Sensitivity Analysis of Model Output with Input Constraints: a Generalized Rationale for Local Methods

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Abstract

In this work, we introduce a generalized rationale for local sensitivity analysis methods that allows to solve the problems connected with input constraints. Several models in use in the risk analysis field are characterized by the presence of deterministic relationships among the input parameters. However, sensitivity analysis issues related to the presence of constraints have been mainly dealt with in a heuristic fashion. We start with a systematic analysis of the effects of constraints. The findings can be summarized in the following three effects. *i*) Constraints makes it impossible to vary one parameter while keeping all others fixed. *ii*) The model output becomes insensitive to a parameter if a constraint is solved for that parameter. *iii*) Sensitivity analysis results depend on which parameter is selected as dependent. The explanation of these effects is found by proposing a result that leads to a natural extension of the local sensitivity analysis rationale introduced in Helton (1993). We then extend the definitions of the Birnbaum, Criticality and the Differential importance measures to the constrained case. In addition, it is introduced a procedure that allows to obtain constrained sensitivity results at the same cost as in the absence of constraints. The application to a non-binary event tree concludes the paper providing a numerical illustration of the above findings.

Keywords: Sensitivity Analysis, Local Importance Measures, Event Trees, Risk Analysis Models.

1 Introduction

Sensitivity analysis (SA) is one of the key steps both in the building and utilization of risk analysis models. In Kessler and McGuire (1999) *"sensitivity runs"* are utilized in the corroboration of the 162 branch event tree model of the Yucca Mountain waste disposal system. Saltelli *et al* (2000) demonstrate the use of SA in the model building process. Saltelli (2002) underlines the role of on global methods for importance assessment. Frey and Patil (2002) and Patil and Frey (2004) present, besides global methods, a thorough overview and application of local methods for deterministic models in food-safety assessment. Iman *et al* (2005a) and (2005b) illustrate the use of uncertainty and SA techniques in the evaluation of hurricane losses. Borgonovo (2006a) compares different global methods in determining uncertainty importance. Although some of the former works deal with local methods, and some with global methods, all the presented techniques rest on a common assumption: no deterministic relationships must be respected by parameters during the sensitivity. Several applications, however, entail the creation of models in which parameters are constrained. Examples are event trees (see Philipson and Wilde, 2000), decision trees (see Clemen, 1997), Markov-chain based models (Rief, 1998.) To study the sensitivity of the output to changes in the probabilities, one needs to account for the fact that probabilities of outcomes of the same node (event trees and decision trees) or in the same row (Markov matrices) must sum to unity. The way one deals with the constraint is usually heuristic. For example, it is customary to express the last probabilities in each node as the complement to unity of the others (see for instance Clemen, 1997, p. 133.) This approach, however, can be subject to criticism. Besides noting that the solve-for-one parameter method cannot be applied in general — if the constraint is not analytically solvable, it is not possible to express one of the parameters as an explicit function of the others, — one is also left with the question of how/whether the sensitivity results change if, instead of solving for the last probability, one solves for any other probabilities in the same constraint.

In this work, we address the issues connected with local SA of model output in the presence of constraints and introduce a result that allows their solution. The first step is the investigation of the differences between a constrained and an unconstrained sensitivity exercise. We summarize the findings of the analysis into three main Effects: *1)* the change in one parameter induces a change in the other parameters bound by the same constraint; *2)* SA results depend on which parameters are selected as dependent in the constraints; *3)* once a parameter is selected as dependent, the sensitivity of the output on that parameter is null. Effect 1 implies that one cannot perform a one-parameter-at-a-time SA in the presence of constraints. Thus, Effect 1 impairs the applicability of techniques as Tornado Diagrams (Clemen, 1997 or Howard, 1988) or Nominal range sensitivity (Frey and Patil, 2002) — as well as its variations, as the ΔLOR method in (Frey and Patil, 2002) — that evaluate the "*effect on model outputs exerted by individually varying only one of the model inputs across its entire range of plausible values, while holding all other inputs at their nominal or base-case values* (Frey and Patil, 2002)." Effects 2 and 3 have the following numerical implications. To inspect the change in results connected with the dependent parameter choice it is, in principle, necessary to apply the sensitivity algorithms as many times as there are available choices of dependent parameters. If the model is computationally intensive, having to repeat the sensitivity runs can become time consuming (if not impossible.)

As the local sensitivity effects of constraints have not been systematically studied (nor have been the global ones, but the focus of this work is on the local results[1]), we are then left with answering the question of the mathematical explanation of the Effects. To explain the Effects, we start with the rationale for local SA proposed in Helton (1993) and look for a generalization of the rationale that allows to perform local SA in the presence of any (solvable and non-solvable) constraints. The result is obtained by nesting the differentiation of the constraints into the differentiation of the model output.

Since several local importance measures descend from Helton's (1993) rationale, one needs to deal with the impact of the generalized rationale on the definition of these local sensitivity indicators. In particular, we determine the modifications induced by constraints on the definitions of the Birnbaum [Birnbaum (1969)], the Criticality [Cheok *et al* (1998), Borgonovo and Apostolakis (2001), Frey and Patil (2002), Patil and Frey (2004)] and the Differential importance measures [Borgonovo and Apostolakis (2001), Borgonovo (2007)].

We are then left with addressing the computational implications of the new approach. We show

[1]In this work, we do not discuss the effects of constraints in global Sensitivity Analysis, which are the subject of future research. We can briefly summarize some of the features: the first effect is that, in performing the global sensitivity exercise, one needs to generate inputs that not only reflect the analyst uncertainty, but that also match the equation of the constraint. This generates several complications, as illustrated in Philipson and Wilde (2000) for the case of probabilities summing to unity. The second effect is that deterministic constraints induce correlations among the inputs. As a consequence, one cannot utilize techniques that rely on independence among the parameters.

that, in virtue of the generalized rationale, no further runs of the sensitivity algorithm are necessary to inspect the effect of changes in the parameter selections, with a computational cost saving of order (at least) n. Furthermore, in the case of probabilistic models, the generalized rationale allows to find constrained sensitivity results through the sole output differentiation, *i.e.*, at the same cost of an unconstrained SA. Finally, we illustrate the implementation of the generalized rationale structured into four steps by means of the application to a non-binary event tree.

The remainder of the paper is organized as follows. In Section 2, we discuss the Effects of constraints and list some examples of models in use in risk analysis and decision making that involve input constraints. In Section 3, we present the rationale for local SA with constraints and discuss how it explains the constrained sensitivity Effects. In Section 4, we illustrate the generalization of the definitions of the Birnbaum, Criticality and Differential importance measures. In Section 5, we define a stepwise implementation of the rationale and illustrate its application to a non-binary event tree. Conclusions are offered in Section 6.

2 Effects of Constraints in Local SA and Examples of Constrained Sensitivity Problems

In this Section, we carry out the first step of the analysis, namely, the identification of the key features of the presence of constraints. We introduce them by means of an example.

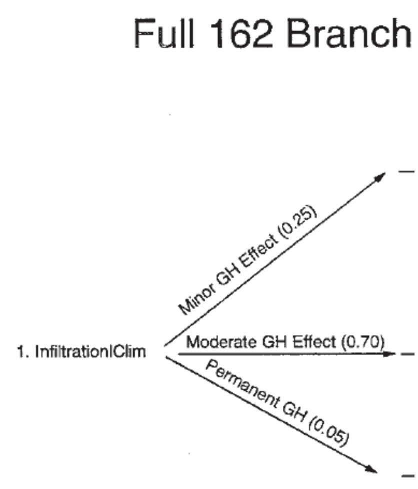

Figure 1: First branhces form the 162 Branch Case of the IMARC model in Kessler and McGuire (1999.)

Example 1 *Figure 1 reproduces the first of the 162 branches of the IMARC logic tree [Figure 3 in Kessler and McGuire (1999).] Suppose that an analyst is interested in the sensitivity of the model output to a change in* $P(\textit{Minor GH Effect})$ *from* 0.25 *to* 0.31. $P(\textit{Minor GH Effect})$ *is in the same node as* $P(\textit{Moderate GH Effect})$ *and* $P(\textit{Permanent GH})$*, whose base case values are* 0.70 *and* 0.05 *respectively. To assess the sensitivity let us apply a Tornado Diagram scheme [see Chapter 5 of Clemen (1997); the name Tornado Diagram is attributed to Howard (1988.)] The definition of Tornado Diagram foresees that one performs the sensitivity by keeping all parameters fixed, but the one of interest. In so doing, one would have:* $P(\textit{Minor GH Effect}) + P(\textit{Moderate GH Effect}) + P(\textit{Permanent GH}) = 1.06$ *thus violating the laws of probability. In fact, it must hold that:*

$$P(\textit{Minor GH Effect}) + P(\textit{Moderate GH Effect}) + P(\textit{Permanent GH}) = 1 \tag{1}$$

The problem is overcome if one, as usual, adopts the solve-for-one-probability approach. For example, if one selects $P(\textit{Moderate GH Effect})$ *as dependent and writes:* $P(\textit{Moderate GH Effect}) = 1 - P(\textit{Minor GH Effect}) - P(\textit{Permanent GH})$*, one would get* $P(\textit{Moderate GH Effect}) = 1 - 0.31 - 0.05 = 0.64$ *and then one can appreciate the sensitivity of the output inserting the values:* $P(\textit{Minor GH Effect}) = 0.31$*,* $P(\textit{Moderate GH Effect}) = 0.64$ *and* $P(\textit{Permanent GH}) = 0.05$*.*

Example 1 shows the first effect generated by the presence of constraints:

Effect 1 the sensitivity on one of the parameters, P(Minor GH Effect) in the example, induces a change in the values of the other parameters that are bound by the same constraint. In our case, the change in P(Minor GH Effect) from 0.25 to 0.31 is rebalanced by the change in P(Moderate GH Effect) that shifts from 0.70 to 0.64 to respect the constraint.

Effect 1 implies that, in the presence of constraints, one cannot perform a one-variable-at a-time SA, *i.e.*, one cannot vary solely the parameter of interest while keeping the other parameters fixed. Thus, techniques as Tornado Diagrams (Clemen, 1997- Ch. 5, or Howard, 1988), nominal range sensitivity (Frey and Patil, 2002) and their variations (Frey and Patil, 2002) cannot be directly applied in the presence of constraints.

A second effect induced by the presence of constraints is revealed by the next Example.

Example 2 *Suppose that, instead of P(Moderate GH Effect) one considers P(Permanent GH) as dependent and solves the constraint writing: $P(\text{Permanent GH}) = 1 - P(\text{Minor GH Effect}) - P(\text{Moderate GH Effect})$. In this case, one would end up with: $P(\text{Permanent GH}) = 1 - 0.31 - 0.7 = -0.1$. That is, the sensitivity is feasible if one solves the constraint for P(Moderate GH Effect), but not feasible if one solves for P(Permanent GH).*

Example 2 is a reflection of the following Effect 2.

Effect 2 In the presence of constraints, SA results depends on the selection of the dependent/independent parameter(s).

It is worth noting that a numerical complication is associated with Effect 2. It originates in an SA application to portfolio management (we refer to Reyes Santos and Haimes, 2004 for an overview of portfolio problems.) In presence of budget constraints the asset weights must sum to unity: $\sum_{i=1}^{n} x_i = 1$. The solution one adopts is to solve the constraint for the n^{th} asset, $x_n = 1 - \sum_{i=1}^{n-1} x_i$. Such an asset, then, has its variation determined by the others, and act only as a rebalancing entity. It is called the pivotal asset. It is of interests to analysts to inspect the effect of the pivotal asset choice [see Borgonovo and Percoco, 2007]. However, this implies to repeat the SA n times[2] — in fact, given n inputs and 1 constraint there are n possible choice of the pivotal asset. — More in general, it could be of interest in the SA of any model to appreciate the effect of a certain selection of the pivotal parameters. Thus, one would need to run the sensitivity algorithm as many times as there are available choices, with the consequence that, if the model is computationally intensive, the SA exercise can become extremely time consuming.

Finally, there is a third effect related to the choice of the dependent variables:

Effect 3 once a parameter is selected as dependent, it disappears from the independent variable list and the sensitivity of the output on that parameter is null.

The above considerations signal that constraints introduce features in a SA of model output that are not encountered if parameters are free.

[2] Quoting directly from Manganelli et al, 2002: *"by changing the pivotal asset, one obtains different sensitivity measures. Computing these sensitivity measures for each single asset of the portfolio, it is possible to compute a matrix of sensitivities analogous to the variance-covariance matrix."*

In the remainder of this Section, we present, without the purpose of being exhaustive, examples of models involving input constraints.

Event Trees. Starting with the work of Kaplan and Garrick (1981,) risk is expressed in terms of triplets of the form $\{s_i, p_i, c_i\}$ where s_i is a scenario, p_i the corresponding probability and c_i the consequence associated with the scenario. In the context of risk assessment of complex technological systems, the calculation of p_i requires to consider *"all possible outcomes of events occurring sequentially* [Papazoglou (1998)]." Diagrams built with the purpose of enabling the assessment of possible consequences and their likelihood are event trees [Papazoglou (1998)]. Let R be the risk metric of interest (examples of risk metrics in nuclear risk assessment are the core damage frequency or the large early release frequency, see Cheok et al, 1998.) R is a function of the probabilities of the sequences of outcomes leading to the consequence of interest: $R = f(p)$.

For the purposes of this discussion it is convenient to use the following notation. Let:

$w = (w_1, w_2, ..., w_n)$ be the set of all outcomes (events) in the event tree;

$p = (p_1, p_2, ..., p_n)$ be the corresponding subjective conditional probabilities;

L be the number of sequences (paths) that lead to consequence c;

$S_l = (w_{1_l}, w_{2_l}, ..., w_{Q_l}) \subseteq \mathcal{W}$ be the subset of outcomes that form the l^{th} sequence leading to consequence c $(l = 1, 2, ...L)$; $K_l = (1_l, 2_l, ..., Q_l)$ be the corresponding set of indices.

By construction (Papazoglou, 1998), the probability of sequence S_l is $P(S_l) = \prod\limits_{k \in K_l} p_k$ and, therefore:

$$R = \sum_{l=1}^{L} P(S_l) = \sum_{l=1}^{L} \left(\prod_{k \in K_l} p_k \right) = f(p) \tag{2}$$

The specification of the constraints is as follows. Let $q = 1, 2, ..., Q$ the number of nodes involved in the event tree. Let n_q the number of outcomes associated with each node with, clearly, $n = n_1 + n_2 + ... + n_Q$. As the (conditional) probabilities of all outcomes of the same node must sum to unity, one has the set of constraints:

$$\begin{cases} \sum_{s=1}^{n_1} p_s = 1 \\ \sum_{s=n_1}^{n_1+n_2} p_s = 1 \\ \dots \\ \sum_{s=n_1+n_2+...+n_{Q-1}}^{n} p_s = 1 \end{cases} \tag{3}$$

Hence, the SA problem can be formulated as:

$$\begin{array}{c}\text{Study the Sensitivity of}\\ R = f(p),\\ \text{subject to (s.t.)}\\ \left\{\begin{array}{c}\sum_{s=1}^{n_1} p_s = 1\\ \sum_{s=n_1}^{n_1+n_2} p_s = 1\\ \ldots\\ \sum_{s=n_1+n_2+\ldots+n_{Q-1}}^{n} p_s = 1\end{array}\right.\end{array} \tag{4}$$

i.e., it is a constrained SA problem with multiple constraints.

Decision Trees. Papazoglou (1998) establishes that the mathematical foundations and formulas that hold for event trees are also applicable to decision trees: "*the same is true for other applications of event trees like decision trees where the consequences of a particular combination of decision and event outcomes are evaluated in a possibly multidimensional consequence space* [Papazoglou, 1998, p. 170.]" Borgonovo and Peccati (2006b) show that for decision making problems represented in the form of influence diagrams and decision trees the SA of the model output is a constrained one.

Markov Chains. As a further example of risk analysis models whose SA is a constrained problem, we mention models supported by discrete time Markov processes. In such models (see Rief, 1988), an $n \times n$ input matrix of the form $P = [p_{ij}] \;\; i,j = 1,2,...,n$ feeds into the model. As the rows of P must sum to unity, one is led to study the sensitivity of a model output with the inputs constraints represented by

$$\sum_{j=1}^{n} p_{ij} = 1, \;\; \forall i = 1,2,..,n. \tag{5}$$

Hence the SA problem discussed in Rief (1998) is indeed a constrained one.

Finally, we also recall that constrained sensitivity problems are encountered in the SA of *portfolio properties* (Borgonovo and Percoco, 2007).

In summary, several models utilized in risk analysis in both the financial and technological realms impose the presence of input constraints. Thus, in performing their local SA, an analyst has to cope with Effects 1, 2 and 3. In the next Section, we present a result that allows to explain the three Effects and to streamline the computation of local SA.

3 Generalized Rationale for Local Sensitivity Analysis

In this section, we provide a result that allows to modify the rationale for local SA to include the presence of constraints and to explain Effects 1, 2 and 3.

Let

$$Y = f(x),\ \ f : X \subseteq \mathbb{R}^n \to \mathbb{R} \tag{6}$$

a differentiable deterministic n-variate model. We denote the base case of the parameters as $x^0 \in X \subseteq \mathbb{R}^n$ and $Y^0 = f(x^0)$ the corresponding base case the model output. Assuming, as usual, that f is differentiable, the change in Y provoked by the (small) changes $dx = \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ ... \\ x_n - x_n^0 \end{bmatrix}$ can be expressed as (see Helton, 1993):

$$Y - Y^0 = d_1Y + d_2Y + ... + d_nY + o(\left\|x - x^0\right\|^2) \tag{7}$$

where

$$d_iY = \frac{\partial f}{\partial x_i}|_{x^0}\ dx_i \tag{8}$$

is the first order contribution of x_i $(i = 1, 2, ..., n)$ to the total output change dY.

Eqs. (7) and (8) represent the Taylor series expansion used by Helton (1993) and provide the *local SA rationale* from which several local indicators can be derived (see also Borgonovo and Apostolakis, 2001.) Eq. (8), however, holds if parameters are free.

Let us now study how the rationale [eqs. (7) and (8)] is modified by the presence of constraints. To illustrate the procedure, we make use of a single constraint in the next derivation. We provide the extension to the case of multiple constraints at the end of this Section. Therefore, we face the following problem:

$$
\begin{gathered}
\text{Study the Sensitivity of} \\
Y = f(x_1, x_2, ..., x_n) \\
\text{s.t.} \\
g(x_1, x_2, ..., x_n) = c
\end{gathered} \tag{9}
$$

Eq. (9) states the constraint as an implicit equation (the most general form). We suppose that $g(x)$ satisfies the hypotheses of the implicit function theorem and by differentiation of both sides, we get:

$$
\frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + .. + \frac{\partial g}{\partial x_k}dx_k + ... + \frac{\partial g}{\partial x_n}dx_n = 0 \tag{10}
$$

Eq. (10) displays the relationship that binds parameter changes, namely, it shows that parameters are no more "free" to vary. In virtue of eq. (10), one cannot perform the sensitivity on x_i while keeping the other parameters fixed. One or more of the other parameters must vary in accordance with eq. (10) to rebalance the change in x_i, so that $g(x) = c$ (the constraint) is satisfied. Thus, eq. (10) offers the explanation of Effect 1.

Now, suppose that one selects parameter x_k as dependent. Provided that $\dfrac{\partial g}{\partial x_k} \neq 0$, eq. (10) can be solved for dx_k:

$$
dx_k = \frac{-\dfrac{\partial g}{\partial x_1}dx_1 - \dfrac{\partial g}{\partial x_2}dx_2 - ... - \dfrac{\partial g}{\partial x_n}dx_n}{\dfrac{\partial g}{\partial x_k}} \tag{11}
$$

Inserting dx_k [eq. (10)] into eq. (8) and rearranging eq. (7), the change in Y becomes:

$$
\begin{aligned}
Y - Y^0 = &\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_k}\frac{\dfrac{\partial g}{\partial x_1}}{\dfrac{\partial g}{\partial x_k}}\right)|_{x^0}\left(x_1 - x_1^0\right) + \left(\frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_k}\frac{\dfrac{\partial g}{\partial x_2}}{\dfrac{\partial g}{\partial x_k}}\right)|_{x^0}\left(x_2 - x_2^0\right) ...+ \\
&... + \left(\frac{\partial f}{\partial x_n} - \frac{\partial f}{\partial x_k}\frac{\dfrac{\partial g}{\partial x_n}}{\dfrac{\partial g}{\partial x_k}}\right)|_{x^0}\left(x_n - x_n^0\right) + o(\left\|x - x^0\right\|^2)
\end{aligned} \tag{12}
$$

Hence, the following holds.

1. Comparing eq. (12) with eq. (7) one infers that, in the presence of constraints, the change in

Y can be written as $Y - Y^0 = \sum_{i=1}^{n} d_i Y + o(\left\| x - x^0 \right\|^2)$ where:

$$d_i Y = \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_k} \frac{\frac{\partial g}{\partial x_i}}{\frac{\partial g}{\partial x_i}} \right) |_{x^0} \, dx_i \tag{13}$$

Eqs. (12) and (13) generalize Helton's (1993) rationale [eq. (7)] in the presence of one constraint.

2. Comparing eqs. (8) and (13), one notes that, in the constrained case, the output rate of change is represented by $\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_k} \left(\frac{\partial g}{\partial x_i} \Big\backslash \frac{\partial g}{\partial x_k} \right)$, which plays the same role as $\frac{\partial f}{\partial x_i}$ in the unconstrained case. We use the notation:

$$f_{i|k} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_k} \left(\frac{\partial g}{\partial x_i} \Big\backslash \frac{\partial g}{\partial x_k} \right) \tag{14}$$

and call $f_{i|k}$ the *constrained derivative* of f w.r.t. x_i, and x_k the *dependent* or *pivotal* parameter.

In $f_{i|k}$, each free partial derivative $\frac{\partial f}{\partial x_i}$ is diminished by a term $[\frac{\partial f}{\partial x_k} \left(\frac{\partial g}{\partial x_i} \Big\backslash \frac{\partial g}{\partial x_k} \right)]$ generated by the presence of the constraint.

We have mentioned above that eq. (10) explains Effect 1. Let us now investigate how the generalized rationale [eqs. (12), (13)] and $f_{i|k}$ [eq. (14)] explain Effects 2 and 3.

Effect 2 Eq. (14) shows that, in the presence of constraints, local SA results depend on the choice of the dependent asset, as $f_{i|k}$ changes as k changes.

Effect 3 Letting $i = k$ in eq. (14), one gets:

$$f_{k|k} = 0 \tag{15}$$

Eq. (15) implies that, once a variable is chosen as dependent, the sensitivity of the output on that variable is null.

In risk assessment models, it is particularly relevant the case of linear constraints, *i.e.*, the case in which $g(x) = c$ can be written as:

$$\sum_{i=1}^{n} x_i = c \tag{16}$$

For linear constraints [eq. (16)], all the partial derivatives $\dfrac{\partial g}{\partial x_i}$ are equal to unity. Substituting $\dfrac{\partial g}{\partial x_s} = 1 \ \forall i = 1, 2, ..., n$ in eq. (14), one gets:

$$f_{i|k} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_k} \tag{17}$$

Eq. (17) states that, for linear constraints, the constrained derivative $f_{i|k}$ is the difference between the rate of change of f w.r.t. the parameter of interest ($\dfrac{\partial f}{\partial x_i}$) and the rate of change of f w.r.t. the dependent parameter ($\dfrac{\partial f}{\partial x_k}$).

In many applications, multiple constraints apply [*e.g.*, eqs. (3)-(5).] The extensions to multiple constraints of the generalized rationale [eq. (12)] and of constrained derivatives [eq. (14)] leads to the following result. Suppose that the parameters can be grouped in Q constraints (Appendix A), and let

$$g(x) = \begin{cases} g^1(x_1, x_2, ..., x_{n_1}) = c^1 \\ g^2(x_{n_1+1}, x_{n_1+2}, ..., x_{n_2}) = c^2 \\ .... \\ g^Q(x_{n_{Q-1}}, x_{n_{Q-1}+1}, ..., x_n) = c^Q \end{cases} . \tag{18}$$

If one reproduces the approach utilized in the derivation of eq. (12) for each of the Q constraints and denotes the pivotal parameter for each constraint as x_{k_q}, one obtains the following result (see Appendix A):

Generalized Rationale in the presence of multiple constraints, the change in model output can be written as:

$$Y - Y^0 = \sum_{q=0}^{Q} \sum_{s=n_q+1}^{n_{q+1}} \left[\frac{\partial f}{\partial x_s} - \frac{\partial f}{\partial x_{k_q}} \left(\frac{\partial g^q}{\partial x_s} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}} \right) \right] dx_s + o(\left\| x - x^0 \right\|^2) \tag{19}$$

and the constrained derivatives as:

$$f_{i|k_q} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{k_q}} \left(\frac{\partial g^q}{\partial x_i} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}} \right) \tag{20}$$

where x_i and x_{k_q} belong to the same constraint.

One notes that eq. (20) maintains the same form as eq. (14) and that there is one dependent parameter for each constraint.

The fact that the rate of change of the output in the presence of constraints is represented by $f_{i|k_q}$ instead of $\frac{\partial f}{\partial x_i}$ has an impact on the definition of all indicators that can be derived from Helton's (1993) rationale. This is the subject of discussion in the next section.

4 Effect of Constraints on Local Importance Measures

In this section, we study the impact of the generalized rationale on the definition of local importance measures.

The local importance measures affected by the change in rationale are: the Birnbaum [Birnbaum, (1969),] the Criticality [Helton (1993), Cheok *et al* (1998),] and the Differential (Borgonovo and Apostolakis, 2001) importance measures. Before introducing the effects of constraints, we formulate the analysis in terms of two "*Settings* [Saltelli and Tarantola (2002)[3]]"

Setting 1 The basis for Setting 1 is represented by Samuelson's definition of local SA. Samuelson (1947) defines SA as the problem of addressing "*the response of our system to changes in certain parameters.*" Thus, Setting 1 concerns the determination of the direction of change of the output in response to input changes.

In terms of eqs. (7) or (12)/(19), Setting 1 consists of determining the sign of d_iY, which coincides with the sign of $\frac{\partial f}{\partial x_i}$ in the unconstrained case, and the sign of $f_{i|k_q}$ in the constrained case.

Setting 2 The basis for Setting 2 is the following definition of SA: "*the study of how the variation in the output of a model can be apportioned to variations in the input* (Tarantola, 2000)." Thus, setting 2 concerns the determination of the parameter contributions to the output change.

In terms of eqs. (7) or (12)/(19), Setting 2 consists in assessing the magnitude of d_iY. Then, x_i is more influential than x_j if $|d_iY| > |d_jY|$.

[3] The formulation of lottery settins for global sensitivity analysis can be found in Tarantola (2000), and later in Saltelli and Tarantola (2002).

Let us start with partial derivatives. Partial derivatives are tantamount in the determination of a model's sensitivity and are at the basis, for example, of the Comparative Statics technique in Economics. In Risk Assessment, the Birnbaum importance measure (B) (Birnbaum, 1969) is the partial derivative of the system unavailability with respect to a basic event probability [see also Cheok *et al* (1998) and Borgonovo (2007).] We write:

$$B_i := \frac{\partial f}{\partial x_i}|_{x^0} \tag{21}$$

The former definition of B [eqs. (21)] holds under the assumption that parameters are free to vary. Based on the results of Section 3, we know that the derivative of an output given a set of constraints is represented by eq. (14), which replaces $\frac{\partial f}{\partial x_i}$. Let x_i belong to the q^{th} constraint. Then the Birnbaum importance of x_i becomes:

$$B_{i|k_q} = \left[\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_i}\bigg\backslash\frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0} \tag{22}$$

Frey and Patil (2002) propose a survey of local and global sensitivity indicators. In presenting the automatic differentiation technique, they underline that "*the values of partial derivatives are a measure of local sensitivity*" [see also Helton (1993).] Indeed, B_i is the rate of change of the output w.r.t. the change in x_i. Therefore, it is always suited for Setting 1. However, it is not always suited for Setting 2. For instance, B cannot be utilized to determine parameter importance when inputs have different dimensions. In fact, if x_i and x_j are denominated in different units, the respective partial derivatives cannot be compared as they have different units too [Borgonovo and Apostolakis (2001), Borgonovo and Peccati (2006a)]. More in general, for partial derivatives to become an appropriate measure for Setting 2, one needs to add the assumption that changes in the parameters are uniform, *i.e.*,

$$\text{Hypothesis 1: } x_i - x_i^0 = dx_i = x_j - x_j^0 = dx_j \ \ \forall i,j. \tag{23}$$

In fact, if eq. (23) is true, then studying $|d_iY| \underset{>}{\overset{<}{=}} |d_jY|$ turns into determining whether the Birnbaum importance of x_i is bigger than the Birnbaum importance of x_j[4].

Several authors [Helton (1993), Cheok *et al* (1998), Frey and Patil (2002), Patil and Frey (2004), Saltelli *et al* (2000), Epstein and Rauzy (2005)] utilize a normalization of partial derivatives which gives rise to the Criticality importance measure (C.) C is also known in Economics under the name of Elasticity (Samuelson, 1947, Takayama, 1993.) The rationale below C as a sensitivity measure for Setting 2 can be found in Helton (1993), who bases the introduction of C on the following normalization of eq. (7):

$$\begin{aligned}\frac{Y-Y^0}{Y^0} &= \frac{\partial Y}{\partial x_1}\frac{x_1^0}{Y^0}\frac{\left(x_1-x_1^0\right)}{x_1^0}+\frac{\partial Y}{\partial x_2}\frac{x_2^0}{Y^0}\frac{\left(x_2-x_2^0\right)}{x_2^0}+...+\frac{\partial Y}{\partial x_n}\frac{x_n^0}{Y^0}\frac{\left(x_n-x_n^0\right)}{x_n^0}= \\ &= C_1\frac{\left(x_1-x_1^0\right)}{x_1^0}+C_2\frac{\left(x_2-x_2^0\right)}{x_2^0}+...+C_n\frac{\left(x_n-x_n^0\right)}{x_n^0}\end{aligned} \tag{25}$$

Then:

$$C_i = \frac{\partial f}{\partial x_i}|_{x^0}\ \frac{x_i^0}{Y^0} \tag{26}$$

Utilizing the generalized version of the rationale [eq. (19) instead of eqs. (7) and (8)] and applying the same normalization, one extends the criticality importance measure to the case of input constraints as follows:

$$C_{i|k_q} = \left[\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_i}\Big\backslash\frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}\ \frac{x_i^0}{Y^0} \tag{27}$$

Now, let us assumes proportional changes in the parameters, *i.e.*,

$$\text{Hypothesis 2:}\quad \frac{\left(x_i-x_i^0\right)}{x_i^0}=\frac{\left(x_j-x_j^0\right)}{x_j^0}\ \forall i,j. \tag{28}$$

Then, $\left|C_{i|k_{q_i}}\right| > \left|C_{j|k_{q_j}}\right| \Longleftrightarrow |d_iY| > |d_jY|$, *i.e.*, x_i is more important than x_j. Hence, C measures the importance of parameters under the assumption of proportional changes [Borgonovo and Apostolakis (2001), Borgonovo and Peccati (2004) and (2006a)].

The above discussion shows that local sensitivity results are dependent on the relative parameters

[4] Let x_i and x_j belong to the q^{th} and r^{th} constraints respectively. $|d_iY| \lesseqgtr |d_jY|$ is equivalent to:

$$\left|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_i}\Big\backslash\frac{\partial g^q}{\partial x_{k_q}}\right)dx_i\right| \lesseqgtr \left|\frac{\partial f}{\partial x_j}-\frac{\partial f}{\partial x_{k_r}}\left(\frac{\partial g^r}{\partial x_j}\Big\backslash\frac{\partial g^r}{\partial x_{k_r}}\right)dx_j\right| \tag{24}$$

If Hypothesis 1 holds, then dx_i and dx_j can be simplified and the above inequality reduces to the comparison of the Birnbaum importances of x_i and x_j.

changes, *i.e.*, on the way inputs are varied. Rheinboldt (1993) provides a geometric interpretation of this fact, stating that local sensitivity results depend on the direction of change. To account for all the possible ways in which parameters are varied one needs to resort to the differential importance measure (D) (Borgonovo and Apostolakis, 2001):

$$D_i = \frac{d_i Y}{dY} = \frac{\frac{\partial f}{\partial x_i}|_{x^0}\, dx_i}{\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}|_{x^0}\, dx_j} \tag{29}$$

D measures the importance of x_i as the ratio of the change in Y provoked by a variation of x_i [$d_i Y$, numerator of eq. (29)] to the change in Y provoked by all inputs [total differential, denominator of eq. (29).]

In view of the generalized rationale, eq. (19), the differential importance measure in the presence of constraints becomes:

$$D_{i|k_q}(x, dx) = \frac{\left[\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_i}\Big\backslash\frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}\, dx_i}{\sum_{q=1}^{Q}\sum_{l=n_{q-1}}^{n_q}\left[\frac{\partial f}{\partial x_l} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_l}\Big\backslash\frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}\, dx_l}. \tag{30}$$

In terms of Setting 1, if dY is greater than 0, than the sign of D_i directly reflects the sign of $d_i Y$. If dY is negative, then the direction of change of Y is opposite to the sign of D.

In terms of Setting 2, we note that no matter what assumption is stated on the parameter changes

$$|D_i| \geq |D_j| \Longleftrightarrow x_i \text{ is more important than } x_j \tag{31}$$

Thus, D is always suited for Setting 2.

We note that:

1. the relationships that hold between B, C and D in the absence of constraints [Borgonovo and Apostolakis (2001), Borgonovo (2007)] still hold in the presence of constraints. Under

Hypothesis 1 (uniform changes) [eq. (23),]

$$D_{i|k_q}(x, dx) = \frac{\left[\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_i} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}}{\sum_{q=1}^{Q}\sum_{l=n_{q-1}}^{n_q}\left[\frac{\partial f}{\partial x_l} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_l} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}} = \frac{B_{i|k_q}}{\sum_{j=1}^{n} B_{j|k_q}} \tag{32}$$

i.e., $D^{(1)}$ induces the same ranking as B.[5] Under Hypothesis 2 (proportional changes) one has:

$$D^{(2)}_{i|k_q} = \frac{\left[\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_i} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}\, x_i^0}{\sum_{q=1}^{Q}\sum_{l=n_{q-1}}^{n_q}\left[\frac{\partial f}{\partial x_l} - \frac{\partial f}{\partial x_{k_q}}\left(\frac{\partial g^q}{\partial x_l} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}}\right)\right]|_{x^0}\, x_l^0} = \frac{C_{i|k_q}}{\sum_{j=1}^{n} C_{j|k_q}} \tag{33}$$

and therefore C and $D^{(2)}$ lead to the same ranking;

2. as the total differential is the sum of the differentials, D shares the additivity property also in the constrained case. *I.e.*, the joint D of a group of parameters is the sum of the importances of the parameters in the group;

3. $$D_{k_q|k_q}(x, dx) = 0 \quad \forall dx_i, dx_j \tag{34}$$

   which implies that once a parameter is chosen as dependent, the importance of such a parameter is null, independently of the direction of change. Thus, in the case of uniform changes, $B_{k_q|k_q} = 0$ and, in the case of proportional changes, $C_{k_q|k_q} = 0$.

The results of this section and of Section 3 explain the Effects of constraints and extend the definitions of importance measures to the presence of constraints. We are now left with discussing how the numerical implications of Effects 1, 2 and 3 can be tackled in view of the generalized rationale. This is presented in the next Section.

5 Numerical Implementation of the Generalized Rationale

5.1 The Steps

The purpose of this Section is to show the numerical implications of the results presented in Section 3. As mentioned in Sections 1, 2 and 3, there are n possible choices of pivotal parame-

[5] We use the symbols $D^{(1)}$ and $D^{(2)}$ to denote that D is computed under assumptions 1 and 2 respectively.

ters. Let us state the problem as follows. Let $z = \{z_1, z_2, .., z_n, z_{n+1}, z_{n+2}, ..., z_{n+m}\}$ denote the set of all variables involved in the problem. Fink and Rheinboldt (1984) maintain that "*it often happens in applications that certain quantities are naturally identified as parameters.*" That is, z can be partitioned into two sets of variables, say, $x = \{x_1 = z_1, x_2 = z_2, .., x_n = z_n\}$ and $Y = \{Y_1 = z_{n+1}, Y_2 = z_{n+2}, .., Y_m = z_{n+m}\}$, in which the second group (output) is seen as dependent on the first (input/parameters). In a constrained SA, the selection of what is a parameter or an output can be "less natural" and more dependent on the analyst's choice than in unconstrained problems. To illustrate the concept, consider the following example. The output of an event tree, z_4, is a function of three probabilities (z_1, z_2, z_3.) It is natural to set $x = \{z_1, z_2, z_3\}$ and $Y = \{z_4\}$. This is the parameter selection (partition) that one obtains if parameters are free. If the three events are emanating from the same node (see for instance Example 1,) then the following constraint applies on x: $z_1 + z_2 + z_3 = 1$. If one solves the constraint for the third probability as dependent on the first two, one is indeed adopting the following partition: $[x = (z_1, z_2),\ Y = (z_4,\ z_3)]$. Similarly, if one solves for z_2, or z_1, one is adopting the following partitions $[x = (z_1, z_3),\ Y = (z_4,\ z_2)]$ and $[x = (z_2, z_3),\ Y = (z_4, z_1)]$ respectively. Since, as proven in Section 3, the sensitivity results depend on the choice of the parameters, to inspect the change in results obtained by changing the selection, one has to repeat the sensitivity calculations $n = 3$ times, in correspondence of each pivotal parameter choice (partition). This could make the sensitivity exercise cumbersome (if not impossible) when the model at hand is time consuming. The results proven in Section 3, however, can be utilized to obtain constrained sensitivity results without having to repeat the sensitivity algorithm in correspondence of each partition. The approach is based on the concept of constrained derivative [eq. (20)] and is illustrated in the following steps.

Step 1) Identify the output of interest and apply a first partition $[x, Y]$ ignoring the constraints, *i.e.*, considering all parameters as free;

Step 2) Apply a differentiation scheme to compute the free derivatives of the output $(\dfrac{\partial f}{\partial x_i})$ and the derivatives of the constraints $(\dfrac{\partial g^q}{\partial x_i})$;

Step 3) Select the pivotal parameters and apply eq. (14) or (20) to obtain the constrained derivatives;

Step 4) Compute the importance measures based on the results of Section 4.

Note that Steps 1 and 2 need not to be repeated, if one changes the selection of the pivotal parameters. Thus, one can examine how the SA results change due to a change in the selection without further model runs. This grants a computational cost saving of order $n \times t$, where n are the available choices of pivotal parameters and t the model runs required by the differentiation algorithm.

We note that linear constraints grant a further numerical simplification. In fact, as all derivatives of the constraints are equal to 1, in Step 2 the differentiation of the constraint becomes unnecessary and one obtains the constrained SA results directly by subtracting $\frac{\partial f}{\partial x_{k_q}}$ from $\frac{\partial f}{\partial x_i}$. This means that constrained SA results are obtained at the same cost of the unconstrained case.

5.2 A Numerical Illustration

In this Section, we illustrate the application of the constrained sensitivity Steps via a numerical discussion. We consider the event tree presented in Papazoglou (1998) and reported in Figure 2.

Each sequence ends into four possible states. The system is working in state I, while states II, III and IV correspond to system damage (c). There are 6 sequences (paths) leading to c (Figure 2): $S_1 = (w_1, w_4, w_{10})$, $S_2 = (w_1, w_5)$, $S_3 = (w_2, w_6, w_{13})$, $S_4 = (w_2, w_6, w_{14})$, $S_5 = (w_2, w_7)$, $S_6 = (w_2, w_3)$. Utilizing the notation of eq. (2) — see also Figure 2, — eq. (4) becomes:

$$\begin{gathered}
\text{Study the Sensitivity of} \\
R = h(p) = p_1 p_4 p_{10} + p_1 p_5 + p_2 p_6 p_{12} + p_2 p_6 p_{13} + p_2 p_7 + p_3 \\
s.t. \\
\begin{cases}
p_1 + p_2 + p_3 = 1 \\
p_4 + p_5 = 1 \\
p_6 + p_7 = 1 \\
p_8 + p_9 + p_{10} = 1 \\
p_{11} + p_{12} + p_{13} = 1
\end{cases}
\end{gathered} \tag{35}$$

We adopt the numerical values for the probabilities displayed in Table 1.

We structure our discussion following the steps proposed in Section 5.1.

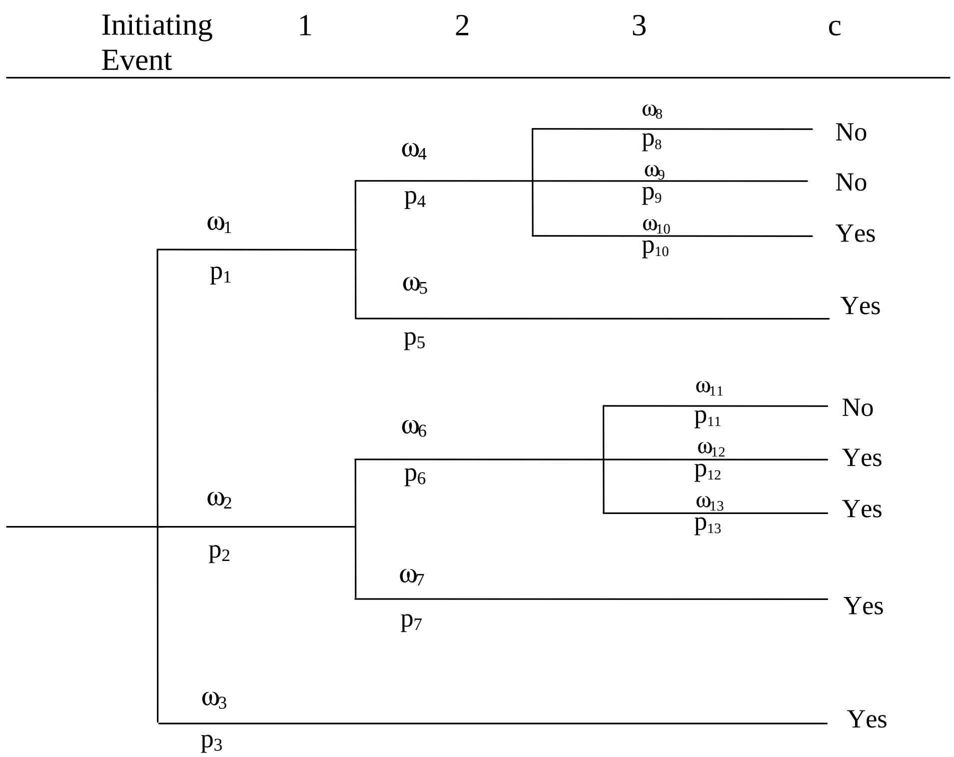

Figure 2: Non-binary Event Tree from Fig. 3 of Papazoglou (1998.)

Step 1) The output of interest is R and the parameters are the probabilities in the event tree. Therefore the partition is $[x = \{p_1, p_2, ..., p_{13}\},\ Y = \{R\}]$.

Step 2) The partial derivatives of R w.r.t. the p_i's are illustrated in Figure 3. The derivatives of the constraint are all equal to unity as $g(x)$ is linear in eq. (35).

As far as the numerical results are concerned, one notes that all the unconstrained derivatives are positive or null ($\frac{\partial R}{\partial p_i} \geq 0,\ i = 1, 2, ..., 13$). This could lead one to conclude that no increase in

Table 1: Values of the probabilities, free derivatives and constrained derivatives for the non-binary event tree.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
Value	0.6	0.25	0.15	0.3	0.7	0.6	0.4	0.1	0.6	0.3	0.5	0.3	0.2
$\frac{\partial R}{\partial p_i}$	0.79	1.3	1	0.18	0.6	0.375	0.25	0	0	0.18	0	0.15	0.15
$R_{i\|k}$	-0.21	0.3	0	-0.42	0	0.125	0	0	0	0.18	0	0.15	0.15

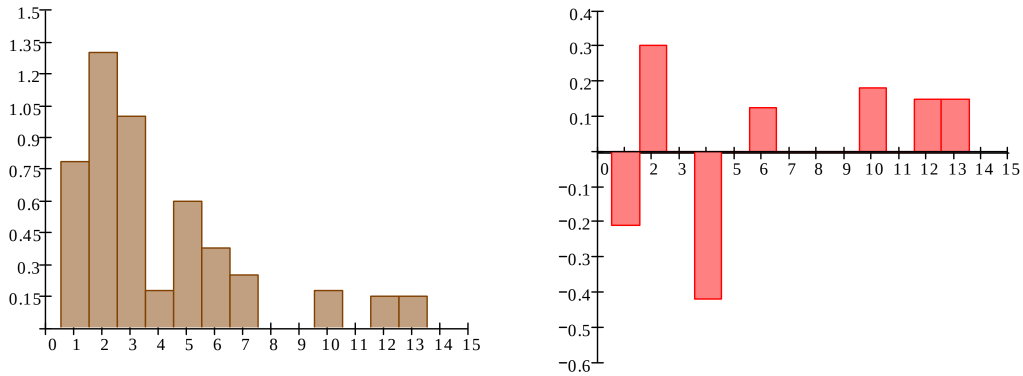

Figure 3: Free partial derivatives ($\frac{\partial R}{\partial p_i}$) (left) vs constrained derivatives ($R_{i|k_q}$) (right).

any of the probabilities can lead to a decrease in risk. This conclusion, however, is not valid, as the p's are constrained and the rate of change is not given by $\frac{\partial R}{\partial p_i}$, but by $R_{i|k}$.

We have $Q = 5$ constraints [eq. (35).] One needs 5 pivotal parameters, x_{k_q}, $q = 1, 2, ..., 5$. Let us inspect the following choice: $x_{k_1} = p_3, x_{k_2} = p_5, x_{k_3} = p_7, x_{k_4} = p_8$, and $x_{k_5} = p_{11}$. Then the selection becomes: $[x = \{p_1, p_2, p_4, p_6, p_9, p_{10}, p_{12}, p_{13}\}$, $Y = \{R, p_3, p_5, p_7, p_8, p_{11}\}]$.

Step 3) One obtains the constrained derivatives by subtracting $\frac{\partial R}{\partial p_{k_q}}$ from $\frac{\partial R}{\partial p_i}$. The numerical values are illustrated in the fourth row of Table 1.

We note that $R_{3|3} = R_{5|5} = R_{7|7} = R_{8|8} = R_{11|11} = 0$ as the probabilities p_3, p_5, p_7, p_8 and p_{11} have bee taken as pivotal. This result is a reflection of Effect 3 and eq. (15).

It is observed that the constrained derivatives are no more all positive or null (Figure 3.) As an example, let us consider $R_{4|5} = -0.42$ (Table 1, Figure 3.) In the presence of constraints, the positive effect of an increase in p_4 alone — if it were free to vary the rate of change would be $\frac{\partial R}{\partial p_4} = 0.18$ — is diminished [eq. (20)] by the effect of p_5 ($\frac{\partial R}{\partial p_5} = 0.6$), as p_5 bounces back in order for the constraint to be satisfied.

Some of the constrained derivatives are indeed equal to the free derivatives, as a result of the choice of the pivotal parameters. In our example, $R_{10|8} = \frac{\partial R}{\partial p_{10}}$ and $R_{12|11} = \frac{\partial R}{\partial p_{12}}$. In fact, p_8 and p_{11} appear in the constraints but R does not depend upon them. Hence, $\frac{\partial R}{\partial p_8} = 0$ and $\frac{\partial R}{\partial p_{11}} = 0$, and thanks to eq. (20), the constrained derivative equals the free derivative.

Let us now interpret the results in the light of Settings 1 and 2 (Section 4.) In terms of Setting 1, Table 1 and Figure (3) show that an increase in p_1 and p_4 leads to a decrease in R, while an increase in p_2, p_6, p_{10}, p_{12}, and p_{13} leads to an decrease in R.

In terms of Setting 2, Figure 4 shows the $D^{(1)}/B$ and $D^{(2)}/C$ importance measures[6].

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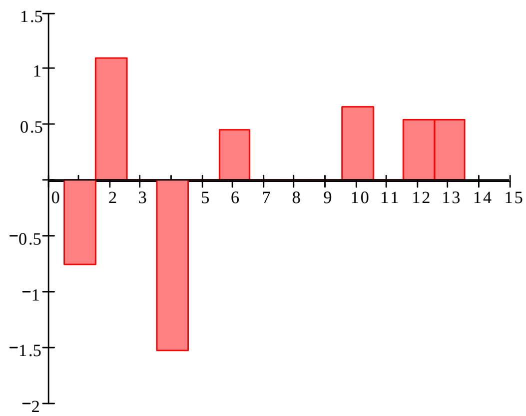

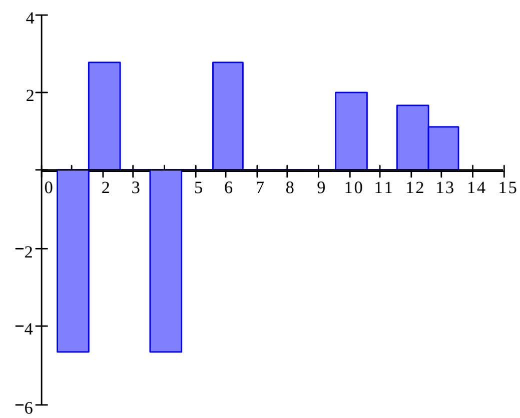

Figure 4: $D^{(1)}/B$ (left) and $D^{(2)}/C$ (right).

Figure 4 shows that probabilities associated with a zero value of the corresponding constrained derivatives have null importance. Regarding ranking, under the assumption of uniform changes, the most important probability is p_4, followed by p_1 and p_2. Suppose now an analyst is interested in the joint effect of a simultaneous changes in the probabilities of node 1 in the event tree, namely p_1, p_2 and p_3 (Figure 2.) The answer is found by utilizing the additivity property of D to get:

$$D^{(1)}_{1,2,3|3} = D^{(1)}_{1|3} + D^{(1)}_{2|3} + D^{(1)}_{3|3} = 0.327 \quad (36)$$

Under the assumption of proportional changes ($D^{(2)}/C$), the importance measure values are displayed in Figure 4 (right diagram.) We observe that the ranking is not the same as the one obtained with $D^{(1)}/B$. The most important probabilities are now p_1 and p_4, followed by p_2 and p_6. This is due to the different definition of the importance measures and has the geometric interpretation proposed in Section 4. Similarly to the case of Hypothesis 1, the importance of simultaneous propor-

[6]We have used the notation $D^{(1)}/B$ and $D^{(2)}/C$ since D produces the same ranking as B under the hypothesis of uniform changes and of C under the assumption of proportional changes.

tional changes in the probabilities related to the first node is: $D_{1,2,3}^{(2)} = D_1^{(2)} + D_2^{(2)} + D_3^{(2)} = -1.889$, in this case.

Finally, it is worth to point out that the current results have been obtained with the selection $[x = \{p_1, p_2, p_4, p_6, p_9, p_{10}, p_{12}, p_{13}\},\ Y = \{R, p_3, p_5, p_7, p_8, p_{11}\}]$. Let us then discuss how the effects of a change in the partition can be determined. If, as it is often done in the practice, one solves the first constraint for $p_3 = 1 - p_2 - p_1$, the second for $p_5 = 1 - p_4$, and the others for p_7, p_8 and p_{11} and substitutes into the model, one obtains R as the following function of the parameters, namely

$$R = q(p_1, p_2, p_4, p_6, p_{10}, p_{12}, p_{13}) = 1 - p_1 p_4 (1 - p_{10}) - p_2 p_6 (1 - p_{12} - p_{13}) \tag{37}$$

Computing the derivatives of q [eq. (37)], one obtains the constrained derivatives in Table 1. As an example, $\dfrac{\partial q}{\partial p_1} = R_{1|3}$. Let us now choose a different partition, say $[x = \{p_1, p_3, p_5, p_7, p_8, p_{10}, p_{11}, p_{13}\}$, $Y = \{R, p_2, p_4, p_6, p_9, p_{12}\}]$. If one were using the solve-for-a-parameter method, one would need to resolve the constraints for $p_2, p_4, p_6, p_9, p_{12}$, express R as a function the selected parameters, obtaining, say, $R = m(p_1, p_3, p_5, p_7, p_8, p_{10}, p_{11}, p_{13})$ and recompute the derivatives $\dfrac{\partial m}{\partial p_i}$. These further calculations are not necessary given the Steps in Section 5.1. In fact, $\dfrac{\partial m}{\partial p_i}$ are determined by the difference between the free derivatives $\dfrac{\partial R}{\partial p_i}$ — displayed in the second row of Table 1, — and the derivatives of the newly selected pivotal parameters.

6 Conclusions

In this work, we have proposed an approach to solve problems connected with the presence of constraints in the local SA of model output. As these problems have not been studied in a systematic way, the first step of our analysis has been the investigation of issues connected with the presence of constraints. We have evidenced three main Effects and studied their implications. In particular, we have seen that constraints do not allow to perform a one-variable-at-a-time SA and that the sensitivity of the output becomes null on a parameter, once a constraint is solved for that parameter. We have then provided a rigorous explanation the Effects. This has been done by introducing a novel approach that, resting on the simultaneous differentiation of the output and of the constraints, allows a generalization of Helton's (1993) rationale for local SA. We have then addressed the impact of the new rationale on the definitions of local SA indicators based on differentiation (the Birnbaum, Criticality and the Differential importance measures.) We have seen that the result allows to generalize the definitions of the indicators to the presence of constraints preserving their properties.

We have next presented a stepwise implementation of the generalized rationale that allows to streamline the numerical complications generated by the presence of constraints. In particular, we have seen that one is enabled to assess the influence of alternative parameter selections without further SA algorithm runs, with a computational cost saving of order (at least) n.

The numerical illustration of the SA of a non-binary event tree has concluded the work, providing a sample application of the above findings.

7 Appendix A: Extension to the Case of Multiple Constraints

Suppose that the parameters are grouped in $Q < n$ groups. We call each group x^q $(q = 1...Q)$. Each group contains n_q parameters (clearly $\sum_{q=1}^{Q} n_q = n$) and is subdivided as follows:

$$\underbrace{x_1\, x_2 \ldots x_{n_1}}_{x^1}\ \underbrace{x_{n_1+1}\, x_{n_1+2} \ldots x_{n_2}}_{x^2} \ldots.\ \underbrace{x_{n_{Q-1}+1}\, x_{n_{Q-1}+2} \ldots x_n}_{x^Q} \tag{38}$$

so that two groups do not have inputs in common ($x^l \bigcap x^m = \varnothing$) and they cover all inputs ($\bigcup\limits_{q=1}^{Q} x^q = x$). Note that the number of parameters in group q is equal to $n_q - n_{q-1}$. Furthermore, we let $n_0 = 0$ and $n_Q = n$. Each parameter group is constrained by:

$$g(x) = \begin{cases} g^1(x_1, x_2, ..., x_{n_1}) = c^1 \\ g^2(x_{n_1+1}, x_{n_1+2}, ..., x_{n_2}) = c^2 \\ .... \\ g^Q(x_{n_{Q-1}}, x_{n_{Q-1}+1}, ..., x_n) = c^Q \end{cases} \tag{39}$$

In eq. (39) there is no loss of generality, as one can consider that the Q^{th} group is the group of unconstrained parameters, with a degenerate constraint. Then, one can differentiate both sides of the constraints $g(x)$

noting that the differential of c^q is zero for all $q = 1, 2, ..., Q$. One gets the set of equations:

$$\begin{cases} \frac{\partial g^q}{\partial x_1} dx_1 + \frac{\partial g^1}{\partial x_2} dx_2 + ... + \frac{\partial g^1}{\partial x_{n_1}} dx_{n_1} = 0 \\ \frac{\partial g^2}{\partial x_{n_1+1}} dx_{n_1+1} + \frac{\partial g^2}{\partial x_{n_1+2}} dx_{n_1+2} + ... + \frac{\partial g^2}{\partial x_{n_1+n_2}} dx_{n_1+n_2} = 0 \\ ... \\ \frac{\partial g^Q}{\partial x_{n_1+n_2+...+n_{Q-1}+1}} dx_{n_1+n_2+...+n_{Q-1}+1} + \frac{\partial g^Q}{\partial x_{n_1+n_2+...+n_{Q-1}+2}} dx_{n_1+n_2+...+n_{Q-1}+2} + ... + \frac{\partial g^Q}{\partial x_n} dx_n = 0 \end{cases} \tag{40}$$

Let us now denote the parameter which is chosen as dependent in each group as $x_{k_q} (x_{k_q} \in x^q)$. Then, provided that $\frac{\partial g^q}{\partial x_{k_q}} \neq 0 \; \forall q$, one can rewrite the equations (40) as:

$$\begin{cases} dx_{k_1} = -\left(\frac{\partial g^1}{\partial x_1} \bigg\backslash \frac{\partial g^1}{\partial x_{k_1}} \right) dx_1 - ... - \left(\frac{\partial g}{\partial x_{n_1}} \bigg\backslash \frac{\partial g^1}{\partial x_{k_1}} \right) dx_{n_1} \\ dx_{k_2} = -\left(\frac{\partial g^2}{\partial x_{n_1+1}} \bigg\backslash \frac{\partial g^2}{\partial x_{k_2}} \right) dx_{n_1+1} - ... - \left(\frac{\partial g}{\partial x_{n_2}} \bigg\backslash \frac{\partial g^1}{\partial x_{k_2}} \right) dx_{n_2} \\ .... \\ dx_{k_Q} = -dx_{n_{Q-1}+1} \left(\frac{\partial g^Q}{\partial x_{n_{Q-1}+1}} \Bigg\backslash \frac{\partial g^Q}{\partial x_{k_Q}} \right) - ... - \left(\frac{\partial g^Q}{\partial x_n} \Bigg\backslash \frac{\partial g^Q}{\partial x_{k_Q}} \right) dx_n \end{cases} \tag{41}$$

Substituting eq. (41) into the differential, one gets:

$$\begin{gathered} dY = \\ \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... - \sum_{\substack{s=1 \\ s \neq k_1}}^{n_1} \left(\frac{\partial g^1}{\partial x_s} \bigg\backslash \frac{\partial g^1}{\partial x_{k_1}} \right) dx_s + ... + \frac{\partial f}{\partial x_{n_1}} dx_{n_1} + \\ + \frac{\partial f}{\partial x_{n_1+1}} dx_{n_1+1} + ... - \sum_{\substack{s=n_1+1 \\ s \neq k_2}}^{n_2} \left(\frac{\partial g^2}{\partial x_s} \bigg\backslash \frac{\partial g^2}{\partial x_{k_2}} \right) dx_s + ... + \frac{\partial f}{\partial x_{n_1+n_2}} dx_{n_1+n_2} + \\ ... \\ + \frac{\partial f}{\partial x_{n_1+n_2+...+n_{Q-1}+1}} dx_{n_1+n_2+...+n_{Q-1}+1} + ... - \sum_{\substack{s=n_1+1 \\ s \neq k_Q}}^{nQ} \left(\frac{\partial g^Q}{\partial x_s} \Bigg\backslash \frac{\partial g^Q}{\partial x_{k_Q}} \right) dx_s + ... + \frac{\partial f}{\partial x_n} dx_n \end{gathered} \tag{42}$$

Rearranging, one obtains:

$$\begin{gathered} dY = \sum_{s=n_0}^{n_1} \left[\frac{\partial f}{\partial x_s} - \frac{\partial f}{\partial x_{k_1}} \left(\frac{\partial g^1}{\partial x_s} \bigg\backslash \frac{\partial g^1}{\partial x_{k_1}} \right) \right] dx_1 + \sum_{s=n_1+1}^{n_2} \left[\frac{\partial f}{\partial x_s} - \frac{\partial f}{\partial x_{k_2}} \left(\frac{\partial g^2}{\partial x_s} \bigg\backslash \frac{\partial g^2}{\partial x_{k_2}} \right) \right] dx_s + .. \\ + \sum_{s=n_{Q-1}}^{n} \left[\frac{\partial f}{\partial x_s} - \frac{\partial f}{\partial x_{k_Q}} \left(\frac{\partial g^Q}{\partial x_s} \Bigg\backslash \frac{\partial g^Q}{\partial x_{k_Q}} \right) \right] dx_s \end{gathered} \tag{43}$$

Eq. (43) shows that the rate of change given the change dx_s that was represented by $f_{i|k}$ [eq. (14)] in the single constraint case, in the case of multiple constraints is represented by:

$$f_{i|k_q} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{k_q}} \left(\frac{\partial g^q}{\partial x_i} \Big\backslash \frac{\partial g^q}{\partial x_{k_q}} \right) \text{ q.e.d..} \tag{44}$$

References

[1] Birnbaum Z.W., 1969: "On the Importance of different Components in a Multicomponent System," in Multivariate Analysis-II, P. R. Krishnaiah Editor, Academic Press, New York, NY, USA, pp. 581-592.

[2] Borgonovo E. and Apostolakis G.E., 2001: 'A New Importance Measure for Risk-Informed Decision-Making', Reliability Engineering and System Safety, 72 (2), 2001, pp. 193-212.

[3] Borgonovo E. and Peccati L., 2004: "Sensitivity Analysis in Investment Project Evaluation," International Journal of Production Economics, 90, pp.17-25.

[4] Borgonovo E., 2006a: "Measuring Uncertainty Importance: Investigation and Comparison of Alternative Approaches," Risk Analysis, 26 (5), p. 1349-1362.

[5] Borgonovo E., 2007: "Differential, Criticality and Birnbaum Importance Measures: an Application to Basic Event, Groups and SSCs in Event Trees and Binary Decision Diagrams," Reliability Engineering and System Safety, 92 (10), pp. 1458–1467.

[6] Borgonovo E. and Peccati L., 2006a: "The Importance of Assumptions in Investment Evaluation," International Journal of Production Economics, 101 (2), pp. 298-311.

[7] Borgonovo E. and Peccati L., 2006b: "Sensitivity Analysis in Decision Making: a Consistent Approach," presented at the 12^{th} International conference on Foundations and Applications of Utility, Risk and Decision Theory, Rome, June 29-July 4 2006.

[8] Borgonovo E. and Percoco M., 2007: "Sensitivity analysis of Portfolio Volatility: Importance of Weights, Sectors and the Impact of Trading Strategies," in Advances in Risk Management, Gregoriou, G. N. (ed.), Palgrave-MacMillan, pp. 47-68.

[9] Cheok M.C., Parry G.W. and Sherry, R.R., 1998: "Use of Importance Measures in Risk-Informed Regulatory Applications," Reliability Engineering and System Safety, 60, pp. 213-226.

[10] Clemen R.T., 1997: "Making Hard Decisions: An Introduction to Decision Analysis," II Edition, Duxbury Press, Pacific Grove, Calif, USA; ISBN: 0534260349.

[11] Epstein S. and Rauzy A., 2005: "Can we trust PRA?," Reliability Engineering and System Safety, 88, pp. 195–205.

[12] Fink J.P. and Rheinboldt W. C., 1984: "Solution Manifolds and Submanifolds of Parameterized Equations and their Discretization Errors," Numerische Matematick, 45, pp. 323-343.

[13] Frey C. H. and Patil S. R., 2002: "Identification and Review of Sensitivity Analysis Methods," Risk Analysis, 22 (3), pp. 553-571.

[14] Helton J.C., 1993: "Uncertainty and Sensitivity Analysis Techniques for Use in Performance Assessment for Radioactive Waste Disposal," Reliability Engineering and System Safety, 42, 327-367.

[15] Howard R. A., 1988: "Decision Analysis: Practice and Promise," Management Science, 34, pp. 679-695.

[16] Kaplan S. and Garrick B. J., 1981: "On the quantitative definition of risk," Risk Analysis, 1(1), pp. 11–27.

[17] Kessler J. H. and McGuire R. K., 1999: "Total System Performance Assessment for Waste Disposal Using a Logic Tree Approach," Risk Analysis, 19(5), pp. 915–931.

[18] Iman R. L., Johnson M. E. and Watson C. C. Jr., 2005a: "Sensitivity Analysis for Computer Model Projections of Hurricane Losses," Risk Analysis, 25 (5), pp.1277-1297.

[19] Iman R. L., Johnson M. E. and Watson C. C. Jr., 2005b: "Uncertainty Analysis for Computer Model Projections of Hurricane Losses," Risk Analysis, 25 (5), pp.1299-1312.

[20] Papazoglou I.A., 1998: "Mathematical Foundations for Event Trees," Reliability Engineering and System Safety, 61, pp. 169–183.

[21] Patil S.R. and Frey C. H., 2004: "Comparison of Sensitivity Analysis Methods Based on Application to a Food Safety Risk Assessment Model," Risk Analysis, 24 (3), pp. 573-585.

[22] Philipson L. L. and Wilde P. D., 2000: "Sampling of uncertain probabilities at event tree nodes with multiple branches", Reliability Engineering and System Safety, 70, pp. 197-203.

[23] Reyes Santos J. and Haimes Y. Y., 2004: "Applying the Partitioned Multiobjective Risk Method to Portfolio Selection," Risk Analysis, 24 (3), pp. 697-713.

[24] Rheinboldt W.C., 1993: "On the Sensitivity of Solutions of Parameterized Equations," SIAM Journal on Numerical Analysis, 30 (2), pp. 305-320.

[25] Rief H., 1998: "Touching on a Zero-Variance Scheme for Solving Linear Equations by Random Walk Processes," Proceedings of the Second International Symposium on Sensitivity Analysis of Model Output, Venice (Italy), 1998.

[26] Saltelli A., Tarantola S. and Campolongo F., 2000: "Sensitivity Analysis as an Ingredient of Modelling", Statistical Science, 19 (4), pp. 377-395.

[27] Saltelli A., 2002: 'Sensitivity Analysis for Importance Assessment', Risk Analysis, 22 (3), p. 579.

[28] Saltelli A. and Tarantola S., 2002: "On the Relative Importance of Input Factors in Mathematical Models: Safety Assessment for Nuclear Waste Disposal," Journal of the American Statistical Association, 97 (459), p. 702-709.

[29] Samuelson P., 1947: "Foundations of Economic Analysis," Harvard University Press, Cambridge, MA.

[30] Takayama A., 1993: "Analytical Methods in Economics," The University of Michigan Press,, MI, USA, ISBN 0-472-10162-5.

[31] Tarantola S., 2000: "Quantifying uncertainty importance when inputs are correlated," in Foresight and Precaution, Cottam Harvey Pape and Tait Editors, Balkema Editions, Rotterdam, ISBN 9058091406.