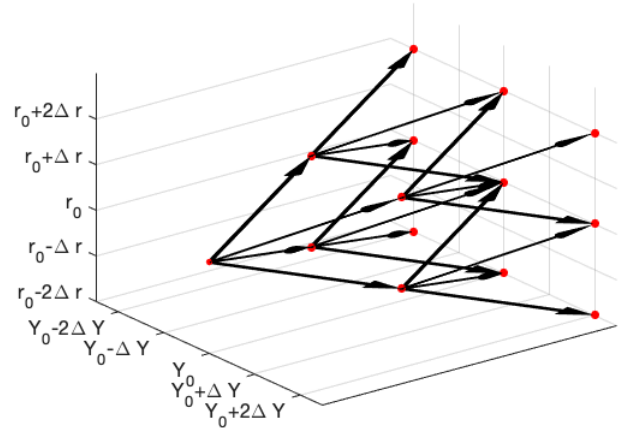


Figure 2: Two steps of the bivariate binomial discretization.

structure to our quadrinomial tree in order to preserve tractability. Setting $\Delta Y^- = -\Delta Y^+ := \Delta Y$ and $\Delta r^- = -\Delta r^+ := \Delta r$ makes the number of different outcomes of our discretization grow quadratically (and non exponentially) in the number of steps. Figure (2) provides a graphical intuition of this trick: starting from (Y_0, r_0) , after two steps the bivariate binomial process may assume nine possible values, namely all the possible ordered couples of $\{Y_0 - 2\Delta Y, Y_0, Y_0 + 2\Delta Y\}$ and of $\{r_0 - 2\Delta r, r_0, r_0 + 2\Delta r\}$. Thus, for a generic number of time steps n , the final possible outcomes of the discretization are $(n + 1)^2$ rather than 2^{n+1} , the number of possible outcomes along a non recombining tree.

We now derive the explicit expressions of the increments and of the transition probabilities of our discretization for the bivariate SDEs (2).

Matching the first two moments of $Y(t)$ and $S(t)$ as well as their cross-variation, neglecting the Δt -second order terms, imposing the proper constraint on the probabilities and imposing a recombining tree as ex-

plained above lead to the following system of eight equations in eight unknowns:

$$\left\{ \begin{array}{l} \mathbb{E}_t[\Delta Y] = (q_{uu} + q_{ud})\Delta Y^+ + (q_{du} + q_{dd})\Delta Y^- \stackrel{!}{=} \mu_Y \Delta t \\ \mathbb{E}_t[\Delta r] = (q_{uu} + q_{du})\Delta r^+ + (q_{ud} + q_{dd})\Delta r^- \stackrel{!}{=} \mu_r \Delta t \\ \mathbb{E}_t[\Delta Y^2] = (q_{uu} + q_{ud})(\Delta Y^+)^2 + (q_{du} + q_{dd})(\Delta Y^-)^2 \stackrel{!}{=} \sigma_S^2 \Delta t \\ \mathbb{E}_t[\Delta r^2] = (q_{uu} + q_{du})(\Delta r^+)^2 + (q_{ud} + q_{dd})(\Delta r^-)^2 \stackrel{!}{=} \sigma_r^2 \Delta t \\ \mathbb{E}_t[\Delta Y \Delta r] = q_{uu}\Delta Y^+ \Delta r^+ + q_{ud}\Delta Y^+ \Delta r^- + \\ \quad + q_{du}\Delta Y^- \Delta r^+ + q_{dd}\Delta Y^- \Delta r^- \stackrel{!}{=} \rho \sigma_S \sigma_r \Delta t \\ \quad \quad \quad q_{uu} + q_{ud} + q_{du} + q_{dd} \stackrel{!}{=} 1 \\ \quad \quad \quad \Delta Y^+ \stackrel{!}{=} -\Delta Y^- \\ \quad \quad \quad \Delta r^+ \stackrel{!}{=} -\Delta r^- \end{array} \right.$$

Imposing $\Delta Y^+ > \Delta Y^-$ and $\Delta r^+ > \Delta r^-$ we get:

$$\begin{aligned} \Delta Y^+ &= \sigma_S \sqrt{\Delta t} = -\Delta Y^- := \Delta Y \\ \Delta r^+ &= \sigma_r \sqrt{\Delta t} = -\Delta r^- := \Delta r \end{aligned} \tag{11}$$

$$\begin{aligned} q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r + \mu_r \Delta Y + (1 + \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S} \\ q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r - \mu_r \Delta Y + (1 - \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S} \\ q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r + \mu_r \Delta Y + (1 - \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S} \\ q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r - \mu_r \Delta Y + (1 + \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S}. \end{aligned} \tag{12}$$

As noted in Nelson and Ramaswamy (1990), the four transition probabilities are not necessarily positive.

In the limit, namely as $\Delta t \rightarrow 0$, we have $\Delta Y, \Delta r \rightarrow 0$ and, therefore, $q_{uu}, q_{dd} \rightarrow \frac{(1+\rho)}{4} > 0$ and $q_{ud}, q_{du} \rightarrow \frac{(1-\rho)}{4} > 0$. For $\Delta t > 0$, some of the four probabilities in (12) may become non-positive. In Appendix A we derive conditions leading to four non-negative probabilities is (12).

Once we obtained a discretization of (2), we can map back the rate of return $Y(t)$ to the level $S(t)$ of the equity. In this way we have a lattice discretization of the solution of (2) and we name this discretization *quadrinomial tree*.

The following theorem shows that our quadrinomial tree converges in distribution to the solution of (2).

Theorem 2 (Convergence of the quadrinomial tree) *The bivariate discrete process $(X_i)_i$ defined in (10) with the parameters in (11) and (12) converges in distribution to $X(t) = (Y(t), r(t))$.*

Proof. See Appendix B. ■

3 American Options

In this section we focus on American equity put (resp. call) options, whose final payoff is $\varphi(S) := (K - S)^+$ (resp. $\varphi(S) := (S - K)^+$). The value at $t \leq T$ of the American equity option with maturity T is:

$$\begin{aligned} V(t) &= \text{ess sup}_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}} \left[\frac{B(t)}{B(\tau)} \varphi(S(\tau)) \middle| \mathcal{F}_t \right] \\ &= \text{ess sup}_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r(s) ds} \varphi(S(\tau)) \middle| \mathcal{F}_t \right] \end{aligned} \quad (13)$$

where τ ranges among all possible stopping times (see for instance Chapter 21 in Björk (2009)). It is well known that Equation (13) admits no explicit formulation even in the standard Black and Scholes market. In our case, since r is stochastic and not independent of S , we cannot split the conditional expected value in (13) into two simpler separate ones. Nevertheless, the following proposition shows that the value of an American option $V(t)$ can be expressed as a deterministic function of time t (or, equivalently, of time to maturity $T - t$) and of the current value of both the underlying asset $S = S(t)$ and the short term interest rate $r = r(t)$. Moreover, this deterministic function inherits the same monotonicity properties with respect to t and S variables as in the constant interest rate environment. We also prove that the American equity put option is decreasing with respect to the current value of the interest rate r , whereas the American equity call option is increasing with respect to r . An increase in r has a direct effect on American equity options via the discounting of future cashflows, that becomes more severe. It has also an indirect effect, channelled through the equity drift, that increases if r increases. For an American equity put option this implies that the likelihood of lower payoffs increases. Thus an increase in r diminishes the value of an American equity put option. On the contrary, for an American equity call option, the drift

increase determined by the increase of r pushes the underlying equity towards higher payoffs' regions, thus potentially increasing the call option value. We show that this positive effect prevails over the negative effect of the increased discounting, and the American call option is actually increasing with respect to r . This result is achieved by a change of the numeraire for the call option, that allows to disentangle the opposite effects of the interest rate's increase on the discounting and the call payoffs' likelihood.

Proposition 3 (*Value of the American option as a deterministic function*) *In the market described by (1), the value of an American put option on S (13) is of the form:*

$$V(t) = F(t, S(t), r(t))$$

with $F : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ given by:

$$F(t, S, r) = \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^{\eta} r(s) ds \right) \cdot \varphi \left(S \exp \left(\int_0^{\eta} r(s) ds - \left(q + \frac{1}{2} \sigma_S^2 \right) \eta + \sigma_S W_S(\eta) \right) \right) \right]. \quad (14)$$

The function F is decreasing with respect to time t , convex with respect to S and increasing (resp. decreasing) in the call (resp. put) case. Moreover, F is increasing (resp. decreasing) in the call (resp. put) case with respect to r .

Proof. See Appendix B. ■

As the American equity option value is a deterministic function of (t, S, r) , at each $t \in [0, T]$, the plane $(S, r) \in \mathbb{R}^+ \times \mathbb{R}$ can be divided into two complementary regions:

- the continuation region $CR(t) = \{(S, r) \in \mathbb{R}^+ \times \mathbb{R} : F(t, S, r) > \varphi(S)\}$, the set of couples (S, r) where it is optimal to continue the option at t ; the r -section of the continuation region at t is $CR_r(t) = \{S \in \mathbb{R}^+ : F(t, S, r) > \varphi(S)\}$;

- the early exercise region $EE R(t) = \{(S, r) \in \mathbb{R}^+ \times \mathbb{R} : F(t, S, r) = \varphi(S)\}$, the set of couples (S, r) where it is optimal to exercise the option at t ; the r -section of the early exercise region at t is $EE R_r(t) = \{S \in \mathbb{R}^+ : F(t, S, r) = \varphi(S)\}$.

The boundary separating the continuation region and the early exercise region as t varies in $[0, T]$ is a surface called *free boundary* in the three-dimensional space (t, S, r) . In Theorem 5 we describe the main features of the free boundary surface, that can be single (the standard case) or double, generalizing the results of Battauz et al. (2015). In the constant interest rate environment, the American equity option value depends only on time t and the equity value S . Thus, the early exercise region and the continuation region are separated by a one-dimensional free boundary, called *critical price*. When the interest rate is positive, there is always a single (if any) free boundary. Consider the put option, in the limiting case where $S = 0$. Its value is $\sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} [e^{-r\eta}(K - 0)^+] = K$. Thus the value of the put option and its immediate payoff coincide at $S = 0$ for any t . Convexity and value dominance of the option value imply that, if early exercise is optimal at t for some underlying value S , then it is optimal to exercise as well for all values of the underlying that are smaller than S . The early exercise region is the area bounded from above by the single critical price $t \mapsto \bar{S}^*(t)$ and is therefore downward connected (see Figure 1 in Battauz et al. (2015)). On the contrary, if $r < 0$, then the American put value when $S = 0$ is $\sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} [e^{-r\eta}(K - 0)^+] = K e^{-r(T-t)} > K$. Thus if early exercise is optimal at t , the exercise region stops strictly above $S = 0$. In a neighborhood of $S = 0$ a deeply in the money continuation region appears, bounded from above by the new lower critical price $t \mapsto \underline{S}^*(t)$ (see Figure 2 in Battauz et al. (2015) and the related comments for a discussion on the economic intuition behind this result). The exercise region for the American put option in this case is no more downward connected.

A similar reasoning applies also in our stochastic interest rate setting. If $S = 0$ the value of the American put option defined in Equation (14) becomes $F(t, 0, r) = \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} [\exp(-\int_0^\eta r(s)ds) \cdot K]$. Suppose that there exist some deterministic η such that $p(0, \eta) > 1$. Then $F(t, 0, r) \geq K \cdot p(0, \eta) > K$. In this case,

if early exercise is optimal at t, r for some value of S , then the early exercise region will be bounded from below by a strictly positive equity value. A non-standard continuation region then appears when the put is most deeply in the money.

The following proposition formalizes this intuition for both American put and call options, and provides necessary conditions for the existence of optimal early exercise opportunities when the current interest rate value determines the existence of a zero-coupon-bond price greater than 1. This is very likely to occur when the current interest rate value is non-positive.

Proposition 4 (*Asymptotic necessary conditions for the existence of optimal early exercise opportunities*) *In the market described by (1), at any point in time t and given the current value of the interest rate $r(t) = r$, suppose that*

$$[NC0] \quad r\alpha - \theta(\alpha + (T - t)) > 0 \text{ with } \alpha = \frac{e^{-\kappa(T-t)} - 1}{\kappa} \leq 0$$

Then the following are jointly necessary conditions for the existence of optimal exercise opportunities at t , for $T - t \approx 0$:

$$[NC1] \quad \text{the dividend yield is non positive, } q \leq 0;$$

$$[NC2] \quad \text{for some } S, \pi_E(t, S, r) = \varphi(S), \text{ where } \pi_E(t, S, r) \text{ is the value of the European put (resp. call) option defined in Proposition 1.}$$

Proof. See Appendix B. ■

The condition $[NC0]$ is very likely satisfied when $r < 0$, as the long-run mean of the interest rate θ is commonly assumed to be positive. $[NC1]$ ensures that the discounted price of the risky security is not a supermartingale. If this was the case, we show in the proof that, under condition $[NC0]$, this would lead automatically to optimal exercise of the American put option at maturity only. For the American put option, if early exercise is optimal under condition $[NC0]$, then EER_r , the early exercise region section at

r , is bounded by below by a strictly positive (non standard) lower boundary. A similar reasoning works for American equity call options. We remark that our results cannot be obtained from standard symmetry results for American options (see Battauz et al. (2015) and the references therein) due to the stochasticity of our interest rates. In the standard Black-Scholes case, the American put-call symmetry swaps the constant interest rate with the constant dividend yield. Being our interest rate stochastic and our dividend yield constant, such symmetry result is not viable.

Under $[NC0]$, $[NC2]$ ensures that the price of the European option $\pi_E(t, S, r)$ does not dominate the immediate payoff value. If this was the case, the American option would dominate the immediate payoff value as well, thus preventing the existence of optimal early exercise opportunities. Although the formal proof of the necessary conditions in Proposition 4 requires the time to maturity to be small enough, we show in the following section that actually those conditions correctly identify nodes on the tree in which a double continuation region appears along the whole lifetime of the option (see Figure 5).

In the following theorem we describe the main properties of the free boundary surface, under the assumption that the early exercise region is non-empty. We distinguish between the standard case of a non-negative interest rate and the case of a negative interest rate, where unusual optimal continuation policies may appear.

Theorem 5 (*The free-boundary surface*)

1. Suppose $r \geq 0$ and assume that $EER_r(\bar{t})$ is non-empty for some $\bar{t} \in (0, T)$.

For the American put option

$$\bar{S}^*(t, r) = \sup \{S \geq 0 : F(t, S, r) = \varphi(S)\} \tag{15}$$

defines the (standard upper) free boundary and early exercise is optimal at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ if $S(t) \leq \bar{S}^(t, r)$. The free boundary $\bar{S}^*(t, r)$ is increasing with respect to $t \geq \bar{t}$ and $r \geq 0$.*

For the American call option

$$\underline{S}^*(t, r) = \inf \{S \geq 0 : F(t, S, r) = \varphi(S)\} \quad (16)$$

defines the (standard lower) free boundary and early exercise is optimal at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ if $S(t) \geq \underline{S}^*(t, r)$. The free boundary $\underline{S}^*(t, r)$ is decreasing with respect to $t \geq \bar{t}$ and increasing with respect to $r \geq 0$.

2. Suppose $r < 0$ and that the necessary conditions of Propositions 4 are satisfied with $q < 0$ and assume that $EER_r(\bar{t})$ is non-empty. Then the segment with extremes $[\underline{S}^*(t, r), \bar{S}^*(t, r)]$ (see Equations (15), (16)) is non-empty for any $t \in [\bar{t}, T]$. The option is optimally exercised at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ whenever $S(t) \in [\underline{S}^*(t, r), \bar{S}^*(t, r)]$. The lower free boundary, $\underline{S}^*(t, r)$, is decreasing with respect to t and the upper free boundary $\bar{S}^*(t, r)$ is increasing with respect to t for any $t \geq \bar{t}$.

For the American put

$$\frac{rK}{q} \leq \underline{S}^*(t, r) < \bar{S}^*(t, r) \leq K.$$

Their limits at maturity are $\lim_{t \rightarrow T} \bar{S}^*(t, r) = K = \bar{S}^*(T, r)$ and $\underline{S}^*(T^-, r) = \lim_{t \rightarrow T} \underline{S}^*(t, r) = \frac{rK}{q} > \underline{S}^*(T, r) = 0$. The lower free boundary, $\underline{S}^*(t, r)$, is decreasing with respect to r and the upper free boundary $\bar{S}^*(t, r)$ is increasing with respect to r .

For the American call

$$K \leq \underline{S}^*(t, r) < \bar{S}^*(t, r) \leq \frac{rK}{q}.$$

Their limits at maturity are $\lim_{t \rightarrow T} \underline{S}^*(t, r) = K = \underline{S}^*(T, r)$ and $\bar{S}^*(T^-, r) = \lim_{t \rightarrow T} \bar{S}^*(t, r) = \frac{rK}{q} < \bar{S}^*(T, r) = +\infty$. The lower free boundary, $\underline{S}^*(t, r)$, is increasing with respect to r and the upper free boundary $\bar{S}^*(t, r)$ is decreasing with respect to r .

3. Suppose $r < 0$ and $q = 0$. Then the early exercise region for the American put option at t is empty. For the American call, suppose $EER_r(\bar{t})$ is non-empty for some $\bar{t} \in (0, T)$. Then early exercise is

optimal at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ if $S(t) \geq \underline{S}^*(t, r)$ (see Equation (16)). The free boundary $\underline{S}^*(t, r)$ is decreasing with respect to $t \geq \bar{t}$ and increasing with respect to $r \geq 0$

Proof. See Appendix B. ■

3.1 Numerical examples

We now present and describe three illustrative numerical examples that show the optimal exercise strategies and the possible characterizations of the continuation region for the American put and call options in the market described by (1), highlighting the free boundary's features derived in Theorem (5).

We exploit our quadrinomial tree to evaluate American options by backward induction. Once the whole quadrinomial tree, namely all the couples (S, r) and the related transition probabilities, have been generated, we start from the values of the state variables S and r at maturity T . At maturity, the American option is exercised in all the nodes in which it is in the money; the resulting payoff is the value of the American option at T . At any other generic instant $t \in \{0, \Delta t, 2\Delta t, \dots, T - \Delta t\}$, and for any couple $(S(t), r(t))$, we compute the immediate payoff $\varphi(S)$ and we compare it to the continuation value of the option. The continuation value is obtained as the discounted (by the current realization of $r(t)$) expected value (according the transition probabilities computed at $(S(t), r(t))$) of the four values of the American option at $t + \Delta t$ connected on the tree to the current node. From the comparison between the immediate exercise and the continuation value, we get the value of the American option in the node $(S(t), r(t))$. Going backward, we finally get the price of the American option at $t = 0$.

Theorem 2 showed that the quadrinomial tree we proposed converges in distribution to the bivariate process that solves (1), as the time step shrinks. Mulinacci and Pratelli (1998) prove that the convergence in distribution of the lattice-based approximation of the underlying state variables implies that the price of the American option evaluated according to the backward procedure described above converges to its theoretical value given by (13).

In all of the three following examples the parameters are: $T = 1$, $n = 125$, $S_0 = K = 1$, $\sigma_S = 0.15$, $r_0 = 0$, $\theta = 0.02$, $\kappa = 1$, $\sigma_r = 0.01$ and $\rho = 0.05$. The dividend yield q of the equity is the only parameter that varies across the examples: in the first one we set $q = 0$, in the second $q = 0.02$ and $q = -0.02$ in the last one.

For each example we:

- compute the value at inception of the European counterpart π_E obtained both with the formula of Proposition 1 and along the quadrinomial tree (the values obtained in the two ways are indistinguishable);
- compute the value at inception of the American option π_A along the quadrinomial tree;
- compute the price of the American option, $\pi_A^{r_0}$, evaluated along the standard binomial tree of Cox et al. (1979) with a deterministic interest rate $r = r_0 = 0^5$. Our aim is to quantify the error that an “unsophisticated” investor would make by evaluating American options within a flat term structure framework rather than within a fluctuating one;
- graphically show how the single, or double (if any), free boundaries look like in the space (t, S, r) . These graphs characterize the optimal exercise policy: at any t , the investor should look at the current values of $(S(t), r(t))$;
- graphically highlight the nodes of the quadrinomial tree where the necessary conditions of Proposition 4 are satisfied.

We first show the numerical results for the American put option that are summed up in Table 1.

⁵we also evaluate the American option with a deterministic interest rate equal to the expected value of r over the investment period; namely, we also set $r = \mathbb{E}^{\mathbb{Q}}[r(T)] = 1.26\%$. This exercise delivers qualitatively similar results. With respect to the last column of Table 1, the relative errors in this case are, respectively, 4.58%, 4.64%, 4.52%.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$ \pi_A - \pi_A^{r_0} /\pi_A$
3	0%	5.620%	5.712%	5.979%	4.67%
4	2%	6.565%	6.570%	6.962%	5.96%
5	-2%	4.763%	5.030%	5.230%	3.97%

Table 1: Results from the three numerical examples for the American put option.

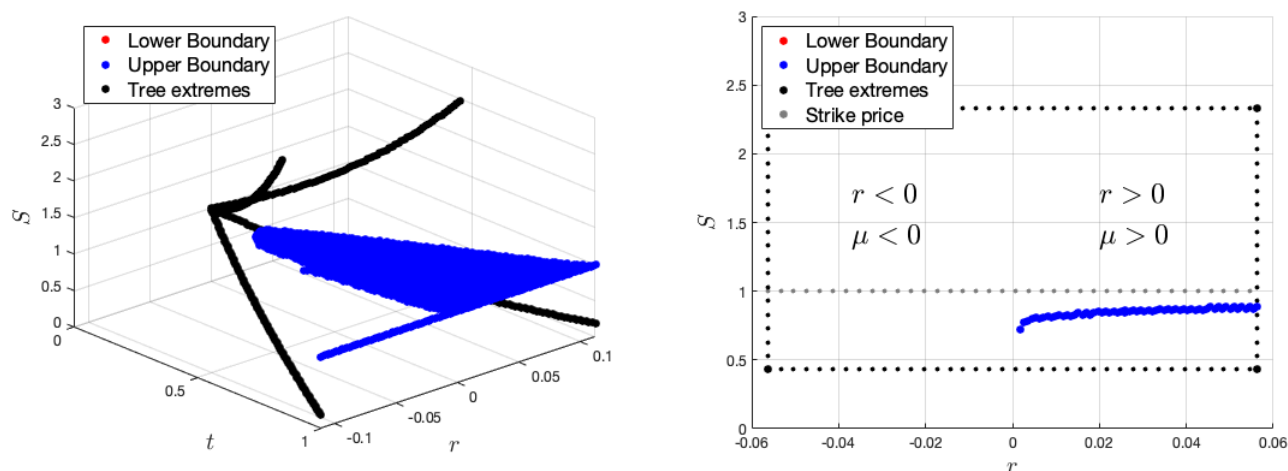


Figure 3: First example, American put: $q = 0\%$.

First example: $q = 0\%$. If the underlying pays no dividend and its volatility is reasonably small, the expected drift of S basically coincides with $r(t) = r$. This splits the domain of r in two complementary regions according to the sign of r , as can be seen in the right panel of Figure 3 (that displays the free boundary section at $t = \frac{T}{2}$). In the left region where r and $\mu = r - q - \frac{\sigma_S^2}{2}$ are both negative, the investor is willing to wait and postpone the exercise as much as possible in order to gain from both the negative discount rate and the implied expected depreciation of S . In the right region, on the contrary, where r and μ are both positive, we have the standard tradeoff between a positive discount rate (that makes the investor willing to exercise the option as soon as possible) and a negative expected drift of S (that makes the investor willing to wait for a larger payoff). This generates the standard upper boundary shown in

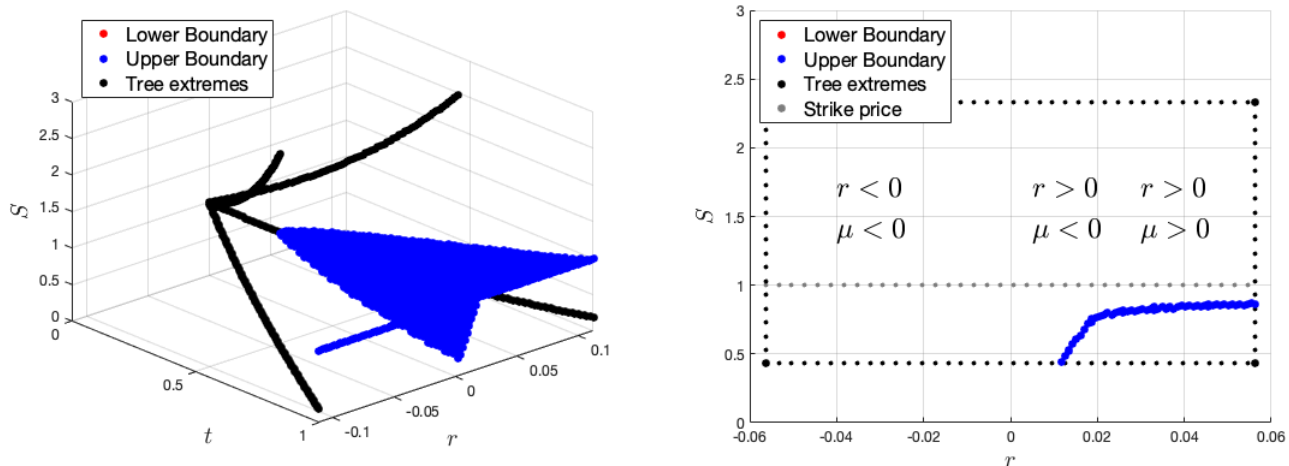


Figure 4: Second example, American put: $q = 2\%$.

the left panel of Figure 3. We notice that the standard upper boundary is increasing with respect to r . Indeed, early exercise is more profitable when r increases and S is likely to appreciate.

The investor who believes that the term structure is flat and evaluates the American put option with a constant discount rate equal to our r_0 makes a relative error almost equal to 5%. This figure is economically significant as it is greater than the maximal error due to suboptimal exercise delay of the option as estimated⁶ in Chockalingam and Feng (2015).

Second example: $q = 2\%$. If the underlying pays (positive) dividends, the drift of S is equal to r plus a negative quantity ($-q - \frac{\sigma_S^2}{2} < 0$). This splits the domain of r into three complementary regions. The first one in which r and μ are both negative, the one in which r is positive but small so that μ is still negative, the last one in which r and μ are both positive. In the first one, the option is optimally exercised at maturity, as before. In the middle region there is a new tradeoff: the investor would like to cash in as soon

⁶Our relative error of 4.67% in the first line of Table 1 corresponds to an absolute pricing error of 27.8 bps. This figure is indeed significant compared to the maximal error obtained in Figure 3 by Chockalingam and Feng (2015). In particular, Figure 3, second row, right column, in Chockalingam and Feng (2015), displays a pricing error of 4 bps, after a rescaling to unit moneyness and with volatility equal to 20%.

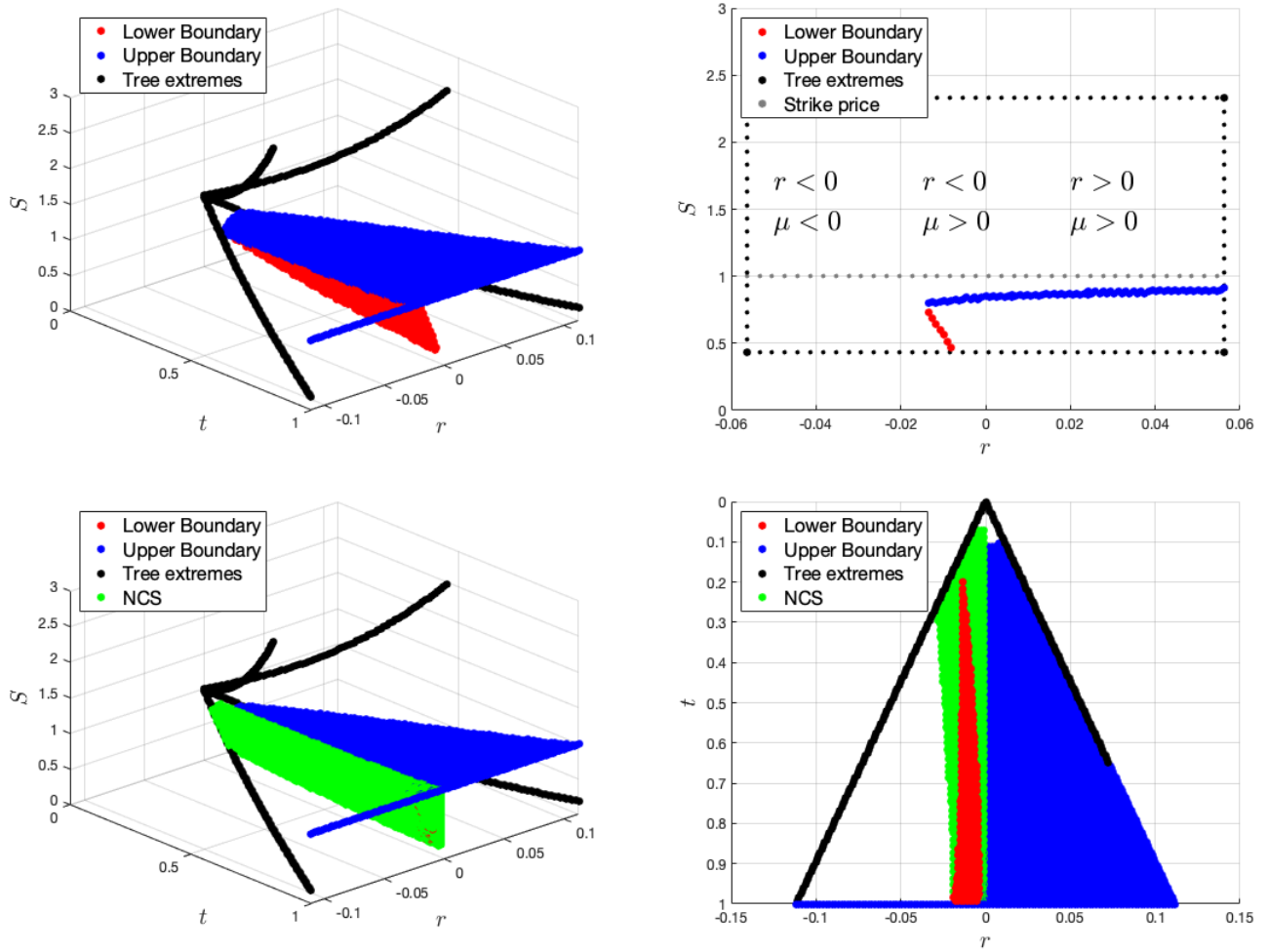


Figure 5: Third example, American put: $q = -2\%$. Green points in the bottom panels show the nodes of the quadrinomial tree in which necessary conditions $[NC0]$, $[NC1]$ and $[NC2]$ of Proposition 4 for a double continuation region hold simultaneously.

as possible due to $r > 0$ but the value of S is expected to decrease as $\mu < 0$. This allows for a standard upper boundary. The critical price below which the investor will exercise, though, becomes smaller as r approaches 0: as r decreases the threat of the positive discount rate weakens and, therefore, the investor would postpone the exercise unless the underlying reaches a very low level. In other words, if the discount is not that strong, the investor prefers to gain the relative high dividend yield keeping the asset as long as

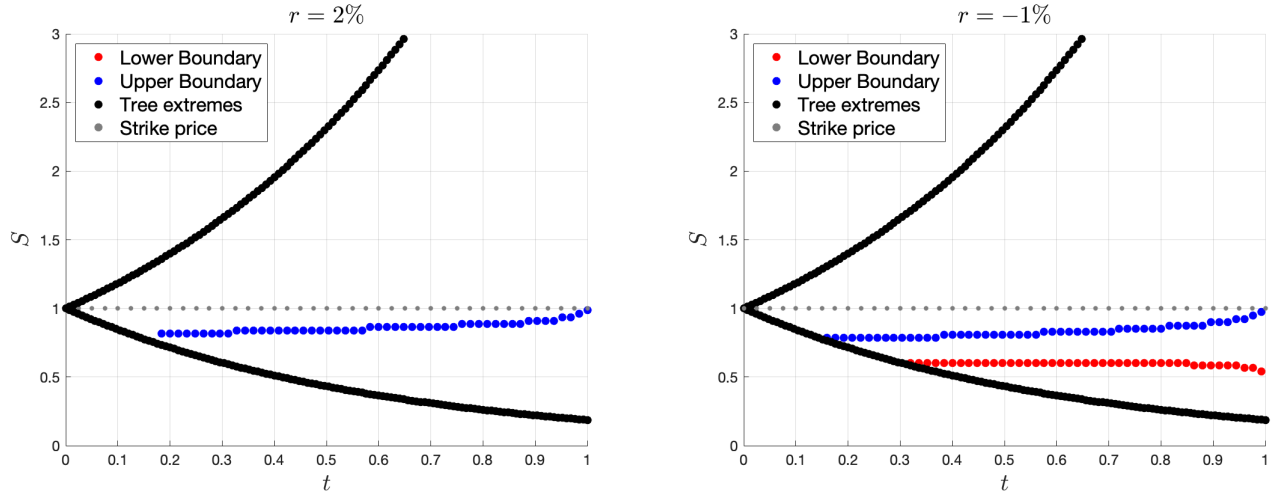


Figure 6: r -sections of free boundaries for the American put option. Left panel $r = 2\%$ and $q = 0\%$. Right panel $r = -1\%$ and $q = -2\%$.

possible. In the last region, we find the standard behaviour already outlined in the first example.

The investor who believes that the term structure is flat and evaluates the American option with a constant interest rate makes here an even higher relative error than before (5.96%).

Third example: $q = -2\%$. In the case of negative dividends⁷, the drift of S is equal to r plus a quantity which is now positive ($-q - \frac{\sigma_S^2}{2} > 0$). As a result, μ may be positive also when r is mildly negative. This splits again the domain of r into three complementary regions, as shown in the top-right panel of Figure 4: the one in which r and μ are both negative, the one in which r is negative but μ is positive and the last one in which r and μ are both positive. In the first region, the option is again optimally exercised at maturity as in the previous examples. In the middle section a double continuation region appears: this is the case in which the necessary conditions in Proposition 4 are satisfied as documented in the bottom panels of Figure 4. To the best of our knowledge, this is the first paper that documents the existence of a non standard double free boundary in a stochastic interest rates framework, generalizing the result

⁷As previously discussed, negative dividends might model storage and insurance cost for commodities such as gold or domestic risk-neutral drifts of foreign equities in quanto options.

obtained in the constant interest rates setting by Battauz et al. (2015). In the last region where both r and μ are positive, we find the standard behaviour already outlined in the first two examples.

We conclude our analysis of the American put option's free boundaries, by displaying in Figure 6 their time-dependence structure. In particular, we show that, for fixed values of r , the upper critical price of the American put is increasing with respect to time t whereas the lower critical price (if any) is decreasing, as already proved in Theorem 5 and documented in the constant interest rate framework by (Battauz et al., 2015).

In Appendix C we also document the impact of the correlation on the American equity options' prices.

We now turn to the American call options. Numerical pricing results for the American call option in the same scenarios analysed above for the American put option are summed up in Table 2. We notice that in all cases the investor who believes that the term structure is flat and evaluates the American call option with a constant discount rate equal to our r_0 makes a relative error between 4% and 5%.

It is well known that American call options on non-paying dividend assets do not display any early exercise premium. This is true under usual market circumstances, i.e. when interest rates are non negative. In fact, in this case, the zero-coupon bonds of any maturity have initial prices that are smaller than one, i.e. $p(0, \tau) < 1$ for any $\tau \in [0, T]$. This implies that the option is optimally exercised at maturity only, as Jensen's inequality implies that

$$\mathbb{E}^{\mathbb{Q}} \left[(S(\tau) - K)^+ e^{-\int_0^\tau r(s) ds} \right] \geq (S(0) - Kp(0, \tau))^+ > (S(0) - K)^+ .$$

The same holds true if S pays a negative dividend yield as $\mathbb{E}^{\mathbb{Q}} \left[S(\tau) e^{-\int_0^\tau r(s) ds} \right] = S(0) e^{-q\tau} > S(0)$.

Within our framework, interest rates are not always positive and zero-coupon bonds may have initial prices larger than one. Thus, early exercise may be optimal under some circumstances as one can indeed see in the following first example.

First example: $q = 0\%$. As explained above, early exercise may be optimal in this case only if zero-coupon

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$ \pi_A - \pi_A^{r_0} /\pi_A$
7	0%	6.339%	6.339%	5.979%	5.69%
8	2%	5.314%	5.396%	5.163%	4.32%
9	-2%	7.511%	7.511%	7.102%	5.44%

Table 2: Results from the three numerical examples for the American call option.

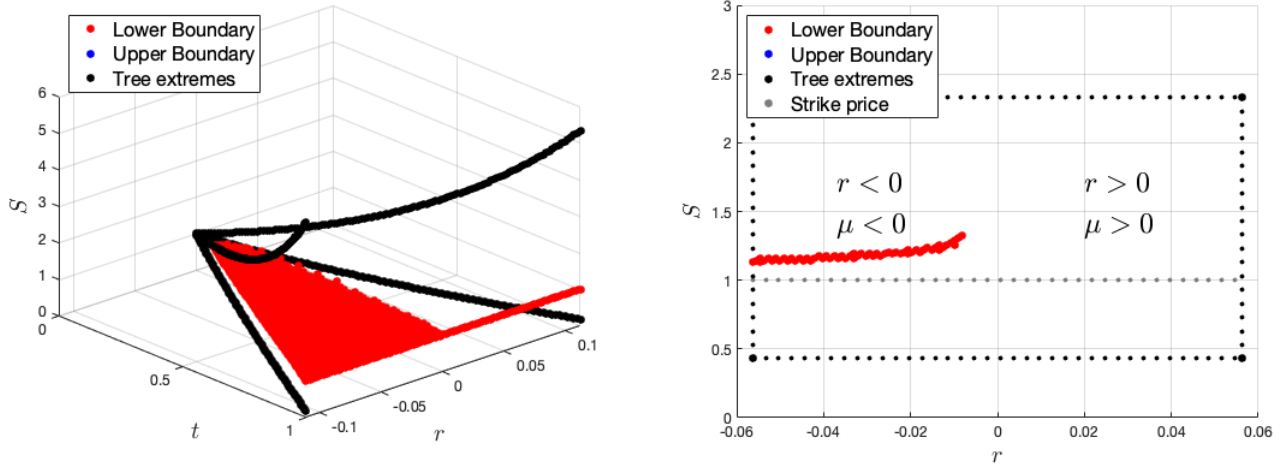


Figure 7: First example, American call: $q = 0\%$.

bonds display initial prices larger than one for some maturity. This is the case portrayed in Figure 7, where a (standard lower) free boundary for the American call option is documented for initial interest rates values smaller than -1% . To our knowledge, this is the first paper that shows the existence of optimal early exercise opportunities for an American call option when the dividend yield is zero. We notice that the critical price, and thus the continuation region, is increasing in r , as the increasing drift μ of S pushes the option towards the in the money region. The impact of these optimal early exercise opportunities on the price of the option, however, is negligible because the risk-neutral probability of the equity price entering the early exercise region is quite small, as one can see from the first row of Table 2.

Second example: $q = 2\%$. When the dividend yield is positive, early exercises of the American call option

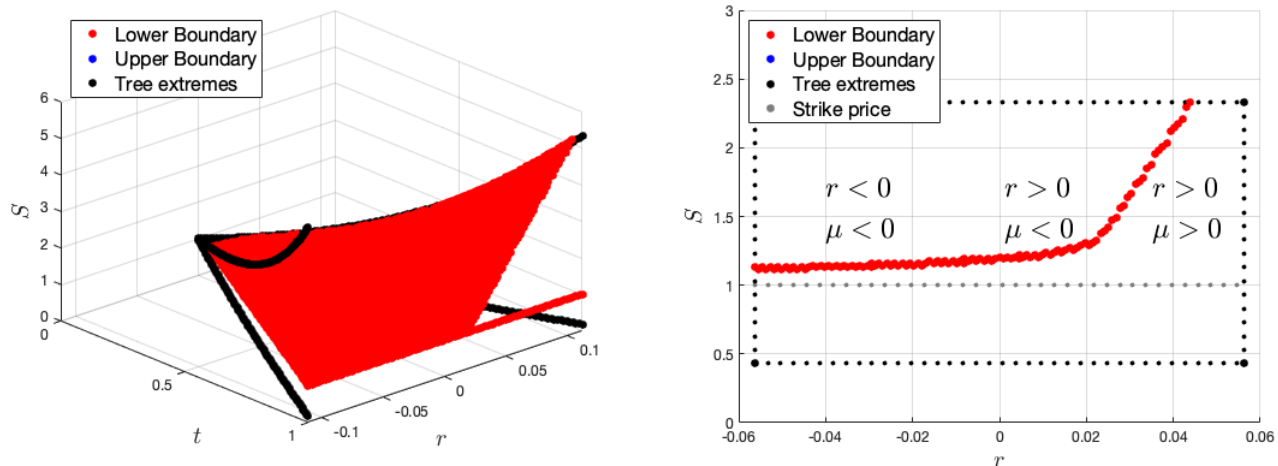


Figure 8: Second example, American call: $q = 2\%$.

become profitable. In Figure 8 we document the existence of a (lower standard) free boundary that is again increasing in r . Interestingly, the slope of the free boundary becomes steeper when μ , the drift of S , turns positive, and the continuation region increases substantially as S is expected to appreciate. Consequently, early exercise in this case is optimal only if S is very deeply in the money.

Third example: $q = -2\%$. As already discussed for the American put option example, when the dividend yield is negative, the instantaneous drift of S , μ , is always positive but for very negative values of r . As a result, early exercise for the American call option is never optimal unless r is very negative. In this case, for negative values of r , a non standard early exercise region appears surrounded by two continuation regions (see the top panels of Figure 9). However, as in the first example with $q = 0\%$, the early exercise premium does not significantly contribute to the price of the American call option because the equity price enters the early exercise region with a very small risk-neutral probability, as one can see from the third row of Table 2. The green dots in the bottom panels of Figure 9 mark the region where our necessary conditions for non standard early exercise of Proposition 4 are satisfied. We notice that this region overlaps very accurately with the area where early exercise is optimal as portrait in the top-left panel of Figure 9. We conclude our analysis of the American call option's free boundaries, by portraying in Figure 10 their

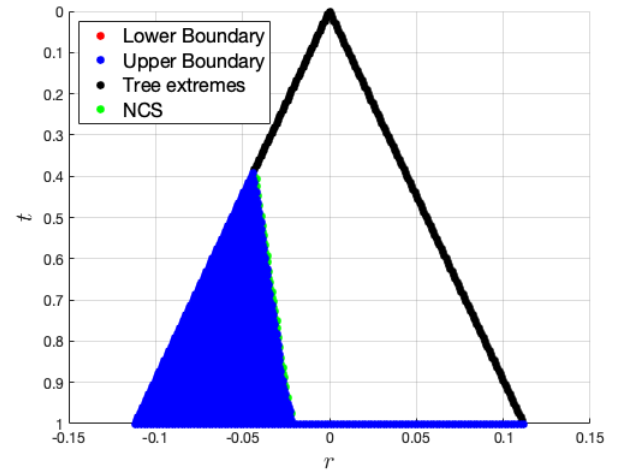
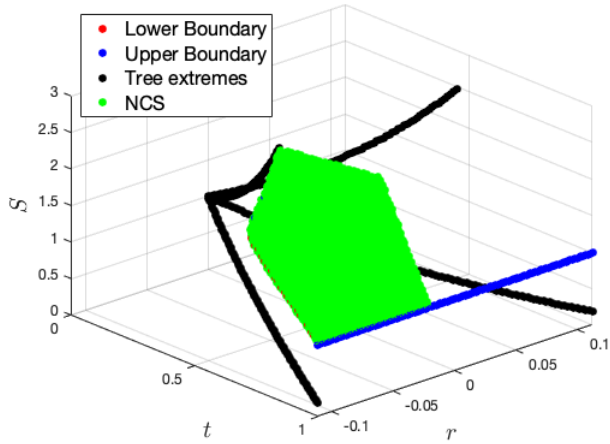
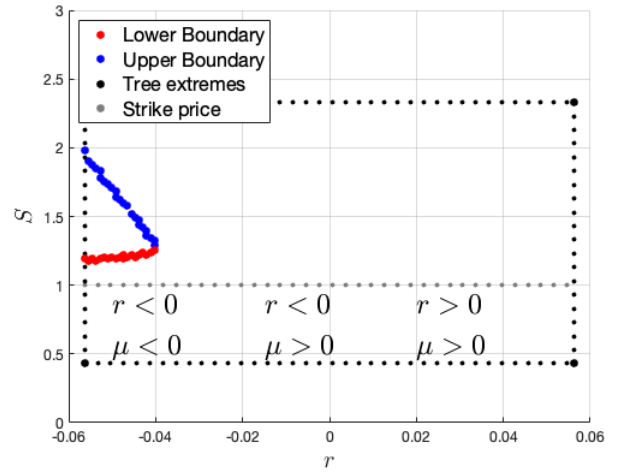
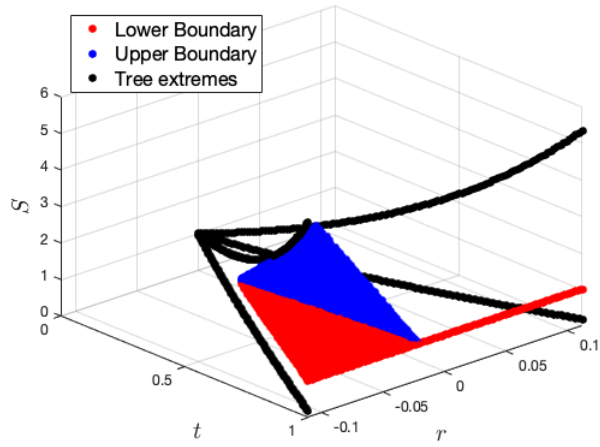


Figure 9: Third example, American call: $q = -2\%$. Green points in the bottom panels show the nodes of the quadrinomial tree in which necessary conditions $[NC0]$, $[NC1]$ and $[NC2]$ of Proposition 4 for a double continuation region hold simultaneously.

time-dependence structure. In particular, we see that for American call options the upper critical price (if any) is increasing with respect to time t whereas the lower critical price is decreasing (see Figure 10), thus confirming the results of Theorem 5 and of (Battauz et al., 2015) in a constant interest rate framework.

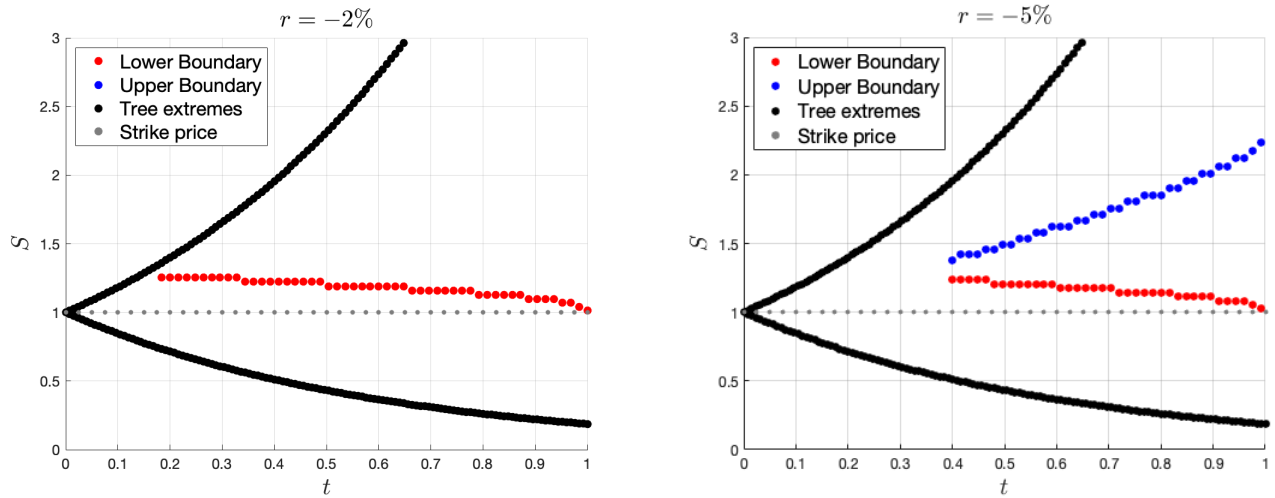


Figure 10: r -sections of free boundaries for the American call option. Left panel $r = -2\%$ and $q = 0\%$. Right panel $r = -5\%$ and $q = -2\%$.

4 Conclusions

In this paper we have studied American equity options in a correlated stochastic interest rate framework of Vasicek (1977) type. We have introduced a tractable lattice-based discretization of the equity price and interest rate processes by means of a quadrinomial tree. Our quadrinomial tree matches the joint discretized moments of the equity price and the stochastic interest rate and converges in distribution to the continuous time original processes. This allowed us to employ our quadrinomial tree to characterize the two-dimensional free boundary for American equity put and call options, that consists of the underlying asset and the interest rate values that trigger the optimal exercise of the option. Our results are in line with the existing literature when interest rates lie in the positive realm. In particular, for the American put options, the higher the dividend yield, the higher the benefits from deferring the option exercise. Moreover, in this case, the exercise region is downward connected with respect to the underlying asset value. On the contrary, when interest rate are likely to assume even mildly negative values, optimal exercise policies change, depending on the tradeoff between the interest rate and the expected rate of return on the equity

price. If such expected rate of return is negative, optimal exercise occurs at maturity only as the option goes (on average) deeper in the money as time goes by and the negative interest rates make the investor willing to cash in as late as possible. If the expected rate of return on the equity asset is positive, the option is expected to move towards the out of the money region. This effect is compensated by the preference to postponement due to negative interest rates. The tradeoff results in a non-standard double continuation region that violates the aforementioned downward connectedness of the exercise region for American put option. We quantified the pricing error that an investor would make assuming a constant interest rate and therefore neglecting the variability (and the related risk) of the term structure. Finally, we documented similar non standard optimal exercise policies also for American call options. In particular, we find that early exercise of the American call option might be optimal even when the equity does not pay any dividend. This results confirm the analytical features of the free boundaries retrived in Theorem 5 for the continuous-time framework described in System (1).

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Appendix

A Bounds of the probabilities in the Quadrinomial Tree

Recall that at each t the four probabilities of an upward/downward movement of r/Y on the tree are:

$$\begin{aligned}
 q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1 + \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S} \\
 q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1 - \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S} \\
 q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1 - \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S} \\
 q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1 + \rho) \sigma_r \sigma_S}{4 \sigma_r \sigma_S}
 \end{aligned} \tag{A1}$$

with $\Delta r^+ = \sigma_r \sqrt{\Delta t}$, $\Delta Y^+ = \sigma_S \sqrt{\Delta t}$, $\mu_Y = r(t) - q - \frac{\sigma_S^2}{2}$ and $\mu_r = \kappa(\theta - r(t))$. From now on we light the notation writing r instead of $r(t)$. Nevertheless, it is crucial to remember that these probabilities are different for each node of the quadrinomial tree.

As already pointed out, the four probabilities sum up to one by construction. Unfortunately, they do not necessarily lie in $(0,1)$. As a first control, we investigate what happens as the length of the time step goes to zero, namely, as $\Delta t \rightarrow 0$. We have

$$\begin{aligned}
 \lim_{\Delta t \rightarrow 0} q_{uu} &= \lim_{\Delta t \rightarrow 0} q_{dd} = \frac{1 + \rho}{4}, \\
 \lim_{\Delta t \rightarrow 0} q_{ud} &= \lim_{\Delta t \rightarrow 0} q_{du} = \frac{1 - \rho}{4}
 \end{aligned}$$

which are all positive quantities (at least as $\rho \in (-1,1)$). Therefore, the problem of having possibly negative probabilities is only due to the discretization procedure.

For instance, with $n = 250$ steps and $T = 1$ (that corresponds to $\Delta t = 0.004$), we need to impose the positivity constraint on all the four numerators in (A1).

Imposing $q_{uu} \geq 0$ and solving with respect to r leads to:

$$A_{uu} r^2 + B_{uu} r + C_{uu} \leq 0 \tag{A2}$$

where:

$$\begin{aligned}
A_{uu} &= \kappa \\
B_{uu} &= -\kappa \left(\theta + q + \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}} \right) - \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{uu} &= -\kappa \theta \left(-q - \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}} \right) - \frac{\sigma_r}{\sqrt{\Delta t}} \left(-q - \frac{\sigma_S^2}{2} \right) - \frac{(1+\rho)\sigma_r\sigma_S}{\Delta t}.
\end{aligned}$$

Provided that the discriminant of Equation (A2) is positive, which surely holds true as $\Delta t \rightarrow 0$, the solution is $\underline{r}_{uu} \leq r \leq \bar{r}_{uu}$, where, of course,

$$\underline{r}_{uu} = \frac{-B_{uu} - \sqrt{B_{uu}^2 - 4A_{uu}C_{uu}}}{2A_{uu}} \quad \text{and} \quad \bar{r}_{uu} = \frac{-B_{uu} + \sqrt{B_{uu}^2 - 4A_{uu}C_{uu}}}{2A_{uu}}.$$

Similarly, we can work out all of the other probabilities.

Imposing $q_{ud} \geq 0$ leads to:

$$A_{ud}r^2 + B_{ud}r + C_{ud} \geq 0$$

where:

$$\begin{aligned}
A_{ud} &= \kappa \\
B_{ud} &= -\kappa \left(\theta + q + \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}} \right) + \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{ud} &= -\kappa \theta \left(-q - \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}} \right) - \frac{\sigma_r}{\sqrt{\Delta t}} \left(q + \frac{\sigma_S^2}{2} \right) + \frac{(1-\rho)\sigma_r\sigma_S}{\Delta t},
\end{aligned}$$

that is solved by $r \leq \underline{r}_{ud} \cup r \geq \bar{r}_{ud}$.

Imposing $q_{du} \geq 0$ leads to:

$$A_{du}r^2 + B_{du}r + C_{du} \geq 0$$

where:

$$\begin{aligned}
A_{du} &= k \\
B_{du} &= -\kappa \left(\theta + q + \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}} \right) - \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{du} &= -\kappa \theta \left(-q - \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}} \right) + \frac{\sigma_r}{\sqrt{\Delta t}} \left(q + \frac{\sigma_S^2}{2} \right) + \frac{(1-\rho)\sigma_r\sigma_S}{\Delta t},
\end{aligned}$$

that is solved by $r \leq \underline{r}_{du} \cup r \geq \bar{r}_{du}$.

Finally, imposing $q_{dd} \geq 0$ leads to:

$$A_{dd}r^2 + B_{dd}r + C_{dd} \leq 0$$

where:

$$\begin{aligned} A_{dd} &= \kappa \\ B_{dd} &= -\kappa \left(\theta + q + \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}} \right) + \frac{\sigma_r}{\sqrt{\Delta t}} \\ C_{dd} &= -\kappa \theta \left(-q - \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}} \right) + \frac{\sigma_r}{\sqrt{\Delta t}} \left(-q - \frac{\sigma_S^2}{2} \right) - \frac{(1 + \rho)\sigma_r\sigma_S}{\Delta t}. \end{aligned}$$

that is solved by $\underline{r}_{dd} \leq r \leq \bar{r}_{dd}$.

Summing up, probabilities in (A1) stay positive as long as r satisfies:

$$\left\{ \begin{array}{l} \underline{r}_{uu} \leq r \leq \bar{r}_{uu} \\ r \leq \underline{r}_{ud} \cup r \geq \bar{r}_{ud} \\ r \leq \underline{r}_{du} \cup r \geq \bar{r}_{du} \\ \underline{r}_{dd} \leq r \leq \bar{r}_{dd} \end{array} \right.$$

The solution to the previous system of inequalities depends on the sign of the correlation ρ . Given the sign of ρ , the eight extremes values $\underline{r}_{uu}, \underline{r}_{ud}, \dots, \bar{r}_{du}, \bar{r}_{dd}$ always satisfy the same chain of inequalities. Furthermore, notice that this eight values depend only on the parameters of the model and not on t .

When $\rho \in (0, 1]$, the only interval on which all of the inequalities hold true is $\bar{r}_{ud} \leq r \leq \underline{r}_{du}$ as it can be conveniently seen in Figure 11.

The intuition here is that when r and S move together and the discretization of r reaches values far away from its long run mean θ , a further movement of r away from θ and in the opposite direction of S is extremely unlikely and, eventually, happens “with a negative probability”.

If, for example, $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$ and $n = 125$, after $m = 100$ steps, namely at $t = m \cdot \Delta t = m \cdot \frac{T}{n} = 0.8$, $r(t)$ spans the interval $[-0.0885, 0.0885]$ and $S(t)$ the interval $[-1.3282, 1.3282]$, both of them assuming $m = 101$ different

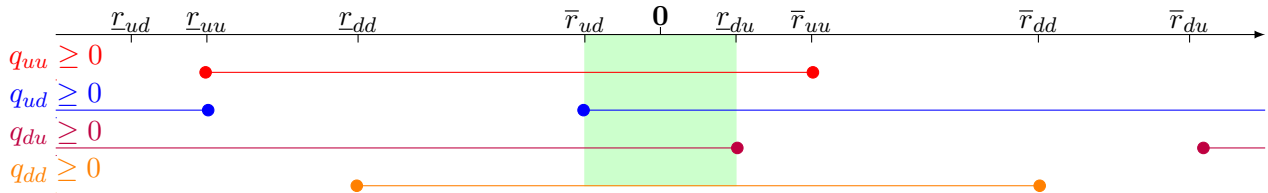


Figure 11: Graphical solution to the system of inequalities when $\rho \in (0, 1]$.

values. Hence, at $t = 0.8$ there are $101^2 = 10201$ possible nodes on tree. As an instance, at the node $(S(t), r(t))_{t=0.8} = (0.5847, -0.0751)$ the four probabilities are:

$$q_{uu} = 0.4885$$

$$q_{ud} = -0.0143$$

$$q_{du} = 0.2780$$

$$q_{dd} = 0.2478.$$

Indeed, with the given parameters, probabilities are all positive as long as $\bar{r}_{ud} = -0.0660 \leq r(t) \leq 0.0861 = r_{du}$, which is not our case. As $r(t)$ is extremely far away from its long-run mean and since $\rho > 0$ implies that r and S are likely to move together in the same direction, q_{ud} , namely the probability that r deviates even further from its long-run mean and also against S , becomes negative. Notice that $q_{ud} > q_{dd}$, meaning that the force that drives r towards its long-run mean prevails on the positive correlation between the two processes.

When such a scenario happens, we adjust the probabilities by setting the negative one to 0 and normalizing to 1 the others. From the example above we would then get:

$$q_{uu} = 0.4816$$

$$q_{ud} = 0$$

$$q_{du} = 0.2741$$

$$q_{dd} = 0.2443.$$

A very similar situation happens when $\rho \in [-1, 0)$ and the four probabilities stay positive as long as $\underline{r}_{dd} \leq r \leq \bar{r}_{uu}$. Figure 12 shows the solution to the system of inequalities in this case. Now q_{uu} or q_{dd} might become negative. This is due to the negative correlation: as r and S are likely to move in the opposite direction, when r is far away from its long-run mean, moving even further in the same direction of S may result in a negative probability. Again, we correct for such a phenomenon with the normalization described above.

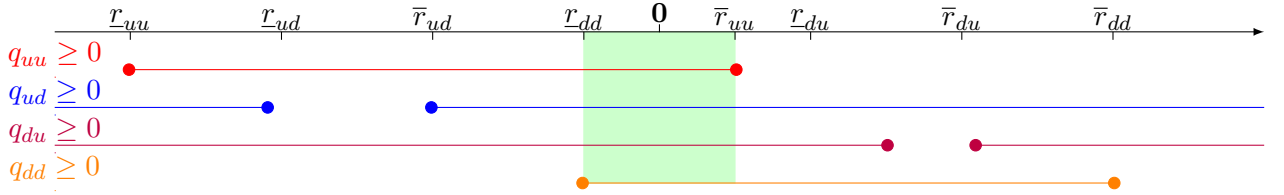


Figure 12: Graphical solution to the system of inequalities when $\rho \in [-1, 0)$.

For sake of completeness, we briefly discuss also the limit of zero correlation between r and S . When $\rho = 0$, $\underline{r}_{uu} = \underline{r}_{ud}$, $\bar{r}_{ud} = \underline{r}_{dd}$, $\bar{r}_{uu} = \underline{r}_{du}$ and $\bar{r}_{du} = \bar{r}_{dd}$. Hence, the two intervals we found for the two previous cases, $\bar{r}_{ud} \leq r \leq \underline{r}_{du}$ when $\rho \in (0, -1]$ and $\underline{r}_{dd} \leq r \leq \bar{r}_{uu}$ when $\rho \in [-1, 0)$, coincide. When $\rho = 0$, probabilities stay positive as long as r belong to that interval.

Since the support of the discretization of $r(t)$ is known at each t , we can retrieve the maximum t before which no normalization of the probabilities is needed.

Given the two thresholds \underline{r} and \bar{r} (where $\underline{r} = \bar{r}_{ud}$, $\bar{r} = \underline{r}_{du}$ if $\rho > 0$ and $\underline{r} = \underline{r}_{dd}$, $\bar{r} = \bar{r}_{uu}$ if $\rho < 0$) we can set \underline{t} and \bar{t} as:

$$\underline{t} := \min_{s \in \{0, \Delta t, 2\Delta t, \dots, T\}} \{r(s) \geq \underline{r}\} \quad \text{and} \quad \bar{t} := \max_{s \in \{0, \Delta t, 2\Delta t, \dots, T\}} \{r(s) \leq \bar{r}\}.$$

Given the binomial structure of the discretization, after m steps we have we have:

$$r(0) - m\Delta r^- = r(0) - m\sigma_r \Delta t \leq r(t) \leq r(0) + m\sigma_r \Delta t = r(0) + m\Delta r^+$$

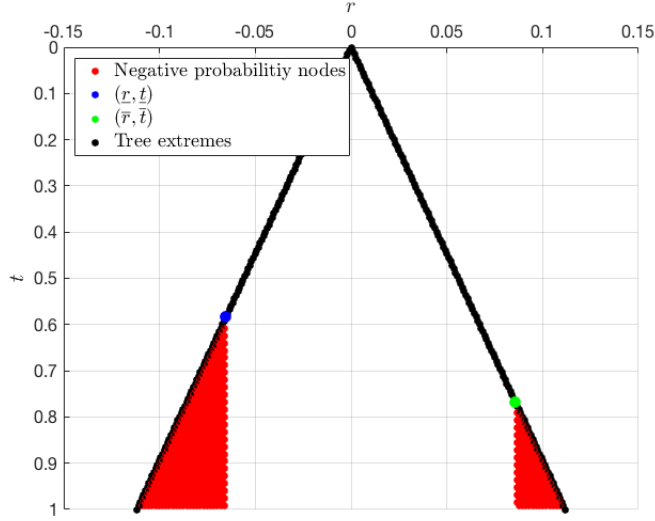


Figure 13: Section of the quadrinomial tree for $S = 0$. Red points indicate nodes at which one transition probability becomes negative. Parameters: $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$, $n = 125$.

and, therefore, from

$$\begin{aligned} r(0) - \underline{m}\sigma_r\Delta t &\geq \underline{r} \\ r(0) + \bar{m}\sigma_r\Delta t &\leq \bar{r} \end{aligned} \tag{A3}$$

we can explicitly compute:

$$\begin{aligned} \underline{t} = \underline{m}\Delta t &= \frac{r(0) - \underline{r}}{\sigma_r\sqrt{\Delta t}}\Delta t = \frac{r(0) - \underline{r}}{\sigma_r}\sqrt{\Delta t} \\ \bar{t} = \bar{m}\Delta t &= \frac{\bar{r} - r(0)}{\sigma_r\sqrt{\Delta t}}\Delta t = \frac{\bar{r} - r(0)}{\sigma_r}\sqrt{\Delta t}. \end{aligned}$$

Of course, neither \underline{r} , \bar{r} nor \underline{t} , \bar{t} are likely to correspond to any node of the discretized process $r(t)$ or to the discretized time line $\{0, \Delta t, 2\Delta t, \dots, T\}$. In this case, we set \underline{r} , \bar{r} and \underline{t} , \bar{t} equal to the smallest values on the grid of $r(t)$ and t that satisfy the constraints in (A3). Going back to the numerical example above, we have that $\underline{t} = 0.5840$ and $\bar{t} = 0.7680$. A section of the quadrinomial tree in this case is displayed in Figure 13.

B Proofs of the Claims

Proof of Proposition 1: Value of the European put/call equity option. We first derive the price of the European put option at $t = 0$. As the payoff of the derivative depends only on the final value of S , the price at any t is obtained straightforwardly thanks to the Markovianity of (S, r) by replacing S_0 by $S(t)$, r_0 by $r(t)$ and T by $T - t$.

In the market described by (1), the risk-neutral price at $t = 0$ of the European put option on S with strike price K and maturity T is given by:

$$\begin{aligned}
 \pi_E^{put}(0) &= \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(s)ds}(K - S(T))^+] \\
 &= \mathbb{E}^{\mathbb{Q}}\left[\frac{(K - S(T))}{B(T)}\mathbf{1}_{\{K - S(T) > 0\}}\right] \\
 &= K\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K - S(T) > 0\}}}{B(T)}\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K - S(T) > 0\}}}{B(T)}\right].
 \end{aligned} \tag{B1}$$

Since $B(T)$ depends on r which is correlated with S , in order to compute the two expected values we would need to know their joint distribution under \mathbb{Q} and then evaluate a double integral. This turns out to be rather complicated. Nevertheless, we can greatly simplify the computation of the two expected values applying a change of numéraire.

We start from the first expectation in (B1). Consider the T -forward measure \mathbb{Q}^T , namely the martingale measure for the numéraire process $p(t, T)$. The Radon-Nikodym derivative of \mathbb{Q}^T with respect to \mathbb{Q} (whose numéraire process is the money market $B(t) = e^{\int_0^t r(s)ds}$) is:

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = L^T(t) = \frac{p(t, T)}{B(t)p(0, T)} \text{ on } \mathcal{F}_t, 0 \leq t \leq T.$$

As $p(T, T) = 1$ and since $p(0, T)$ is a scalar, we get:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)} \right] &= p(0, T) \mathbb{E}^{\mathbb{Q}} \left[\frac{p(T, T)}{B(T)p(0, T)} \mathbf{1}_{\{K-S(T)>0\}} \right] \\
&= p(0, T) \mathbb{E}^{\mathbb{Q}} [L^T(T) \mathbf{1}_{\{K-S(T)>0\}}] \\
&= p(0, T) \mathbb{E}^{\mathbb{Q}^T} [\mathbf{1}_{\{K-S(T)>0\}}] \\
&= p(0, T) \mathbb{Q}^T (S(T) < K) \\
&= p(0, T) \mathbb{Q}^T \left(\frac{S(T)}{p(T, T)} < K \right).
\end{aligned}$$

By definition, under the T-forward measure \mathbb{Q}^T the discounted process of the risky asset, when accounting for the dividend, $Z_{0,T}(t) := \frac{S(t)e^{qt}}{p(t, T)}$ is a martingale. Applying the multidimensional Itô's Lemma to $Z_{0,T}(t)$ under \mathbb{Q} we get:

$$dZ_{0,T}(t) = (\dots) dt + (\nu_S + B(t, T)\nu_r) \cdot dW^{\mathbb{Q}}(t)$$

where, $\nu_S = [\sigma_S \ 0]$, $\nu_r = [\sigma_r \rho \ \sigma_r \sqrt{1-\rho^2}]$ and $W^{\mathbb{Q}}(t) = [W_1^{\mathbb{Q}}(t) \ W_2^{\mathbb{Q}}(t)]'$ is standard two-dimensional Brownian motion under \mathbb{Q} . Since the volatility process $\sigma_{0,T}(t) := \nu_S + B(t, T)\nu_r$ is constant, we can apply a suitable change of measure from \mathbb{Q} to \mathbb{Q}^T to get rid of the deterministic drift of $Z_{0,T}(t)$. Under the T-forward measure we get:

$$dZ_{0,T}(t) = +\sigma_{0,T}(t) \cdot dW^{\mathbb{Q}^T}(t)$$

as we expected. The process $Z_{0,T}(t)$ is now a geometric Brownian motion driven by a bi-dimensional Wiener process. Hence, its solution is:

$$\begin{aligned}
Z_{0,T}(t) &= Z_{0,T}(0) \exp \left\{ -\frac{1}{2} \int_0^t \|\sigma_{0,T}^2(s)\| ds + \int_0^t \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^T}(s) \right\} \\
&= \frac{S(0)}{p(0, T)} \exp \left\{ -\frac{1}{2} \int_0^t \|\sigma_{0,T}^2(s)\| ds + \int_0^t \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^T}(s) \right\}.
\end{aligned}$$

Notice that, due to Itô's Isometry,

$$-\frac{1}{2} \int_0^t \|\sigma_{0,T}^2(s)\| ds + \int_0^t \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^T}(s) \sim \mathcal{N} \left(-\frac{1}{2} \Sigma_{0,T}^2(t), \Sigma_{0,T}^2(t) \right)$$

where:

$$\begin{aligned}
\Sigma_{0,T}^2(t) &:= \int_0^t \|\sigma_{0,T}^2(s)\| ds \\
&= \int_0^t \sigma_S^2 + 2\sigma_S\sigma_r\rho B(s,T) + B(s,T)^2\sigma_r^2 ds \\
&= \sigma_S^2 t + 2\sigma_S\sigma_r\rho \left(\frac{\kappa t - e^{-\kappa T}(-1 + e^{\kappa t})}{\kappa^2} \right) + \sigma_r^2 \left(\frac{e^{-2\kappa T}(-1 + e^{2\kappa t} + 4e^{\kappa T} - 4e^{\kappa(t+T)} + 2e^{2\kappa T}\kappa t)}{2\kappa^3} \right).
\end{aligned}$$

When $t = T$, the expression above simplifies to:

$$\Sigma_{0,T}^2(T) = \sigma_S^2 T + 2\sigma_S\sigma_r\rho \left(\frac{-1 + e^{-\kappa T} + \kappa T}{\kappa^2} \right) + \sigma_r^2 \left(-\frac{3 + e^{-2\kappa T} - 4e^{-\kappa T} - 2\kappa T}{2\kappa^3} \right).$$

Finally, we can compute the T-forward probability that the put option closes in the money as:

$$\begin{aligned}
\mathbb{Q}^T \left(\frac{S(T)}{p(T,T)} < K \right) &= \mathbb{Q}^T \left(\frac{S(T)e^{qT}}{p(T,T)} < Ke^{qT} \right) \\
&= \mathbb{Q}^T (Z_{0,T}(T) < Ke^{qT}) \\
&= \mathbb{Q}^T \left(\frac{S(0)}{p(0,T)} \exp \left\{ -\frac{1}{2} \int_0^T \|\sigma_{0,T}\|^2(s) ds + \int_0^T \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^T}(s) \right\} < Ke^{qT} \right) \\
&= \mathbb{Q}^T \left(\mathcal{N} \left(-\frac{1}{2} \Sigma_{0,T}^2(T), \Sigma_{0,T}^2(T) \right) < \ln \frac{p(0,T)Ke^{qT}}{S(0)} \right) \\
&= N(-\tilde{d}_2)
\end{aligned}$$

with $\tilde{d}_2 = \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}} \left(\ln \frac{S(0)}{p(0,T)K} - \frac{1}{2} \Sigma_{0,T}^2(T) - qT \right)$. Hence,

$$\mathbb{E}^{\mathbb{Q}} \left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)} \right] = p(0,T)N(-\tilde{d}_2).$$

We now turn to the second expected value in (B1). Consider the martingale measure \mathbb{Q}^S with numéraire process $S(t)e^{qt}$. The Radon-Nikodym derivative of \mathbb{Q}^S with respect to \mathbb{Q} is:

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = L^S(t) = \frac{S(t)e^{qt}}{S(0)B(t)} \text{ on } \mathcal{F}_t, 0 \leq t \leq T. \tag{B2}$$

As both $S(0)$ and e^{qT} are scalars, we have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left[\frac{S(T) \mathbf{1}_{\{K-S(T)>0\}}}{B(T)} \right] &= S(0) e^{-qT} \mathbb{E}^{\mathbb{Q}} \left[\frac{S(T) e^{qT}}{S(0) B(T)} \mathbf{1}_{\{K-S(T)>0\}} \right] \\
&= S(0) e^{-qT} \mathbb{E}^{\mathbb{Q}} [L^S(T) \mathbf{1}_{\{K-S(T)>0\}}] \\
&= S(0) e^{-qT} \mathbb{E}^{\mathbb{Q}^S} [\mathbf{1}_{\{K-S(T)>0\}}] \\
&= S(0) e^{-qT} \mathbb{Q}^S(S(T) < K).
\end{aligned}$$

Under \mathbb{Q}^S , the process $Y_{0,T}(t) := \frac{p(0,t)}{S(t)e^{qt}}$ is a martingale. Notice that $Y_{0,T}(t) = Z_{0,T}(t)^{-1}$. Then, Itô's Lemma tells us immediately that:

$$dY_{0,T}(t) = (\dots) dt - (\nu_S + B(t,T)\nu_r) \cdot dW^{\mathbb{Q}}(t)$$

and after a suitable change of measure we get that under \mathbb{Q}^S :

$$dY_{0,T}(t) = -\sigma_{0,T}(t) \cdot dW^{\mathbb{Q}^S}(t).$$

As before, we get:

$$\begin{aligned}
Y_{0,T}(t) &= Y_{0,T}(0) \exp \left\{ -\frac{1}{2} \int_0^t \|\sigma_{0,T}^2(s)\| ds - \int_0^t \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^2}(s) \right\} \\
&= \frac{p(0,T)}{S(0)} \exp \left\{ -\frac{1}{2} \int_0^t \|\sigma_{0,T}^2(s)\| ds - \int_0^t \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^S}(s) \right\}.
\end{aligned}$$

and, since again $p(T, T) = 1$, the \mathbb{Q}^S probability that the put option closes in the money is:

$$\begin{aligned}
\mathbb{Q}^S(S(T) < K) &= \mathbb{Q}^S\left(\frac{1}{S(T)} > \frac{1}{K}\right) \\
&= \mathbb{Q}^S\left(\frac{p(T, T)}{S(T)e^{qT}} > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^S\left(Y_{0,T}(T) > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\frac{p(0, T)}{S(0)} \exp\left\{-\frac{1}{2}\int_0^T \|\sigma_{0,T}^2(s)\| ds - \int_0^T \sigma_{0,T}(s) \cdot dW^{\mathbb{Q}^T}(s)\right\} > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}\left(-\frac{1}{2}\Sigma_{0,T}^2(T), \Sigma_{0,T}^2(T)\right) > \ln \frac{S(0)}{p(0, T)Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}(0, 1) > \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}}\left(\ln \frac{S(0)}{p(0, T)K} + \frac{1}{2}\Sigma_{0,T}^2(T) - qT\right)\right) \\
&= N(-\tilde{d}_1)
\end{aligned}$$

where $\tilde{d}_1 = \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}}\left(\ln \frac{S(0)}{p(0, T)K} + \frac{1}{2}\Sigma_{0,T}^2(T) - qT\right) = \tilde{d}_2 + \sqrt{\Sigma_{0,T}^2(T)}$. Hence,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] = S(0)e^{-qT}N(-\tilde{d}_1).$$

Finally, putting everything together we find the initial price of the put option:

$$\begin{aligned}
\pi_E^{put}(0) &= K\mathbb{E}^{\mathbb{Q}}[p(0, T)\mathbf{1}_{\{K-S(T)>0\}}] - \mathbb{E}^{\mathbb{Q}}[p(0, T)S(T)\mathbf{1}_{\{K-S(T)>0\}}] \\
&= Kp(0, T)N(-\tilde{d}_2) - S(0)e^{-qT}N(-\tilde{d}_1).
\end{aligned}$$

The price of the related call option can be derived by the put-call parity that at $t \in [0, T]$ reads

$$\pi_E^{call}(t) = S(t)e^{-q(T-t)} - Kp(t, T) + \pi_E^{put}(t)$$

as $e^{-q(T-t)}$ units of $S(t)$ at time t lead to one unit of S at time T by continuously investing the dividends in S itself.

Proof of Theorem 2: Convergence of the quadrinomial tree. We now need to show that the bivariate discrete process $(X_i)_i$ defined in (10) with the parameters in (11) and (12) converges in distribution to $X(t) = (Y(t), r(t))$ that solves (7). With the notation of the general case in (9) and exploiting the result of Section 11.3 of Stroock and Varadhan (1997), the desired result holds true if the following four conditions are met:

(A1) the functions $\mu(x, t)$ and $\sigma(x, t)$ are continuous and $\sigma(x, t)$ is non negative;

(A2) with probability 1 a solution $(X_t)_t$ to the SDE:

$$X_t = X_0 + \int_0^t \mu(X_s, s) ds + \int_0^t \sigma(X_s, s) \cdot dW(s)$$

exists for $0 < t < +\infty$ and it is unique in law;

(A3) for all $\delta, T > 0$

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup_{\|x\| \leq \delta, 0 \leq t \leq T} |\Delta Y^\pm| &= 0 \\ \lim_{n \rightarrow +\infty} \sup_{\|x\| \leq \delta, 0 \leq t \leq T} |\Delta r^\pm| &= 0; \end{aligned}$$

(A4) let $X_{i,j}$ indicate the j -th entry of X_i and let $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ be the filtration generated by the discrete bivariate process (X_i) . Define:

$$\mu_i(x, t) := \begin{bmatrix} \mu_{i,1}(x, t) \\ \mu_{i,2}(x, t) \end{bmatrix} \text{ and } \sigma_i^2(x, t) := \begin{bmatrix} \sigma_{i,1}^2(x, t) \\ \sigma_{i,2}^2(x, t) \end{bmatrix}$$

where $\mu_{i,j}(x, t) = \frac{\mathbb{E}^{\mathbb{Q}}[X_{i+1,j} - X_{i,j} | \mathcal{F}_i]}{\frac{T}{n}}$ and $\sigma_{i,j}^2(x, t) = \frac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,j} - X_{i,j})^2 | \mathcal{F}_i]}{\frac{T}{n}}$ for $j = 1, 2$. Let $\rho_i(x, t) = \frac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,1} - X_{i,1})(X_{i+1,2} - X_{i,2}) | \mathcal{F}_i]}{\frac{T}{n}}$ and $\rho(x, t) = \sigma_1(x, t) \cdot \sigma_2(x, t)'$ where $\sigma_j(x, t)$ is the j -th row of $\sigma(x, t)$. Then, for all $\delta, T > 0$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup_{\|x\| \leq \delta, 0 \leq t \leq T} \|\mu_i(x, t) - \mu(x, t)\| &= 0 \\ \lim_{n \rightarrow +\infty} \sup_{\|x\| \leq \delta, 0 \leq t \leq T} \|\sigma_i^2(x, t) - \sigma^2(x, t) \cdot \mathbf{I}_2\| &= 0 \end{aligned}$$

$$\lim_{n \rightarrow +\infty} \sup_{\|x\| \leq \delta, 0 \leq t \leq T} |\rho_i(x, t) - \rho(x, t)| = 0$$

where \mathbf{I}_n is the column vector with all of the n entries equal to one.

For our quadrinomial tree we have $X_t = [Y(t), \quad r(t)]'$,

$$\mu(X_t, t) = \begin{bmatrix} r(t) - q - \frac{\sigma_S^2}{2} \\ \kappa(\theta - r(t)) \end{bmatrix} \quad \text{and} \quad \sigma(X_t, t) = \begin{bmatrix} \sigma_S & 0 \\ \sigma_r \rho & \sigma_r \sqrt{1 - \rho^2} \end{bmatrix}.$$

Assumption (A1) trivially holds true.

Assumption (A2) holds true if the standard conditions for the existence and the uniqueness of the solution to an SDE are met. According, e.g., to Proposition 5.1 in Björk (2009), it is sufficient to show that there exists a constant K such that the following are satisfied for all $x_i = [y_i, \quad r_i]'$, $i = 1, 2$ and t :

$$\|\mu(x_1, t) - \mu(x_2, t)\| \leq K \|x_1 - x_2\|,$$

$$\|\sigma(x_1, t) - \sigma(x_2, t)\| \leq K \|x_1 - x_2\|,$$

$$\|\mu(x_1, t)\| + \|\sigma(x_1, t)\| \leq K (1 + \|x_1\|).$$

Notice that the second and the third conditions involve the operator norm of a matrix $A \in \mathbb{R}^n$ defined as

$$\|A\| := \sup_{\|x\|=1} \{\|A \cdot x\| : x \in \mathbb{R}^n\}.$$

As $\|\mu(x_1, t) - \mu(x_2, t)\| = \sqrt{1 + \kappa^2} |r_1 - r_2|$ and $(r_1 - r_2)^2 \leq \|x_1 - x_2\|^2$, the first condition is surely satisfied for any $K \geq \sqrt{1 + \kappa^2}$. As $\sigma(x_i, t)$ is actually constant and independent of x_i and t , $\|\sigma(x_1, t) - \sigma(x_2, t)\| = 0$ and the second condition is surely satisfied for any $K \geq 0$. Finally, as

$$\|\sigma(x_1, t)\| = \sigma_S^2 + \rho^2 \frac{\sigma_r^2}{2} + |\rho| \frac{\sigma_r}{2} \sqrt{4\sigma_S^2 + \sigma_r^2}$$

is constant and as

$$\|\mu(x_1, t)\| = \sqrt{\left(r_1 - q - \frac{\sigma_S^2}{2}\right)^2 + \kappa^2(\theta - r_1)^2}$$

can be bounded from above by $\sqrt{2(1 + \kappa^2)r_1^2}$, we have

$$\|\mu(x_1, t)\| + \|\sigma(x_1, t)\| \leq \sqrt{2(1 + \kappa^2)} \|x_1\| + \|\sigma(x_1, t)\| \leq K(1 + \|x_1\|)$$

for any $K \geq \max\{\sqrt{2(1+\kappa^2)}, \|\sigma(x_1, t)\|\}$. As the three conditions hold true simultaneously for any $K \geq \max\{\sqrt{2(1+\kappa^2)}, \|\sigma(x_1, t)\|\}$, assumption (A2) is satisfied.

As the increments of the bivariate discrete process $\Delta Y^\pm = \pm\sigma_S\sqrt{\Delta t} = \pm\sigma_S\sqrt{\frac{T}{n}}$, $\Delta r^\pm = \pm\sigma_r\sqrt{\Delta t} = \pm\sigma_r\sqrt{\frac{T}{n}}$ are constant and do not depend neither on x_i , $i = 1, 2$, nor on t ,

$$\sup_{\|x\| \leq \delta, 0 \leq t \leq T} |\Delta Y^\pm| = |\Delta Y^\pm| = \sigma_S \sqrt{\frac{T}{n}},$$

$$\sup_{\|x\| \leq \delta, 0 \leq t \leq T} |\Delta r^\pm| = |\Delta r^\pm| = \sigma_r \sqrt{\frac{T}{n}}.$$

As both of the sups are infinitesimal with respect to n , (A3) holds true as well.

As the parameters in (11) and (12) of the bivariate discretization $X_i = (Y_i, r_i)$ are chosen in order to match the first two moments and the cross-variation of $X(t) = (Y(t), r(t))$, we have $\mu_i(x, t) = \mu(x, t)$, $\sigma_i^2(x, t) = \sigma^2(x, t) \cdot \mathbf{I}_2$ and $\rho_i(x, t) = \rho(x, t)$. Hence, assumption (A4) is satisfied by construction.

Theorem 11.3.3 of Stroock and Varadhan (1997) allows us to conclude.

Proof of Proposition 3: Value of the American option as a deterministic function. Let $\eta := \tau - t$. Then we can rewrite the value of the American option (13) as:

$$V(t) = \text{ess sup}_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{t+\eta} r(s) ds} \varphi(S(t+\eta)) \middle| \mathcal{F}_t \right]. \quad (\text{B3})$$

Recalling the dynamics of the equity price conditional on $S(t) = S$ we can further rewrite $\tilde{V}(t)$ as:

$$V(t) = \text{ess sup}_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{t+\eta} r(s) ds \right) \cdot \varphi \left(S \exp \left(\int_t^{t+\eta} r(s) ds - \left(q + \frac{1}{2} \sigma_S^2 \right) \eta + \sigma_S (W_S(t+\eta) - W_S(t)) \right) \right) \middle| \mathcal{F}_t \right].$$

Therefore, V depends on the expectation of two random variables: the Brownian increment $W_S(t+\eta) - W_S(t)$ and the integral $\int_t^{t+\eta} r(s) ds$, which appears both in the drift part of the underlying and in the discount factor. The first of the two random variables is \mathcal{F}_t -independent by definition:

$$W_S(t+\eta) - W_S(t) \perp \mathcal{F}_t,$$

and, moreover,

$$W_S(t + \eta) - W_S(t) \stackrel{\mathcal{Q}}{\sim} W_S(\eta).$$

We now show that also $\int_t^{t+\eta} r(s)ds$ is independent of \mathcal{F}_t and that $\int_t^{t+\eta} r(s)ds \stackrel{\mathcal{Q}}{\sim} \int_0^\eta r(s)ds$ as well. Recalling the solution to the SDE driving the short term interest rate conditional on $r(t) = r$ we have:

$$\begin{aligned} \int_t^{t+\eta} r(s)ds &= \int_t^{t+\eta} \left[r e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) + \sigma_r \int_t^s e^{-\kappa(s-y)} dW_r(y) \right] ds \\ &= -\frac{e^{-\kappa\eta}}{\kappa}(r - \theta) + \frac{r - \theta}{\kappa} + \theta\eta + \sigma_r \int_t^{t+\eta} \int_t^s e^{-\kappa(s-y)} dW_r(y) ds. \end{aligned}$$

The constant $\alpha := -\frac{e^{-\kappa\eta}}{\kappa}(r - \theta) + \frac{r - \theta}{\kappa} + \theta\eta$ does not depend on t . Exploiting the definition of stochastic integral with $\{t_i\}_{i=1, \dots, N}$ such that $t_0 = t$, $t_N = t + \eta$ and $\|\{t_i\}\| \rightarrow 0$, we get:

$$\int_t^{t+\eta} r(s)ds = \alpha + \sigma_r \int_t^{t+\eta} \sum_{i=0}^{N-1} e^{-\kappa(s-t_i)} (W_r(t_{i+1}) - W_r(t_i)) ds.$$

Since $t_{i+1} > t_i > t$ for any $i = 1, \dots, N - 1$, $W_r(t_{i+1}) - W_r(t_i) \perp \mathcal{F}_t$ by definition and for any value of s .

Hence,

$$\sum_{i=0}^{N-1} e^{-\kappa(s-t_i)} (W_r(t_{i+1}) - W_r(t_i)) \perp \mathcal{F}_t \quad \forall s.$$

Since the sum is independent of \mathcal{F}_t for any s , the outer integral in ds preserves such independence. As a result,

$$\int_t^{t+\eta} r(s)ds \perp \mathcal{F}_t.$$

Furthermore, we need to show that the distribution of

$$\int_t^{t+\eta} r(s)ds$$

does not depend on t . Recalling that:

$$\int_t^{t+\eta} r(s)ds = \alpha + \sigma_r \int_t^{t+\eta} \int_t^s e^{-\kappa(s-y)} dW_r(y) ds$$

and setting $a := s - t$ in the outer integral in ds , we get:

$$\int_t^{t+\eta} r(s)ds = \alpha + \sigma_r \int_0^\eta \int_t^{a+t} e^{-\kappa(a+t-y)} dW_r(y) da.$$

The argument of the inner stochastic integral is deterministic in y and, therefore:

$$\begin{aligned} \int_t^{a+t} e^{-\kappa(s-y)} dW_r(y) &\stackrel{\mathbb{Q}}{\approx} \mathcal{N}\left(0, \int_t^{a+t} e^{-2\kappa(s-y)} dy\right) \\ &\stackrel{\mathbb{Q}}{\approx} \mathcal{N}\left(0, \frac{1}{2\kappa} (1 - e^{-2\kappa a})\right), \end{aligned}$$

which does not depend on t . Thanks to a little abuse of notation we see that:

$$\int_t^{t+\eta} r(s) ds = \alpha + \sigma_r \int_0^\eta \mathcal{N}\left(0, \frac{1}{2\kappa} (1 - e^{-2\kappa a})\right) da,$$

where the right-hand side of the equation does not depend on t . Hence:

$$\int_t^{t+\eta} r(s) ds \stackrel{\mathbb{Q}}{\approx} \int_0^\eta r(s) ds.$$

We now go back to the original expression (B3). Thanks to the independence of \mathcal{F}_t , the conditional expected value turns into the unconditional one:

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{t+\eta} r(s) ds} \varphi(S(t+\eta)) \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{t+\eta} r(s) ds} \varphi(S(t+\eta)) \right],$$

and setting $S(0) = S(t) = S$, $r(0) = r(t) = r$,

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{t+\eta} r(s) ds} \varphi(S(t+\eta)) \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\eta r(s) ds} \varphi(S(\eta)) \right].$$

Therefore, the value on an American option on S defined in (13) reduces to

$$V(t) = F(t, S(t), r(t))$$

with

$$\begin{aligned} F(t, S, r) &= \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_0^\eta r(s) ds\right) \cdot \right. \\ &\quad \left. \cdot \varphi\left(S \exp\left(\int_0^\eta r(s) ds - \left(q + \frac{1}{2}\sigma_S^2\right)\eta + \sigma_S W_S(\eta)\right)\right) \right]. \end{aligned}$$

where t enters only the upper bound of η , namely the time to maturity of the option. From this last expression it is immediate to see that F enjoys the same monotonicity properties of φ w.r.t. S , and that it

is decreasing w.r.t. t , and convex w.r.t. S . For the put option we show now that F is decreasing in r . To this aim we rewrite

$$F(t, S, r) = \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\eta r(s) ds} \left(K - S e^{\int_0^\eta r(s) ds - (q + \frac{1}{2} \sigma_S^2) \eta + \sigma_S W_S(\eta)} \right)^+ \right] \quad (\text{B4})$$

where $r = r(0)$. If $r' > r$ then $F(t, S, r') \leq F(t, S, r)$. In fact, $\int_0^\eta r(s) ds$ started at $r(0) = r' > r$ is larger than $\int_0^\eta r(s) ds$ started at $r(0) = r$. As the object of the expectation in (B4) is a decreasing function of $\int_0^\eta r(s) ds$, we conclude that $F(t, S, r)$ is decreasing in r .

To show that the American call option is increasing with respect to r , we apply a change of numeraire to isolate the effect of the interest rate in the underlying drift only (as under the original risk neutral measure an increase in r has opposite effects in the discount factor and in the call's payoff).

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[(S(\tau) - K)^+ e^{-\int_0^\tau r(s) ds} \right] &= \mathbb{E}^{\mathbb{Q}} \left[\frac{S(\tau) e^{q\tau}}{S(0) B(\tau)} \left(\frac{1}{K} - \frac{1}{S(\tau)} \right)^+ K e^{-q\tau} S(0) \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[L^S(\tau) \left(\frac{1}{K} - \frac{1}{S(\tau)} \right)^+ K e^{-q\tau} S(0) \right] \end{aligned}$$

where $L^S(\tau)$ is the Radon-Nikodym derivative of \mathbb{Q}^S with respect to \mathbb{Q} defined in (B2). Thus the call option is a put option under the new measure on K/S with strike $S(0)$ and interest rate q

$$\mathbb{E}^{\mathbb{Q}} \left[(S(\tau) - K)^+ e^{-\int_0^\tau r(s) ds} \right] = \mathbb{E}^{\mathbb{Q}^S} \left[\left(S(0) - \frac{K}{S(\tau)} \right)^+ e^{-q\tau} \right]$$

Recalling the dynamics of the equity price and of the interest rate under \mathbb{Q} ,

$$\begin{cases} \frac{dS(t)}{S(t)} = (r(t) - q)dt + [\sigma_S \ 0] \cdot dW^{\mathbb{Q}}(t) \\ dr(t) = \kappa(\theta - r(t))dt + [\sigma_r \rho \ \sigma_r \sqrt{1 - \rho^2}] \cdot dW^{\mathbb{Q}}(t) \end{cases} \quad (\text{B5})$$

Girsanov's theorem implies that $dW^{\mathbb{Q}}(t) = dW^{\mathbb{Q}^S}(t) + [\sigma_S \ 0]' dt$ and, therefore, (B5) becomes

$$\begin{cases} \frac{dS(t)}{S(t)} = (r(t) - q + \sigma_S^2)dt + [\sigma_S \ 0] \cdot dW^{\mathbb{Q}^S}(t) \\ dr(t) = \kappa(\theta - r(t) + \frac{\rho \sigma_S \sigma_r}{\kappa})dt + [\sigma_r \rho \ \sigma_r \sqrt{1 - \rho^2}] \cdot dW^{\mathbb{Q}^S}(t) \end{cases} \quad (\text{B6})$$

Ito's formula implies that

$$d \left(\frac{1}{S(t)} \right) = \frac{1}{S(t)} \left((q - r(t))dt - [\sigma_S \ 0] \cdot dW^{\mathbb{Q}^S}(t) \right)$$

and therefore the new underlying

$$d\left(\frac{K}{S(t)}\right) = \frac{K}{S(t)} \left((q - r(t))dt - [\sigma_S \ 0] \cdot dW^{\mathbb{Q}^S}(t) \right)$$

has drift $q - r(t)$. Thus the call option is a put option whose underlying under the new measure is

$$\frac{K}{S(t)} = \frac{K}{S(0)} e^{\int_0^\eta (q - r(s))ds - \frac{1}{2}\sigma_S^2\eta - \sigma_S W_1^{\mathbb{Q}^S}(\eta)}$$

Thus, if the process r starts at $r(0) = r' > r$ the factor $\int_0^\eta (q - r(s))ds$ is smaller than the one started at $r(0) = r$, and thus the put's payoff is larger, and the value of the American option larger as well. This shows that for the call option $r' > r$ implies $F(t, S, r') > F(t, S, r)$.

Proof of Proposition 4: Asymptotic necessary conditions for the existence of a double continuation region. When the interest rate is constant, the value at t of the American put option defined in Equation (14) becomes a deterministic function $G(t, S) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$ of the time and the current value of the underlying:

$$G(t, S) = \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\eta} \left(K - S \exp \left(\left(r - q - \frac{\sigma_S^2}{2} \right) \eta + \sigma_S W_S(\eta) \right) \right)^+ \right]. \quad (\text{B7})$$

As Battauz et al. (2015) show in Section 2, necessary conditions for the double continuation region to appear for a put option at a generic t are that the drift of S is positive and $G(t, 0) > K$. This last condition is verified if there exists some η such that $p(0, \eta) > 1$, as discussed in the comments before Proposition 4. We now deduce [NC0] by imposing $p(0, \eta) > 1$ for some $\eta \in [0, T - t]$. Exploiting Jensen's inequality and the uniform integrability of $r(s)$, we get:

$$\mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^\eta r(s)ds \right) \right] \geq \exp \left(- \mathbb{E}^{\mathbb{Q}} \left[\int_0^\eta r(s)ds \right] \right) = \exp \left(- \int_0^\eta \mathbb{E}^{\mathbb{Q}} [r(s)] ds \right).$$

As before, thanks to (3), we have:

$$\mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^\eta r(s)ds \right) \right] \geq \exp \left(- \int_0^\eta r e^{-\kappa s} + \theta(1 - e^{-\kappa s})ds \right) = \exp(r\alpha - \theta(\alpha + \eta))$$

where we set $\alpha := \frac{e^{-\kappa\eta} - 1}{\kappa}$. Notice that $\alpha \leq 0$ for any κ and $\eta \in [0, T - t]$.

If $r\alpha - \theta(\alpha + \eta) > 0$, then $F(t, 0, r) > K$.

For the American put option, under $[NC0]$, if $[NC1]$ is not satisfied, i.e. $q > 0$, than the discounted risky security \tilde{S} is driven by

$$d\tilde{S}(t) = -qdt + \sigma_S dW_S^{\mathbb{Q}}(t),$$

and \tilde{S} is a supermartingale. Thus, for any $t < \tau < T$,

$$\mathbb{E}^{\mathbb{Q}} \left[S(\tau) e^{-\int_t^\tau r(s) ds} \middle| \mathcal{F}_t \right] \leq S(t)$$

and, by Jensen's inequality,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[(K - S(\tau))^+ e^{-\int_t^\tau r(s) ds} \middle| \mathcal{F}_t \right] &\geq \left(K \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r(s) ds} \middle| \mathcal{F}_t \right] - S(t) e^{-q(\tau-t)} \right)^+ \\ &\geq (K - S(t))^+, \end{aligned}$$

where the last inequalities holds under $[NC0]$. This shows that, for the American put option, under $[NC0]$, if $[NC1]$ is violated, early exercise is never optimal at t .

We deal now with the American call option. For $0 < \tau < T$, we have by Jensen's inequality,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[(S(\tau) - K)^+ e^{-\int_t^\tau r(s) ds} \middle| \mathcal{F}_t \right] &\geq \left(S(t) e^{-q(\tau-t)} - K \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r(s) ds} \middle| \mathcal{F}_t \right] \right)^+ \\ &= \left(S(t) e^{-q(\tau-t)} - K p(t, \tau) \right)^+ \\ &\geq (S(t) - K)^+, \end{aligned}$$

if $q \leq 0$ and $p(t, \tau) \leq 1$. Therefore, to ensure the existence of optimal early exercise opportunities for the American call option, we must assume that $q > 0$, or $q \leq 0$ and $p(t, \tau) > 1$ for some τ .

Under $[NC0]$, if $[NC2]$ is not satisfied, then $\pi_A(t, S, r) \geq \pi_E(t, S, r) > (K - S)^+$, that means that early exercise is never optimal at t .

Proof of Theorem 5. The case $r \geq 0$ is standard (see Detemple, 2014), and therefore we focus on $r < 0$. The continuity, the monotonicity of the r -sections of the put option's free boundaries with respect to t and S and their limits as $t \rightarrow T^-$ follow by adapting the proof of Theorem 2.3 in (Battauz et al., 2015) where now the operator \mathcal{L} becomes $\mathcal{L}F = \frac{\partial F}{\partial S}S(r - q) + \frac{\partial F}{\partial r}\kappa(\theta - r) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma_S^2S^2 + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma_r^2 + \frac{\partial^2 F}{\partial r \partial S}\rho\sigma_r\sigma_S$. The monotonicity properties of the free boundaries with respect to r follow from the monotonicity properties of F . In fact, let $r' > r$, and assume $S \in EER_r$. Then $(K - S)^+ \leq F(t, S, r') \leq F(t, S, r) = (K - S)^+$, where the first inequality follows from value dominance and the second one from the fact that F is decreasing in r . Thus if $S \in EER_r$, then $S \in EER'_r$, and $EER'_r \supseteq EER_r$. By passing to the infimum (resp. supremum) we conclude that the lower (resp. upper) free boundary is decreasing (resp. increasing) with respect to r . For the call option, we start from the monotonicity properties of the free boundaries with respect to r . As the call option is increasing in r , we have that if $r' > r$ and $S \in EER'_r$ then $(S - K)^+ \leq F(t, S, r) \leq F(t, S, r') = (S - K)^+$, where the first inequality follows from value dominance and the second one from the fact that F is increasing in r . This means that $EER'_r \subseteq EER_r$. By passing to the infimum (resp. supremum) we conclude that the lower (resp. upper) free boundary is increasing (resp. decreasing) with respect to r .

For the other call option's properties, we cannot simply adapt the proof of Theorem 3.3 in (Battauz et al., 2015), as it relies on a symmetry result in a constant interest rate environment that fails to be applicable to our setting. The monotonicity properties of \underline{S}^* and \overline{S}^* with respect to t follow from the fact that F is decreasing with respect to t , similarly to the put case. We then prove the inequalities satisfied by the free boundaries. In the EER the function F satisfies

$$\frac{\partial F}{\partial t} + \mathcal{L}F \leq rF \tag{B8}$$

On the EER in the call case $F(t, S, r) = S - K$ and therefore Equation (B8) simplifies to $1 \cdot S(r - q) \leq r(S - K)$. Thus $-Sq \leq -rK$ for all $S \in EER_r$, i.e. $S \leq \frac{r}{q}K$ for all $S \in EER_r$, as $q < 0$. By passing to the supremum we get $K \leq \underline{S}^*(t, r) < \overline{S}^*(t, r) \leq \frac{rK}{q}$.

At maturity $\underline{S}^*(T, r) = K$ and $\overline{S}^*(T, r) = +\infty$, as the option is exercised at T whenever $S(T) \geq K$.

We now show that $\underline{S}^*(T^-, r) = K$ and $\overline{S}^*(T^-, r) = \frac{rK}{q}$. By construction $\underline{S}^*(t, r) \geq K$ for all $t \in (\bar{t}; T)$, and hence $\underline{S}^*(T^-, r) \geq K$. Suppose by contradiction that $\underline{S}^*(T^-, r) > K$. The set $(\bar{t}; T) \times (K; \underline{S}^*(T^-, r)) \subset CR_r$ and therefore $(\mathcal{L} - r)F = -\frac{\partial}{\partial t}F \geq 0$, as F is decreasing w.r.t. t . As $t \uparrow T$ we have $(\mathcal{L} - r)F \rightarrow (\mathcal{L} - r)(S - K) = -qS + rK$ for $S \in (K; \underline{S}^*(T^-, r))$. This implies $-qS + rK \geq 0$ for $S \in (K; \underline{S}^*(T^-, r))$ and passing to the supremum over $S \in (K; \underline{S}^*(T^-, r))$ this delivers $\underline{S}^*(T^-, r) \geq \frac{rK}{q}$ which is a contradiction. We deal now with the upper free boundary limit. Suppose (by contradiction) that $\overline{S}^*(T^-, r) < \frac{rK}{q}$. But then the set $(\bar{t}; T) \times (\overline{S}^*(T^-, r); \frac{rK}{q}) \subset CR_r$ and $(\mathcal{L} - r)F = -\frac{\partial}{\partial t}F \geq 0$ for $S \in (\overline{S}^*(T^-, r); \frac{rK}{q})$. As $t \uparrow T$ we have $(\mathcal{L} - r)F \rightarrow (\mathcal{L} - r)(S - K) = -qS + rK$ for $S \in (\overline{S}^*(T^-, r); \frac{rK}{q})$ (here the limits are in distribution). Then $-qS + rK \geq 0$ for all $S \in (\overline{S}^*(T^-, r); \frac{rK}{q})$ and therefore also for the infimum $-q\overline{S}^*(T^-, r) + rK \geq 0$ that implies the contradiction $\overline{S}^*(T^-, r) \geq \frac{rK}{q}$.

C Additional numerical analysis of the free boundary

In this section we provide additional numerical analysis of American equity options in our Vasicek interest rate framework.⁸ Tables 3 and 4 document the impact of the correlation between equity and interest rate on the pricing of American equity options.

⁸Medvedev and Scaillet (2010) introduce an analytical approach to price American options using a short-maturity asymptotic expansion. They perform a throughout numerical investigation for American call and put options with both stochastic CIR interest rates and stochastic underlying's volatility. Analogously, Boyarchenko and Levendorskiĭ (2013) consider a stochastic volatility equity and stochastic interest rates depending on two CIR factors, allowing for non-zero correlations between all the underlying processes. They provide a sophisticated iterative algorithm to price American derivatives that exploits a sequence of embedded perpetual options and their pricing results are in line with those of the Longstaff and Schwartz method and the asymptotics of Medvedev and Scaillet (2010). Pressacco et al. (2008) perform a throughout comparison between lattice and analytical approximation of the early exercise premium in the Black-Scholes framework.

ρ	q	π_E	π_A	$\pi_A^{r_0}$	$ \pi_A - \pi_A^{r_0} /\pi_A$
American put option					
-50%	0%	5.540%	5.674%	5.979%	5.37%
	2%	6.486%	6.499%	6.962%	7.12%
	-2%	4.683%	5.002%	5.230%	4.55%
-5%	0%	5.606%	5.705%	5.979%	4.79%
	2%	6.551%	6.557%	6.962%	6.17%
	-2%	4.748%	5.025%	5.230%	4.07%
0%	0%	5.613%	5.709%	5.979%	4.72%
	2%	6.558%	6.563%	6.962%	6.08%
	-2%	4.755%	5.028%	5.230%	4.01%
5%	0%	5.620%	5.712%	5.979%	4.67%
	2%	6.565%	6.570%	6.962%	5.96%
	-2%	4.763%	5.030%	5.230%	3.97%
50%	0%	5.672%	5.745%	5.979%	4.06%
	2%	6.629%	6.630%	6.962%	5.00%
	-2%	4.827%	5.053%	5.230%	3.50%

Table 3: Results from the three numerical examples for the American put option.

ρ	q	π_E	π_A	$\pi_A^{r_0}$	$ \pi_A - \pi_A^{r_0} /\pi_A$
American call option					
-50%	0%	6.269%	6.271%	5.979%	4.66%
	2%	5.235%	5.356%	5.163%	3.61%
	-2%	7.432%	7.432%	7.102%	4.44%
-5%	0%	6.335%	6.336%	5.979%	5.64%
	2%	5.230%	5.390%	5.163%	4.22%
	-2%	7.497%	7.497%	7.102%	5.26%
0%	0%	6.342%	6.343%	5.979%	5.75%
	2%	5.307%	5.393%	5.163%	4.27%
	-2%	7.505%	7.505%	7.102%	5.37%
5%	0%	6.339%	6.339%	5.979%	5.68%
	2%	5.314%	5.396%	5.163%	4.32%
	-2%	7.511%	7.511%	7.102%	5.45%
50%	0%	6.415%	6.415%	5.979%	6.80%
	2%	5.378%	5.431%	5.163%	4.94%
	-2%	7.576%	7.576%	7.102%	6.25%

Table 4: Results from the three numerical examples for the American call option.