American Options and Stochastic Interest Rates

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2. **American Equity Options**: represent the majority of exchange-traded derivatives and their valuation may be challenging;
3. **Do (possibly) negative stochastic interest rates** affect the optimal exercise policies of American equity call and put options?
   - Battauz et al. (2015) find that constant negative interest rates impact on American call and put options

Novelties and Results

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2. The characterization of the optimal exercise policy of American equity options under (possibly negative) stochastic interest rates;

Section 2

The Market

The Market

- As in Vasicek (1977), we assume a mean reverting process for the short-term interest rate. Under a risk-neutral measure $\mathbb{Q}$:

$$\mathrm{d}r(t) = \kappa\left(\theta - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t)$$

  - $r(0) = r_0 \in (-1, +\infty)$,
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- A risky equity is traded. Under $\mathbb{Q}$:

$$\mathrm{d}S(t) = S(t)\left(r(t) - q\right)\mathrm{d}t + S(t)\sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t)$$

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    - q constant annual dividend yield,
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- S and r are correlated: $Cov(\mathrm{d}W_S^{\mathbb{Q}}, \mathrm{d}W_r^{\mathbb{Q}}) = \rho \mathrm{d}t$.

The market

- The investor can trade frictionlessly in the following assets:
  - the lognormal risky security S,

$$S(t) = S_0 \exp\left[\int_0^t r(\tau)\mathrm{d}\tau - \left(q + \frac{\sigma_S^2}{2}\right)t + \sigma_S W_S^{\mathbb{Q}}(t)\right];$$

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  - a family of zero coupon bonds with maturity τ up to T:

$$p(t,\tau) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{B(\tau)}\right] = exp\left[A(t,\tau) - B(t,\tau)r(t)\right].$$

more...

American options

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- The value of the American option at t is:

$$\pi_A(t) = ess \sup_{t \leq \tau \leq T} \mathbb{E}_t^{\mathbb{Q}}\left[\varphi(S(\tau))\frac{B(t)}{B(\tau)}\right]$$

Section 3

The Quadrinomial Tree

The quadrinomial tree

- Apply Itô's Lemma to $Y(t) := \ln(S(t)/S_0)$, we get:

$$\begin{cases} \mathrm{d}Y(t) & = \mu_Y \mathrm{d}t + \sigma_S \mathrm{d}W_S(t) \\ \mathrm{d}r(t) & = \mu_r \mathrm{d}t + \sigma_r \mathrm{d}W_r(t) \end{cases}$$

where $\mu_Y := (r(t) - q - \sigma_S^2/2)$ and $\mu_r := \kappa(\theta - r(t))$.

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where $\mu_Y := (r(t) - q - \sigma_S^2/2)$ and $\mu_r := \kappa(\theta - r(t))$.

- Consider the discretizations:

$$Y(t+\Delta t) \begin{array}{l} \nearrow \; Y(t) + \Delta Y^+ \\ \searrow \; Y(t) + \Delta Y^- \end{array} \qquad r(t+\Delta t) \begin{array}{l} \nearrow \; r(t) + \Delta r^+ \\ \searrow \; r(t) + \Delta r^- \end{array}$$

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- We discretise the processes $Y(t) = \ln S(t)/S_0$ and $r(t)$ with $\{Y_i, r_i\}_{i=0,\ldots,n}$ such that

$$(Y_{i+1}, r_{i+1}) = \begin{cases} (Y_i + \Delta Y^+, r_i + \Delta r^+) & \text{with probability } q_{uu} \\ (Y_i + \Delta Y^+, r_i + \Delta r^-) & \text{with probability } q_{ud} \\ (Y_i + \Delta Y^-, r_i + \Delta r^+) & \text{with probability } q_{du} \\ (Y_i + \Delta Y^-, r_i + \Delta r^-) & \text{with probability } q_{dd} \end{cases}$$

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- There are eight parameters to set: q_{uu}, q_{ud}, q_{du}, q_{dd}, $\Delta Y^\pm$ and $\Delta r^\pm$.

The quadrinomial discretization

- Therefore, we impose 8 conditions:

$$
\begin{cases}
\mathbb{E}_t[\Delta Y] = (q_{uu} + q_{ud})\Delta Y^+ + (q_{du} + q_{dd})\Delta Y^- \overset{!}{=} \mu_Y \Delta t \\
\mathbb{E}_t[\Delta r] = (q_{uu} + q_{du})\Delta r^+ + (q_{ud} + q_{dd})\Delta r^- \overset{!}{=} \mu_r \Delta t \\
\mathbb{E}_t[\Delta Y^2] = (q_{uu} + q_{ud})(\Delta Y^+)^2 + (q_{du} + q_{dd})(\Delta Y^-)^2 \overset{!}{=} \sigma_Y^2 \Delta t \\
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\mathbb{E}_t[\Delta Y \Delta r] = q_{uu}\Delta Y^+ \Delta r^+ + q_{ud}\Delta Y^+ \Delta r^- + \\
\qquad + q_{du}\Delta Y^- \Delta r^+ + q_{dd}\Delta Y^- \Delta r^- \overset{!}{=} \rho \sigma_S \sigma_r \Delta t \\
q_{uu} + q_{ud} + q_{du} + q_{dd} \overset{!}{=} 1 \\
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- The solution is:

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$$\begin{aligned} q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S} \\ q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S} \\ q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S} \\ q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S}. \end{aligned}$$

with $\mu_Y := (r(t) - q - \sigma_S^2/2)$ and $\mu_r := \kappa(\theta - r(t))$.

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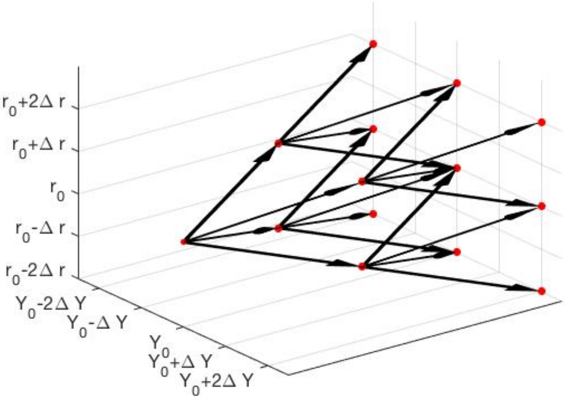
  with $\mu_Y := (r(t) - q - \sigma_S^2/2)$ and $\mu_r := \kappa(\theta - r(t))$.

- $\Delta t \to 0 \implies q_{uu}, q_{dd} \to \frac{(1+\rho)}{4} > 0$ and $q_{ud}, q_{du} \to \frac{(1-\rho)}{4} > 0$.

  more...

A graphical intuition



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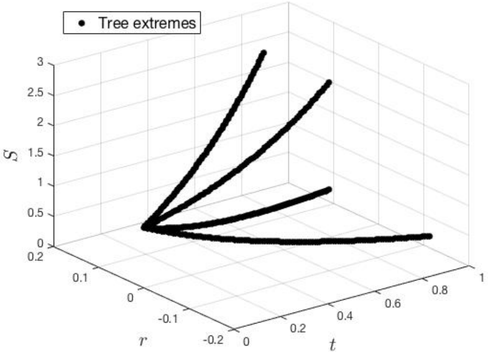

$$T = 1, \; n = 125$$

$$S_0 = 1, \; \sigma_S = 0.05, \; r_0 = 0, \; \kappa = 0.7, \; \theta = 0.02 \; \sigma_r = 0.01, \; \rho = 0.5$$

- The computational complexity is quadratic in the number of steps. (e.g. $125^2 = 15'625 <<< 2^{125} \sim 10^{38}$ possible final values)

Convergence results

- Following Stroock and Varadhan (1997), based on Nelson and Ramaswamy (1990):

Convergence of the bivariate approximation

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- Exploiting Mulinacci and Pratelli (1998):

Convergence of the Snell envelope

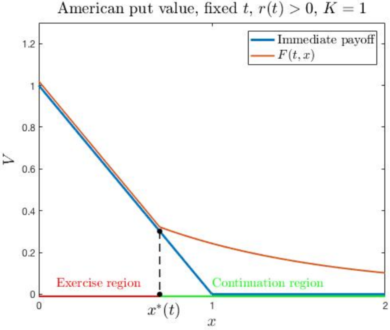
If the lattice-approximated underlying converges in distribution to a stochastic process, so does the American option value on the underlying.

Section 4

American Options

The American Put Option, r constant, $r > 0$

$$\pi_A(t) = \sup_{0 \le \tau \le T-t} \mathbb{E}^{\mathbb{Q}}\left[(K - S(\tau))^+ e^{-r\tau}\right] = F(t,x), \quad x = S(t)$$



$$r \ge 0 \implies F(t,0) = K$$

The American Put Option, r constant, $r < 0$

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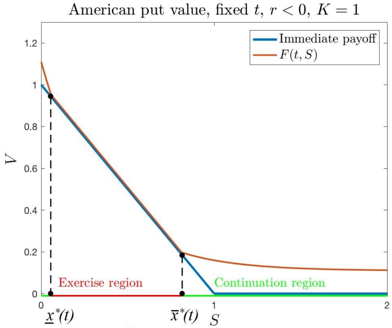

$$r < 0 \implies F(t, 0) > K$$

The Free Boundaries, $r < 0$

- $\mu > 0 \iff q < r - \sigma_S^2/2$: a tradeoff triggers optimal early exercise.
- The functions $t \mapsto \bar{x}^*(t), \underline{x}^*(t)$ are called the upper/lower free boundaries.

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- The functions $t \mapsto \bar{x}^*(t), \underline{x}^*(t)$ are called the upper/lower free boundaries.
- At any $t \leq T$ the investor look at the current value of S, $S(t)$:
  - if $S(t) \leq \underline{x}^*(t) \cup S(t) \geq \bar{x}^*(t) \to$ wait
  - if $\underline{x}^*(t) < S(t) < \bar{x}^*(t) \to$ exercise

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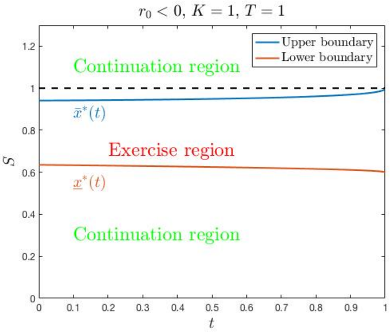

Section 5

Stochastic r

Stochastic r

The value of an American put option in the market with stochastic r:

$$\pi^A(t) = ess \sup_{t \leq \tau \leq T} \mathbb{E}_t^{\mathbb{Q}} \left[(K - S(\tau))^+ e^{-\int_t^\tau r(z) \mathrm{d}z} \right]$$

$$S(\tau) = S(t) \exp\left(\int_t^\tau r(z) \mathrm{d}z - (q + \sigma_S^2/2)\tau + \sigma_S (W_S(\tau) - W_S(t)) \right).$$

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V^A is a deterministic function

$$\pi^A(t) =: F(t, x, r)$$

where $x = S(t)$ and $r = r(t)$.

more...

Stochastic r: properties of $F(t, x, r)$

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Stochastic r: properties of $F(t, x, r)$

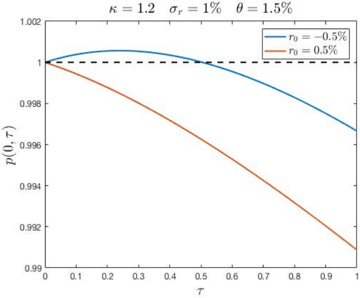
- Fix t and r. $F(t, x, r)$ is still $\geq (K-x)^+$ and it is still decreasing and convex in x.
- For $x = 0$

$$F(t,0,r) = \sup_{0\leq\eta\leq T-t} \mathbb{E}^{\mathbb{Q}}\left[Ke^{-\int_0^\eta r(z)\mathrm{d}z}\right]$$
$$= \sup_{0\leq\eta\leq T-t} Kp(0,\eta)$$

Hence, $F(t,0,r) > K \iff p(0,\eta) > 1$ for some $\eta \in [0, T-t]$.

Stochastic r: properties of $F(t, x, r)$

Plotting $p(0, \tau)$



r stochastic

Jointly necessary conditions for the existence of a double continuation region at t and given $r(t) = r$:

[NC0] the current level of the interest rate r is such that $r\alpha - \theta_\varphi(\alpha + (T - t)) > 0$ with $\alpha = \frac{e^{-\kappa(T-t)}-1}{\kappa} \leq 0$ where $\theta_\varphi = \theta$ for the American put (resp. $\theta_\varphi = \theta + \frac{\sigma_r \sigma_S \rho}{\kappa}$ for the American call);

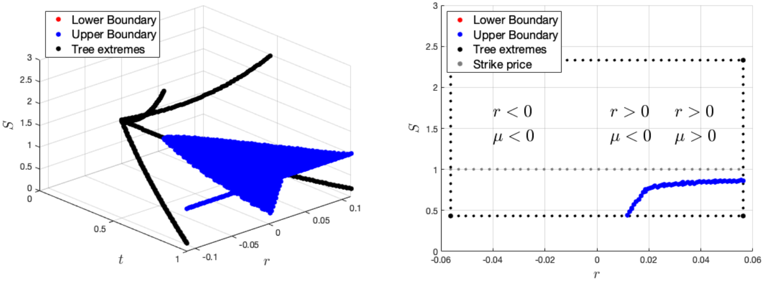
[NC1] the dividend yield is non positive, $q \leq 0$;

[NC2] for some S, $\pi_E(t, S, r) = \varphi(S)$, where $\pi_E(t, S, r)$ is the value of the European option, and $\varphi(S)$ is the immediate option payoff.

more...

American put, r stochastic, $q = 2\%$

$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$

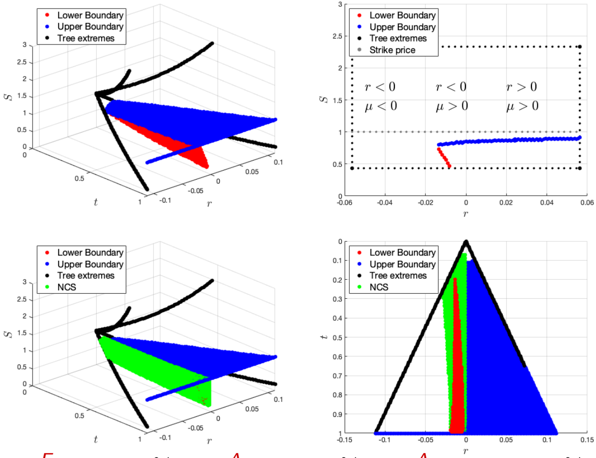


$$\pi^E = 6.565\%, \quad \pi^A = 6.973\%, \quad \pi^A_{r=0\%} = 6.570\%$$

The Early Exercise Region is increasing w.r.t. r

American put, r stochastic, $q = -2\%$

$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$

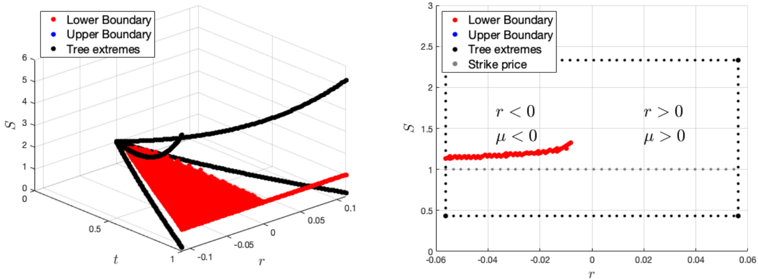


$\pi^E = 4.763\%, \quad \pi^A = 5.240\%, \quad \pi^A_{r=0\%} = 5.030\%$

more...

American call, r stochastic, $q = 0\%$

$$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$$

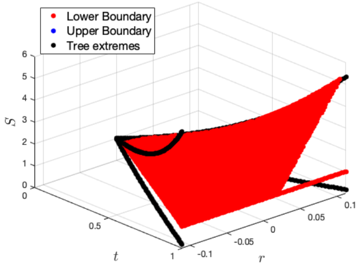


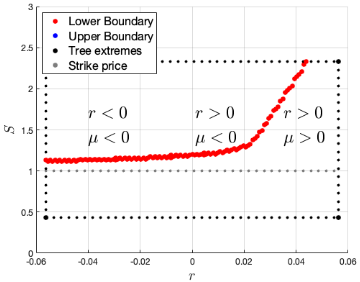
$$\pi^E = 6.339\%, \quad \pi^A = 6.339\%, \quad \pi^A_{r=0\%} = 5.979\%$$

The free boundary is increasing w.r.t. r

American call, r stochastic, $q = 2\%$

$$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$$

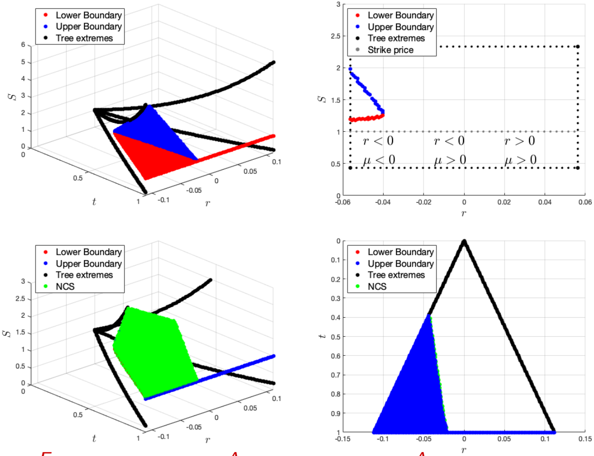




$$\pi^E = 5.314\%, \quad \pi^A = 5.396\%, \quad \pi^A_{r=0\%} = 5.163\%$$

American call, r stochastic, $q = -2\%$

$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$



$\pi^E = 7.511\%, \quad \pi^A = 7.511\%, \quad \pi^A_{r=0\%} = 7.102\%$

Section 6

Conclusions

Our results:

- We developed the quadrinomial tree, a new lattice-based approach to discretize a financial market with a risky equity correlated with a stochastic interest rate.
- We investigated the properties of Americanoptions in the stochastic interest rate framework $\rightsquigarrow$ double continuation region, early exercise of American call options on zero-dividend equity.
- We characterize the optimal exercise policy of such options for different sets of parameters.

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Thanks for your attention!

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Thanks for your attention!

The bond pays 1 to its holder at T and its price at $t \in (0, T)$ is labelled with $p(t, T)$. By no arbitrage valuation, we have

$$p(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left[-\int_0^t r(s)\mathrm{d}s\right]\middle|\mathcal{F}_t\right],$$

that admits a closed formula solution as derived in Section 3.2.1 of Brigo, Mercurio (2007):

$$p(t, T) = e^{A(t,T)-B(t,T)r(t)}$$

where:

$$\begin{aligned} B(t, T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \\ A(t, T) &= \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right)\left(B(t, T) - (T - t)\right) - \frac{\sigma_r^2 B^2(t, T)}{4\kappa}. \end{aligned}$$

back

The probabilities

- Recall that $\mu_Y := (r(t) - q - \sigma_S^2/2)$ and $\mu_r := \kappa(\theta - r(t))$ and:

$$
\begin{aligned}
q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S} \\
q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S} \\
q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S} \\
q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_Y}{4\sigma_r \sigma_S}
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\end{aligned}
$$

- Each inequality is quadratic in $r(t)$.

The probabilities

- At any t, the four inequalities are satisfied when:

$$\begin{cases} \underline{r}_{uu} \leq r \leq \overline{r}_{uu} \\ r \leq \underline{r}_{ud} \cup r \geq \overline{r}_{ud} \\ r \leq \underline{r}_{du} \cup r \geq \overline{r}_{du} \\ \underline{r}_{dd} \leq r \leq \overline{r}_{dd} \end{cases}$$

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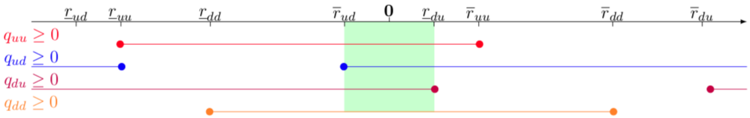

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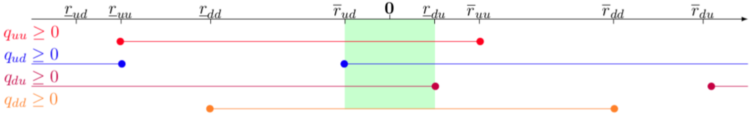

- q_{ud} and q_{du} may become negative!

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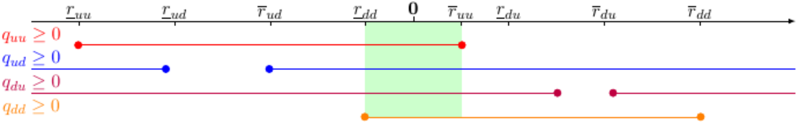

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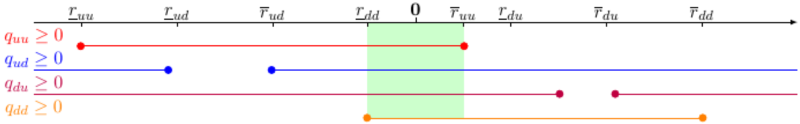
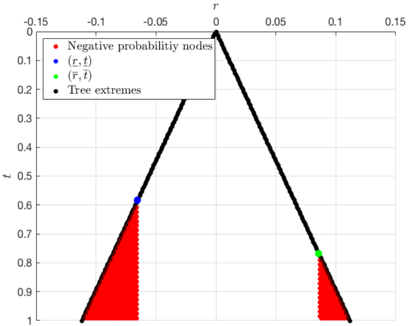


- q_{uu} and q_{dd} may become negative!

The probabilities

- When might negative probabilities appear?

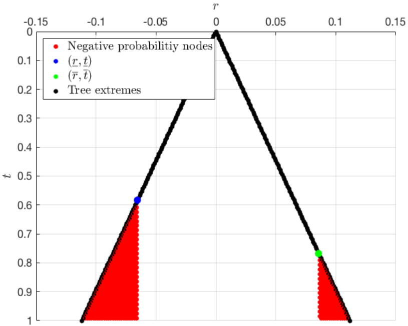


Section at $S = 0$; $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$, $n = 125$

back

The probabilities

- When might negative probabilities appear?



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- Nevertheless, $\mathbb{Q}$(“reaching the blue point”) $\approx 10^{-6}$.

back

Convergence results

Convergence of the discretization

Let A1-A4 hold. Then $X_i := (Y_i, r_i) \xrightarrow{d} X(t)$.

A1 the functions $\mu(x,t)$ and $\sigma(x,t)$ are continuous and $\sigma(x,t)$ is non negative;

A2 with probability 1 a solution $(X_t)_t$ to the SDE:

$$X_t = X_0 + \int_0^t \mu(X_s, s)\mathrm{d}s + \int_0^t \sigma(X_s, s) \cdot \mathrm{d}W(s)$$

exists for $0 < t < +\infty$ and it is unique in law;

A3 for all δ, $T > 0$

$$\lim_{n \to +\infty} \sup_{||x|| \leq \delta, 0 \leq t \leq T} |\Delta Y^{\pm}| = 0$$

$$\lim_{n \to +\infty} \sup_{||x|| \leq \delta, 0 \leq t \leq T} |\Delta r^{\pm}| = 0;$$

Convergence results

A4 let $X_{i,j}$ indicate the j-th entry of X_i and let $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ be the filtration generated by the discrete bivariate process (X_i). Define:

$$\mu_i(x,t) := \begin{bmatrix} \mu_{i,1}(x,t) \\ \mu_{i,2}(x,t) \end{bmatrix} \text{ and } \sigma_i^2(x,t) := \begin{bmatrix} \sigma_{i,1}^2(x,t) \\ \sigma_{i,2}^2(x,t) \end{bmatrix}$$

where $\mu_{i,j}(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[X_{i+1,j} - X_{i,j}|\mathcal{F}_i]}{\frac{T}{n}}$ and $\sigma_{i,j}^2(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,j} - X_{i,j})^2|\mathcal{F}_i]}{\frac{T}{n}}$ for $j = 1, 2$. Let $\rho_i(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,1} - X_{i,1})(X_{i+1,2} - X_{i,2})|\mathcal{F}_i]}{\frac{T}{n}}$ and $\rho(x,t) = \sigma_1(x,t) \cdot \sigma_2(x,t)'$ where $\sigma_j(x,t)$ is the j-th row of $\sigma(x,t)$. Then, for all $\delta, T > 0$,

$$\lim_{n \to +\infty} \sup_{||x|| \le \delta, 0 \le t \le T} ||\mu_i(x,t) - \mu(x,t)|| = 0$$

$$\lim_{n \to +\infty} \sup_{||x|| \le \delta, 0 \le t \le T} ||\sigma_i^2(x,t) - \sigma^2(x,t) \cdot \mathbf{I}_2|| = 0$$

$$\lim_{n \to +\infty} \sup_{||x|| \le \delta, 0 \le t \le T} |\rho_i(x,t) - \rho(x,t)| = 0$$

where $\mathbf{I}_n$ is the column vector with all of the n entries equal to one.

back

The European Put Option

The price at $t = 0$ of an European put option on S with strike K is equal to

$$\pi^E(0) = Kp(0,T)N(-\tilde{d}_2) - S_0 e^{-qT} N(-\tilde{d}_1)$$

with:

$$\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma^2_{0,T}}}\left(\ln \frac{S_0}{Kp(0,T)} - \frac{1}{2}\Sigma^2_{0,T} - qT\right), \\
\tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma^2_{0,T}}, \\
\Sigma^2_{0,T} &= \sigma_S^2 T + 2\sigma_S \sigma_r \rho \left(\frac{-1 + e^{-\kappa T} + \kappa T}{k^2}\right) + \\
&\quad -\sigma_r^2 \left(\frac{3 + e^{-2\kappa T} - 4e^{-\kappa T} - 2\kappa T}{2k^3}\right).
\end{aligned}$$

The European Call Option

The price at $t = 0$ of an European put option on S with strike K is equal to

$$\pi^E(0) = S_0 e^{-qT} N(\tilde{d}_1) - Kp(0,T) N(\tilde{d}_2)$$

with:

$$\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma^2_{0,T}}} \left(\ln \frac{S_0}{Kp(0,T)} - \frac{1}{2} \Sigma^2_{0,T} - qT \right), \\
\tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma^2_{0,T}}, \\
\Sigma^2_{0,T} &= \sigma_S^2 T + 2\sigma_S \sigma_r \rho \left(\frac{-1 + e^{-\kappa T} + \kappa T}{k^2} \right) + \\
&\quad - \sigma_r^2 \left(\frac{3 + e^{-2\kappa T} - 4e^{-\kappa T} - 2\kappa T}{2k^3} \right).
\end{aligned}$$

back

V^A as a deterministic function

In the market described above, the value of an American put option on S is of the form:

$$V^A = F(t, S(t), r(t))$$

with $F : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ given by:

$$F(t, x, r) = \sup_{0 \le \eta \le T-t} \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_0^{\eta} r(s) \mathrm{d}s \right) \cdot \right.$$

$$\left. \cdot \left(K - x \exp\left(\int_0^{\eta} r(s) \mathrm{d}s - \left(q + \frac{1}{2}\sigma_S^2 \right) \eta + \sigma_S W_S(\eta) \right) \right)^+ \right]$$

and $r(0) = r$.

back

Proof of necessary conditions [NC0], [NC1], [NC2]

[NC0] for the put option $\iff F(t,0,r) > K$ iff

$$\sup_{0\leq\eta\leq T-t} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s\right)\right] > 1.$$

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s\right)\right] \geq \ldots$$

$$\geq \exp\left(-\int_0^\eta re^{-\kappa s} + \theta(1-e^{-\kappa s})\mathrm{d}s\right) = \exp\left(r\alpha - \theta(\alpha+\eta)\right) \geq 1$$

where $\alpha := \dfrac{e^{-\kappa\eta}-1}{\kappa}$.

For $\kappa \cdot \eta \approx 0$ the last inequality holds if

$$\left(-1+\frac{\kappa\eta}{2}\right)(r-\theta) - \theta > 0.$$

- for $r > \theta$ if $T - t > \dfrac{2}{\kappa}\cdot\dfrac{r}{r-\theta}$
- for $r < \theta$ if $\kappa r < 0$ i.e. $r < 0$.

Proof of necessary conditions [NC0], [NC1], [NC2]

Proof of [NC0] for the call option.

$$\mathbb{E}^{\mathbb{Q}}\left[(S(\tau)-K)^{+}\,e^{-\int_0^\tau r(s)\mathrm{d}s}\right]=\mathbb{E}^{\mathbb{Q}^S}\left[\left(\frac{1}{K}-\frac{1}{S(\tau)}\right)^{+}Ke^{-q\tau}S(0)\right]$$
$$\geq\left(S(0)e^{-q\tau}-K\mathbb{E}^{\mathbb{Q}^S}\left[\frac{e^{-q\tau}S(0)}{S(\tau)}\right]\right)^{+}$$

with

$$\mathbb{E}^{\mathbb{Q}^S}\left[\frac{e^{-q\tau}S(0)}{S(\tau)}\right]=\mathbb{E}^{\mathbb{Q}^S}\left[e^{-\int_0^\tau r(s)\mathrm{d}s}\right]=p_{\varphi}(0,\tau)$$

Proof of necessary conditions [NC0], [NC1], [NC2]

since with respect to th S-martingale measure $\mathbb{Q}^S$

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} & = (r(t) - q + \sigma_S^2)\mathrm{d}t + [\sigma_S \quad 0] \cdot \mathrm{d}W^{\mathbb{Q}^S}(t) \\ \mathrm{d}r(t) & = \kappa(\theta - r(t) + \frac{\rho\sigma_S\sigma_r}{\kappa})\mathrm{d}t + [\sigma_r\rho \quad \sigma_r\sqrt{1-\rho^2}] \cdot \mathrm{d}W^{\mathbb{Q}^S}(t) \end{cases}$$

and Ito's formula delivers

$$\mathrm{d}\left(\frac{1}{S(t)}\right) = \frac{1}{S(t)}\Big((q - r(t))\mathrm{d}t - [\sigma_S \quad 0] \cdot \mathrm{d}W^{\mathbb{Q}^S}(t)\Big)$$

back

Free boundaries w.r.t. t

$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$

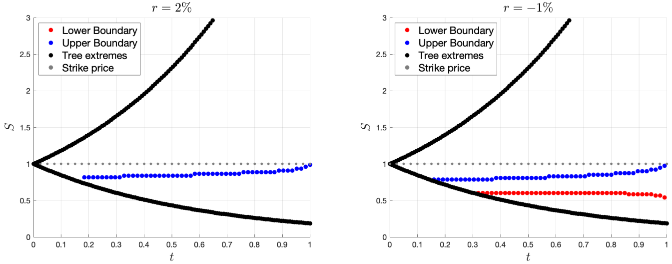

Figure: $r-$sections of free boundaries for the American put option. Left panel $r = 2\%$ and $q = 0\%$. Right panel $r = -1\%$ and $q = -2\%$.

Free boundaries w.r.t. t

$T = 1, n = 125, S_0 = K = 1, \sigma_S = 10\%, r_0 = 0\%, \theta = 2\%, \kappa = 1, \sigma_r = 2\%, \rho = 0.05$

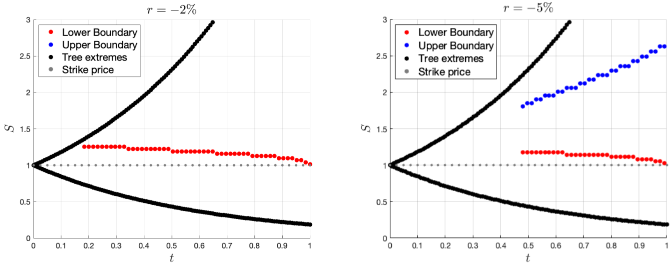

Figure: $r-$sections of free boundaries for the American call option. Left panel $r = -2\%$ and $q = 0\%$. Right panel $r = -5\%$ and $q = -2\%$.

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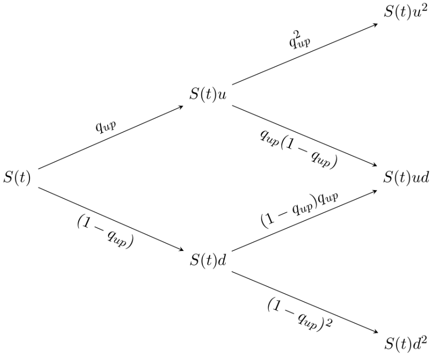
The binomial tree of CRR

In their seminal work Cox, Ross, Rubinstein (1979) provide a binomial discretization of the lognormal security $S(t)$.



where $u = e^{\sigma\sqrt{\Delta t}}$, $\quad d = u^{-1}$, and $q_{up} = \frac{e^{(r-q)\Delta t} - d}{u - d}$.