Finite Change Comparative Statics for Risk-Coherent Inventories

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Abstract

This work introduces a comprehensive approach to the sensitivity analysis (SA) of risk-coherent inventory models. We address the issues posed by i) the piecewise-defined nature of risk-coherent objective functions and ii) by the need of multiple model evaluations. The solutions of these issues is found by introducing the extended finite change sensitivity indices (FCSI's). We obtain properties and invariance conditions for the sensitivity of risk-coherent optimization problems. An inventory management case study involving risk-neutral and conditional value-at-risk (CVaR) objective function illustrates our methodology. Three SA settings are formulated to obtain managerial insights. Numerical findings show that risk-neutral decision-makers are more exposed to variations in exogenous variables than CVaR decision-makers.

Keywords: Inventory Management; Sensitivity Analysis; Comparative Statics; Finite Change Sensitivity Indices; Coherent Risk Measures

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1 Introduction

Recent works have demonstrated the use of coherent measures as a novel and effective way to manage risk in inventory problems [Ahmed et al (2007), Gotoh and Takano (2007), Borgonovo and Peccati (2009a)]. The convexity of the objective functions insures feasibility in a broad variety of applications. However, an explicit expression of the solution is generally not available. This prevents a direct interpretation of model results and a straightforward derivation of managerial insights.

The need to explain "*what it was about the inputs that made the outputs come out as they did* (Little (1970); p. B469)" is underlined in Little's seminal paper on the creation and utilization of decision-support models for managers. Eshenbach (1992) underlines the need of identifying the "*most critical factors"* on which to focus "*managerial attention during implementation* (Eschenbach, 1992; p. 40-41)." Works as Rabitz and Alis (1999), Wallace (2000), Saltelli et al (2000), Saltelli and Tarantola (2002), Saltelli et al (2004) have established the awareness that these questions are answered only by a systematic application of sensitivity analysis (SA).

Wallace (2000) and Higle and Wallace (2003) address the use of SA in examining management science model output. They underline the key-issue of establishing consistency between the managerial questions and the SA method selected for the analysis. In linear programming, Jansen et al (1997), Koltay and Terlaki (2000), Koltay and Tatay (2008) discuss the differences in the mathematical and managerial interpretation of SA results. Saltelli et al (2008) (p. 24) recognize that "*a poor definition of the objectives of a sensitivity analysis can lead to confused or inconclusive results.*" The works by Saltelli and Tarantola (2002), Saltelli et al (2004) and Saltelli et al (2008) demonstrate that these issues are solved by SA settings. A setting is "*a way of framing the sensitivity quest in such a way that the answer can be confidently entrusted to a well-identified sensitivity measure* [Saltelli et al (2008), p. 24]."

Purpose of this work is to establish a comprehensive and consistent approach to the SA of risk-coherent inventory problems. To achieve this goal, we proceed as follows. We first address the specific (1) technical and (2) result communication issues. Technical issues are posed by the piecewise-defined character of risk-coherent objective functions [Borgonovo and Peccati (2009b)]. This non-smootheness makes comparative statics and differential approaches not applicable. We show that the integral function decomposition at the basis of the finite change sensitivity indices (FCSI) provides the required generality and solves the technical issues. Result communication issues are posed by the multi-item nature of the problem and, more in general, by the presence of multiple outputs of interests to the decision-maker. We introduce two alternative ways for dealing with result communication. The utilization of the norm of the optimal policy and the technique of the Savage Score correlation coefficients [Iman and Conover (1987)]. We highlight advantages and drawbacks of each approach.

The second step is to enrich information further by enabling a deeper exploration of the exogenous variable space. In SA practice, decision-makers assess a set of efficient scenarios [Tietje (2005)]. The model is tested at each scenario. Since, in previous inventory management works one [in perturbation approaches Bogataj and Cibej (1994), in comparative statics Borgonovo (2008)] or two points [Borgonovo (2010)] were explored, we need to formalize the application of FCSI's in the presence of multiple scenarios. We show that this is achieved by applying the finite-change decomposition at each model jump. As a result, plentiful information is obtained on the behavior of the decision criteria and on the determinants of the problem. We synthesize this information in sensitivity measures called extended FCSI's. By the extended FCSI's one obtains insights on both the magnitude and direction of impact and on the importance of the exogenous variables. Flexibility in assessing the effect of individual variables and groups is offered by the approach.

The third step is to derive general properties of extended FCSI's in risk-coherent problems. We show that, if the loss function of the system at hand (not necessarily an inventory system) is separable in a group of exogenous variables, then: *i*) the optimal risk-coherent policy is insensitive on that group; *ii*) the value of the risk-measure at the optimum is sensitive and responds additively to changes in the parameters of the group.

We then discuss the SA settings that allow one to interpret numerical results and obtaining managerial insights consistence with Eschenbach's and Little's questions.

We apply the proposed methodology to a stochastic inventory problem with risk-neutral and conditional value at risk (CVaR) objective functions. Numerical results confirm the theoretical expectations on the behavior of the sensitivity measures. We discuss managerial insights in the light of the SA settings. Comparison of the numerical findings for risk-neutral and CVaR decision-makers show that both the CVaR optimal policy and value-at-the-optimum are less sensitive to exogenous variable changes than the corresponding risk-neutral optimal policy and expected loss.

The remainder of this work is organized as follows. Section 2 discusses technical aspects and the choice of the sensitivity measures. Section 3 formalizes the notion of extended FCSI's. Section 4 proves relevant properties of the sensitivity of risk-coherent problems under separability conditions of the loss function. Section 5 discusses the SA settings for gaining managerial insights. Section 6 presents the case study and illustrated numerical findings. Conclusions are offered in Section 7.

2 Comparative Statics in Risk-Coherent Problems: Issues and Solutions

In this section, we address technical aspects associated with the piecewise-definite nature of risk-coherent objective functions [Borgonovo and Peccati (2009b).]

We start with a deterministic inventory system as in Borgonovo (2008), to examine the conditions under which comparative statics is applicable. Let $\mathbf{y} \in Y \subseteq \mathbb{R}^m$, $\mathbf{x} \in X \subseteq \mathbb{R}^n$, $Z(\mathbf{y};\mathbf{x})$, $Z : Y \times X \to \mathbb{R}$, S denote choice (endogenous) variables, exogenous variables, loss function of the inventory system and feasible set, respectively [see Table 1 for notation.]

The optimal policy $\mathbf{y}^*$ solves the problem

$$\mathcal{P}_1 = \left\{ \min_{\mathbf{y}\in S} Z(\mathbf{y};\mathbf{x}) \right. \tag{1}$$

Under the regularity condition $Z(\mathbf{y};\mathbf{x}) \in C^1(X)$, by Dini's implicit function theorem, the solution

Table 1: Notation and Symbols used throughout this work

Symbol	Meaning
$\boldsymbol{\omega} = \{\omega_1, \omega_2, ..., \omega_s\}$	Vector of stochastic variable in the risk coherent problems
$(\boldsymbol{\omega}, \otimes, F)$	Measure Space
$\mathbf{y} = \{y_1, y_2, ..., y_m\}$	Endogenous variables (model output)
m	Number of endogenous variables
I	Number of Inventoried Items
Z	Loss Function
$\rho(\cdot)$	Coherent risk-measure
$\mathbf{x} = \{x_1, x_2, ..., x_n\}$	Exogenous variables (parameters)
n	Number of exogenous variables
g, $[\mathbf{y} = g(\mathbf{x})]$	Exogenous-Endogenous variable relationship
$\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, ..., \gamma_Q\}$	Vector of parameter groups
Q	Number of Parameter Groups
$\mathbf{y}^*$	Optimal order policy
m	Number of choice variables
p	Size of the range partitions (number of scenarios)
$CVaR^*_\alpha$	CVaR at the point of optimum
$\mathbb{E}[Z^*]$	Expected loss at the optimum
$\xi^s_{i_1,i_2,...,i_s}$	Group finite change sensitivity index of order s (FCSI)
$\varphi^k_{i_1,i_2,...,i_k}$	Parameter Finite change sensitivity index of order k for
ξ^T_i	Group total order FCSI
$a = \{a_1, a_2, ..., a_I\}$	Unit fixed costs per inventoried item
$r = \{r_1, r_2, ..., r_I\}$	Unit revenues per inventoried item
$c = \{c_1, c_2, ..., c_I\}$	Unit holding costs per inventoried item per unit time

of $\mathcal{P}_1$ defines the differentiable function

$$\mathbf{y}^* = g(\mathbf{x}^*) : X \subseteq \mathbb{R}^n \to \mathbb{R}^m \tag{2}$$

Therefore, comparative statics can be applied. Borgonovo (2008) shows that the differential importance of exogenous variable x_i in respect of choice variable y_j (D_s^j) is given by:

$$D_s^j(\mathbf{x}^*) = \frac{\mathrm{d}_s y_j}{\mathrm{d} y_j}|_{\mathbf{x}^*} = \frac{\frac{\partial y_j(\mathbf{x}^*)}{\partial x_s}\mathrm{d}x_s}{\sum_{k=1}^{n}\frac{\partial y_j(\mathbf{x}^*)}{\partial x_k}\mathrm{d}x_k} \tag{3}$$

where $\mathrm{d}_s y_j$ is the partial differential of y_j, $\frac{\partial y_j}{\partial x_s}$ is the partial derivative of y_j with respect to x_s and $\mathrm{d}x_s$ the infinitesimal change in x_s. $D_s^j(\mathbf{x}^*)$ generalizes comparative statics sensitivity measures [Borgonovo (2008)]. If $Z(\mathbf{y};\mathbf{x})$ is twice continuously differentiable, by the fundamental theorem of comparative statics, $D_s^j(\mathbf{y}^*, \mathbf{x}^*)$ is expressed in terms the Hessian matrix of the Lagrangian function of $\mathcal{P}_1$ [Borgonovo (2008)]. We now show that these regularity conditions are not satisfied in stochastic optimization problems with risk-coherent objective functions, generally.

In the presence of uncertainty, the economic consequences depend on $\mathbf{y}$, $\mathbf{x}$ and on the stochastic quantities ($\boldsymbol{\omega}$) involved in the problem [Ruszczynski and Shapiro (2005), Ahmed et al (2007).] For instance, in Grubbstroem (2008), Ahmed et al (2007), Borgonovo and Peccati (2009a), Gotoh and Takano (2007), ω is random demand. More in general, we let $\boldsymbol{\omega}$ be a random vector encompassing the stochastic variables of the problem and denote by $(\Omega, \mathcal{B}(\Omega), F)$ the probability space, with $\boldsymbol{\omega} \in \Omega, \subseteq \mathbb{R}^w$. One has $Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}) : X \times Y \times \Omega \to H \subseteq \mathbb{R}$. Consider the function $\rho = \rho(Z) : H \to \mathbb{R}$, with $\rho[Z] > 0$, $\forall Z \neq 0$. ρ is a coherent risk-measure, if it satisfies the translational invariance, positive homogeneity and subadditivity axioms of Artzner et al (1999). Correspondingly, the optimal policy solves the problem [Ruszczynski and Shapiro (2005)]

$$\mathcal{P}_2 = \begin{cases} \min\limits_{\mathbf{y} \in S(\mathbf{x})} \rho[Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x})] \end{cases} \tag{4}$$

$\mathcal{P}_2$ is a non-linear stochastic program. Under suitable convexity conditions [Ruszczynski and Shapiro (2005)], $\mathcal{P}_2$ is feasible. We let $\mathbf{y}^*(\mathbf{x})$ represent a solution of $\mathcal{P}_2$.

We refer to Artzner et al (1999) for a complete description of the axioms and implications of coherent measures of risk. In inventory management, coherent risk measures have been applied for the first time in Ahmed et al (2007). Gotoh and Takano (2007) generalize the problem presented in Ahmed et al (2007) to a multi-item version. Borgonovo and Peccati (2009a) compare the optimal policies ($\mathbf{y}^*$) implied by different coherent risk-measures for the same inventory system. In Chen et al (2007), a thorough analysis of risk aversion in inventory management is presented.

Borgonovo and Peccati (2009b) note that piecewise-definiteness is a characteristic feature of risk-coherent problems. They set forth in detail the conditions under which a piecewise-defined function is differentiable. For our purposes, it suffices to say that the objective function in $\mathcal{P}_2$ might not be twice differentiable at x, even if $Z(y,\boldsymbol{\omega},x)$ is smooth. Therefore, the assumptions of comparative statics are not satisfied by $\mathcal{P}_2$, in general.

A mathematical background that does not rest on differentiability assumptions is represented by the decomposition of a finite change in a finite number of terms discussed in Borgonovo (2010). We present the framework in its most general form, i.e., allowing for the sensitivity on groups of exogenous variables [Borgonovo and Peccati (2009c)]. To do so, we partition the set of exogenous variables ($\mathbf{x}$) in Q groups

$$\underbrace{x_1\, x_2 \,..\, x_{s_1}}_{\gamma_1}\ \underbrace{x_{s_1+1}\, x_{s_1+2} \,..\, x_{s_2}}_{\gamma_2} ... \underbrace{x_{s_{kQ-1}+1}\, x_{s_{Q-1}+2} \,..\, x_n}_{\gamma_Q} \tag{5}$$

with $\gamma_s \cap \gamma_l = \varnothing$, and $\cup_{s=1}^{Q}\gamma_s = \mathbf{x}$. Let $\boldsymbol{\gamma} = \left\{\gamma_1, \gamma_2, ..., \gamma_Q\right\}$ denote the vector of groups. Borgonovo and Peccati (2009c) show that, if g is a measurable function, then any finite change in g can be decomposed as follows:

$$\Delta g = g(\boldsymbol{\gamma}^1) - g(\boldsymbol{\gamma}^0) = \sum_{i=1}^{Q} \Delta_{\gamma_i} g + \sum_{i<j}^{Q} \Delta_{\gamma_i,\gamma_j} g + ... + \Delta_{\gamma_{i_1},\gamma_{i_2},...,\gamma_{i_Q}} g \tag{6}$$

where $\boldsymbol{\gamma}^1$ and $\boldsymbol{\gamma}^0$ are two possible values of the exogenous variables and

$$\begin{cases} \Delta_{\gamma_i} g = g(\gamma_i^1; \gamma_{(-i)}^0) - g(\gamma^0) \\ \Delta_{\gamma_i,\gamma_j} g = g(\gamma_i^1, \gamma_j^1; \gamma_{-(i,j)}^0) - \Delta_{\gamma_i} g - \Delta_{\gamma_j} g - g(\gamma^0) \\ ... \end{cases} \tag{7}$$

In eq. (6) the notation $(\gamma_i^1; \boldsymbol{\gamma}_{(-i)}^0)$ means that all groups are kept at $\boldsymbol{\gamma}^0$ but γ_i. The terms $\Delta_i g$ equal the change in g due to the variation in γ_i alone [eq. (7)]. The terms $\Delta_{i,j} g$ are called second order terms and represent the effect on g of the interaction between γ_i and γ_j. From eq. (7), in fact, $\Delta_{i,j} g$ is obtained by subtracting from the change in g due to the simultaneous changes in γ_i and γ_j $[g(\gamma_i^1, \gamma_j^1; \boldsymbol{\gamma}_{-(i,j)}^0) - g(\boldsymbol{\gamma}^0)]$ the individual effects of the changes in γ_i and γ_j $[-\Delta_i g - \Delta_j g]$. The higher order terms share a similar interpretation.

By eq. (6), one obtains the finite change sensitivity indices (FCSI) defined as follows [Borgonovo (2010), Borgonovo and Peccati (2009c)]

$$\begin{cases} \xi_{\gamma_i}^1 := \Delta_{\gamma_i} g \\ \xi_{\gamma_{i_1},\gamma_{i_2},...,\gamma_{i_k}}^k := \Delta_{\gamma_{i_1},\gamma_{i_2},...,\gamma_{i_k}} g \\ \xi_{\gamma_i}^T := \Delta_{\gamma_i} g + \sum_{i \neq j} \Delta_{\gamma_i,\gamma_j} g + ... + \Delta_{\gamma_1,\gamma_2,...,\gamma_Q} g \end{cases} \tag{8}$$

$\xi_{\gamma_i}^1$ quantifies the individual impact of group γ_i; $\xi_{\gamma_{i_1},\gamma_{i_1},...,\gamma_{i_1}}^k$ the interaction effect among groups $\gamma_{i_1}, \gamma_{i_2}, ..., \gamma_{i_k}$; $\xi_{\gamma_i}^T$ the total effect of γ_i.

These definitions apply in particular when the exogenous variable partition [eq. (5)] coincides with $\mathbf{x}$ itself, namely, $\gamma_i = x_i$, $Q = n$. In this case, the FCSI's concern individual parameters. We shall utilize the notation φ_i^1 for individual FCSI's, instead of $\xi_{\gamma_i}^1$. The relationship among $\xi_{\gamma_i}^1$ and the individual FCSI's contained in group γ_i are proven in Theorem 2 of Borgonovo and Peccati (2009c). The following result is relevant in the remainder of this work. Let $x_{i_1}, x_{i_2}, ..., x_{i_k}$ the parameters in group γ_i, [$\gamma_i = (x_{i_1}, x_{i_2}, ..., x_{i_k})$.] Then, it holds that

$$\xi_{\gamma_i}^1 = \sum_{s=1}^{k} \varphi_{i_1,i_2,...,i_s}^s \tag{9}$$

Eq. (9) means that the first order FCSI of a group of k parameters ($\xi_{\gamma_i}^1$) equals the sum of the FCSI's of all orders from 1 to k of the parameters contained in the group. Therefore, $\xi_{\gamma_i}^1$ entails all individual and interaction effects of the parameters in group γ_i.

Eq. (6) requires the sole measurability of g. Thus, the FCSI's can be computed in all those risk coherent problems in which the implicit relationship between $\mathbf{y}$ and the exogenous variables is measurable. Let us allow g to be a smooth function, for the moment. One can then study the limiting properties of the FCSI as the changes in exogenous variables become small — we consider one endogenous variable for notation simplicity. — In Borgonovo (2010) it is shown that

$$\lim_{\Delta\mathbf{x}\mapsto 0} \frac{\varphi_i^T}{\Delta x_i} = \lim_{\Delta\mathbf{x}\mapsto 0} \frac{\varphi_i^1}{h_i} = \frac{\partial g}{\partial x_i} \tag{10}$$

and

$$\lim_{\Delta\mathbf{x}\mapsto 0} \frac{\varphi_i^T}{\Delta g} = \frac{\varphi_i^1}{\Delta g} = D_i \tag{11}$$

Eqs. (10) and (11) suggest that as $\Delta x \to 0$, if g is smooth, the FCSI's tend to the comparative statics indicators (partial derivative and differential importance, respectively). This provides the required generalization of the comparative statics framework.

In summary, the FCSI's allow to solve the methodological issues generated by the presence of a non-smooth objective function in $\mathcal{P}_2$. In addition, they allow one to obtain sensitivity measures for generic changes (finite and infinitesimal) and simultaneous variations in the exogenous variables (groups). However, up to now, FCSI's have been utilized to decompose a single model output jump across two scenarios.

In the next section, we present the framework to accommodate the exploration of the input parameter space at several points.

3 Extended Finite Change Sensitivity Indices

In the common practical situations decision-makers deal with model results obtained on multiple scenarios. The first is model result monitoring. At $t = 0$, information allows us to set the exogenous variables at $\mathbf{x}^0$. $\mathbf{y}_0^* = \mathbf{y}^*(\mathbf{x}^0)$ and $\rho_0^* = \rho^*(\mathbf{x}^0)$ are the corresponding optimal policy and risk-measure value at the optimum. As time goes by, exogenous variables evolve to the values $\mathbf{x}^1$ (at $t = 1$), next at $\mathbf{x}^2$, etc.. (The sequence of values of the exogenous variables form a set of historical scenarios; geometrically they represent a trajectory in the exogenous variable space; see Figure 1.) The decision-maker adjusts the optimal policy in correspondence of the exogenous variable changes, obtaining the sequences $\mathbf{y}_0^*, \mathbf{y}_1^*, \mathbf{y}_2^*, ...$ and $\rho_0^*, \rho_1^*, \rho_2^*, ...$ Little's question then becomes relevant: what has made the outputs come out as they did?

The second is interpretation of model results in scenario analysis. In this case, a predictive utilization of the model is of interest. The decision-maker selects a set of plausible scenarios. [Scenario selection methods have been widely studied in the literature; see the works of O'Brien (2004) and Tietje (2005). For the selected scenarios to be representative of the decision-maker's view on possible future states-of-the-world, the process involves cognitive and psychological aspects (Jungermann and Thuring (1988)). The ideal selection method leads to a set of *consistent*, *different*, *reliable* and *efficient* scenarios (Tietje (2005), *p.* 421).] After the set of scenarios has been selected, the model is assessed at each scenario. In this perspective application of the model, Eschenbach's question becomes relevant: what are the factors on which managers need to focus attention (and resources) during implementation?

To answer these questions, some formalism is needed. Both in the retrospective and in the perspective modes, our information consists of a sequence of model output values at multiple points in the exogenous variable space. Let $[x_i^-, x_i^+]$ denote the range assigned by the decision-maker to exogenous variable x_i $(i = 1, 2, ..., n)$ (Figure 1). Correspondingly, X is the Cartesian product $X = [x_1^-, x_1^+] \times [x_2^-, x_2^+] \times ... \times [x_n^-, x_n^+]$ (see also Van Groenendaal and Kleijnen (2002), Tietje (2005).) Consider then a partition of each one-dimensional interval into p subintervals [called levels in design of experiments; see Van Groenendaal and Kleijnen (2002), Saltelli et al (2009)] — For notation simplicity, we set a unique p, but a different p_i for each x_i can be used. — X is then the union of p^n hypercubes $X_{r_1, r_2, ..., r_n} = [x_{1_{r_1 - 1}}, x_{i_{r_1}}] \times [x_{i_{r_2 - 1}}, x_{i_{r_2}}] \times ... \times [x_{i_{r_n - 1}}, x_{i_{r_n}}]$, with $r_i = 1, 2, ..., p$. Figure 1 show a three exogenous variable problem.

In Figure 1, each exogenous variable variation range is divided into $p = 2$ subintervals. X is the union of 2^3 cubes. Geometrically, a scenario is a corner of one of the cubes in Figure 1. Selecting a set of scenarios corresponds to obtaining multiple model output values (Figure 1). With reference to Figure 1, the three points $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2$ represent three possible scenarios. If the points are explored in this sequence, $\mathbf{x}^0 \to \mathbf{x}^1, \mathbf{x}^1 \to \mathbf{x}^2$, one has the trajectory in Figure 1. In this case, there are two model output finite changes $\Delta g^{0 \to 1}$ and $\Delta g^{0 \to 2}$. Each of them can be decomposed by eq. (6). The corresponding FCSI's are then computed. Generalizing, let p be the number of levels and $\Delta g^{0 \to 1}$, $\Delta g^{1 \to 2}$, ..., $\Delta g^{p-1 \to p}$. One obtains p sets of FCSI's as follows:

$$\begin{cases} \xi_i^{1,s} = \Delta_i^{s-1 \to s} g \\ \xi_{i_1, i_2, ..., i_k}^{k,s} := \Delta_{i_1, i_2, ..., i_k}^{s-1 \to s} g \\ \xi_i^{T,s} = \Delta_i^{s-1 \to s} g + \sum_{i \neq j} \Delta_{\gamma_i, \gamma_j}^{s-1 \to s} g + ... + \Delta_{i_1, i_2, ..., i_n}^{s-1 \to s} g \end{cases} \qquad s = 1, 2, ..., p \tag{12}$$

$\xi_i^{1,s}$ is the fraction of the change in y across scenarios $s - 1$ and s associated with x_i. It coincides with the difference in model predictions caused by the change in x_i from its value in scenario $s - 1$

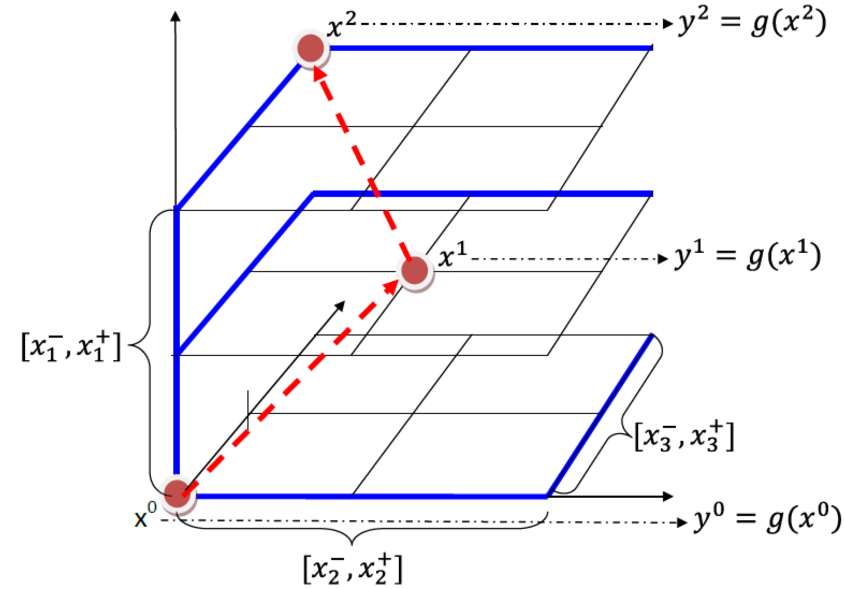

Figure 1: A trajectory in the input parameter space. Points x^0,x^1 and x^2 represent three possible scenarios. One obtains three corresponding model output values.

to the value it assumes in scenario s. $\xi^{k,s}_{i_1,i_2,...,i_k}$ quantifies the portion of the change $\Delta g^{s-1\rightarrow s}$ due to the interaction of exogenous variables $i_1, i_2, ..., i_k$. $\xi_i^{T,s}$ is the fraction of the change in model results associated with x_i individually and in its interactions with all remaining exogenous variables, when x_i shifts across scenarios $s-1 \rightarrow s$.

By considering the set of FCSI's, a decision-maker has a full dissection of model behavior across the selected scenarios. In principle, $(2^n - 1)$ indices per jump are available, leading to a total of $(2^n - 1) \cdot (p - 1)$ sensitivity measures. This figure equals the number of points at which the exogenous variable space is explored. In result communication, such plentiful details need to be synthesized. The design of experiments realm has provided rigorous approaches for aggregating sensitivity measures [Myers and Montgomery (1995), Van Groenendaal and Kleijnen (2002), Saltelli et al (2009).] By drawing from this literature, we proceed as follows. We define the across scenario averages of the sensitivity measures. We write:

$$\widetilde{\xi}^1_i = \frac{\sum_{s=1}^p \xi_i^{1,s}}{p}\; \widetilde{\xi}^k_{i_1,i_2,...,i_k} = \frac{\sum_{s=1}^p \xi^{k,s}_{i_1,i_2,...,i_k}}{p}\;\; \widetilde{\xi}^T_i = \frac{\sum_{s=1}^p \xi_i^{T,s}}{p},\; i = 1, 2, ..., n \tag{13}$$

where $\widetilde{\xi}^1_i$, $\widetilde{\xi}^k_{i_1,i_2,...,i_k}$ and $\widetilde{\xi}^T_i$ are now called extended FCSI's. By the sensitivity measures in eq. (13), an analyst gains information on the average magnitude and direction of impact of the exogenous variable variations across scenarios. However, in estimating the importance of a parameter, retaining the sign of the sensitivity measures, can lead to "Type II" errors [Saltelli et al (2009)]. These error is connected with a sensitivity measure assuming both positive and negative values. Its average can be null as a result of compensations. Nonetheless, the variable might be very influential. This limitation is overcome by averaging the absolute values of the sensitivity measures [Saltelli et al (2009)]. We write:

$$\widetilde{\left|\xi_i^1\right|} = \frac{\sum_{s=1}^p \left|\xi_i^{1,s}\right|}{p} \quad \widetilde{\left|\xi_{i_1,i_2,...,i_k}^k\right|} = \frac{\sum_{s=1}^p \left|\xi_{i_1,i_2,...,i_k}^{k,s}\right|}{p} \quad \widetilde{\left|\xi_i^T\right|} = \frac{\sum_{s=1}^p \left|\xi_i^{T,s}\right|}{p}, \; i = 1, 2, ..., n \tag{14}$$

$\widetilde{\left|\xi_i^1\right|}$ then is called the individual effect of exogenous variable x_i, $\widetilde{\left|\xi_{i_1,i_2,...,i_k}^k\right|}$ the effect of the interactions among exogenous variables $i_1, i_2, ..., i_k$, and $\left|\xi_i^T\right|$ the total effect of variable x_i.

Finally, we note that the extreme values of the ranges define the two scenarios, $\mathbf{x}^- = (x_1^-, x_1^-, ..., x_n^-)$ and $\mathbf{x}^+ = (x_1^+, x_1^+, ..., x_n^+)$. Letting $\mathbf{x}^0 = \mathbf{x}^-$ and $\mathbf{x}^1 = \mathbf{x}^+$, and restricting attention to these two points only, one finds back the framework of the FCSI's in Borgonovo (2010) and Borgonovo and Peccati (2009c).

In the next section, we derive general properties of the extended finite change sensitivity indices for risk-coherent problems.

4 Properties of Finite Change Sensitivity Indices in Risk-Coherent Problems

In this section, we introduce general SA properties of risk coherent problems. As we are to see, the structure of the loss function plays a central role. We propose the following definition.

Definition 1 *We say that:*

1. *the loss function, $Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x})$, is separable in respect of parameter group γ_i, if Z can be written as*

$$Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}) = H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)}) + T(\gamma_i) \tag{15}$$

2. *the loss function is additively separable, if it can be written as:*

$$Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}) = H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)}) + \sum_{l:x_l \in \gamma_i} T_l(x_l) \tag{16}$$

where the sum $\sum_{l:x_l \in \gamma_i}$ is extended over all the parameters in group γ_i.

By the above definition, one obtains that the objective function in $\mathcal{P}_2$ is separable.

Lemma 1 1. *If $Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x})$ is separable in γ_i, then ρ is.*

2. *If $Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x})$ is additively separable in γ_i, then ρ is.*

Proof. Point 1. By Definition 1 and by translational invariance of a coherent risk-measure, one obtains

$$\rho(Z) = \rho[H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)}) + \sum_{l:x_l \in \gamma_i} T_l(x_l)] = \rho[H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)})] + T(\gamma_i) \tag{17}$$

Point 2. Similarly,

$$\rho = \rho[H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)})] + \rho = \rho[H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)})] + \sum_{l: x_l \in \gamma_i} T_l(x_l) \tag{18}$$

■

The next two results prove invariance and additivity conditions for the endogenous variables of generic risk-coherent problems.

Theorem 1 *In $\mathcal{P}_2$, let Z be separable. Consider $\mathbf{y}^*$ as endogenous variable of interest. Then, for any set of changes,* $\widetilde{\xi^1_{\gamma_i}} = \widetilde{\xi^k_{\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k}}} = \widetilde{\xi^T_{\gamma_i}} = 0$ *and* $\left|\widetilde{\xi^1_{\gamma_i}}\right|, \left|\widetilde{\xi^k_{\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k}}}\right| = \left|\widetilde{\xi^T_{\gamma_i}}\right| = 0.$

Proof. By Lemma 1, the objective function of the optimization problem is written as $\rho[H(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}_{(-\gamma_i)})] + T(\gamma_i)$. As a consequence, $\mathbf{y}^*$ does not depend on γ_i, but only on $\mathbf{x}_{(-\gamma_i)}$. ■

Lemma 1 and Theorem 1 state that, if Z is separable on γ_i, then the optimal policy does not depend on γ_i. Therefore all sensitivity measures are null. In particular, the following holds.

Corollary 1 *Suppose that the objective function in $\mathcal{P}_2$ is smooth and comparative statics is applicable to $\mathbf{y}(\mathbf{x}^*)$. Then, if Z is separable in γ_i, all comparative statics sensitivity measures of the parameters in γ_i are null.*

Proof. If Z is separable in γ_i, then it is separable in $x_l \in \gamma_i$. Hence, by Lemma 1 and Theorem 1 $\widetilde{\varphi^1_l} = \widetilde{\varphi^k_l} = \widetilde{\varphi^T_l} = 0$ for any change and for any scenario shift. By eqs. (10) and (11) one obtains $\dfrac{\partial \mathbf{y}^*_j}{\partial x_l} = 0$ and $D^j_l(\mathbf{x}^*) = 0$. ■

Theorem 1 and Corollary 1 state that, in a risk-coherent optimization problem, separability of the objective function on a group of parameters makes the optimal policy insensitive to that group of exogenous variables. This happens for all scenarios, any type of change (finite or infinitesimal) and any sensitivity measure.

The next result concerns the effect of separability on the value of the risk-measure at the optimum.

Theorem 2 *1. Let $\rho^*$the endogenous variable in $\mathcal{P}_2$ under examination. Let Z be separable. Then,*

$$\begin{aligned}
&1)\widetilde{\xi^k_{\gamma_i, \gamma_{i_2}, \ldots, \gamma_{i_k}}} = 0 \ \forall k = 2, \ldots, n \\
&2)\widetilde{\xi^1_{\gamma_i}} = \widetilde{\xi^T_{\gamma_i}} = \widetilde{\Delta T} \\
&3)\left|\widetilde{\xi^1_{\gamma_i}}\right| = \left|\widetilde{\xi^T_{\gamma_i}}\right| = \widetilde{|\Delta T|}
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&\widetilde{\Delta T} = \textstyle\sum_{s=1}^p \dfrac{T_s(\gamma_s) - T_s(\gamma_{s-1})}{s} \text{ and} \\
&\widetilde{|\Delta T|} = \textstyle\sum_{s=1}^p \dfrac{T_s(\gamma_s) - T_s(\gamma_{s-1})}{s}
\end{aligned} \tag{20}$$

2. *If Z is additively separable, then*

$$\widetilde{\xi^1_{\gamma_i}} = \widetilde{\xi^T_{\gamma_i}} = \sum_{l:x_l \in \gamma_i} \widetilde{\Delta T_l} = \sum_{l:x_l \in \gamma_i} \widetilde{\varphi^1_l} \text{ and } \left|\widetilde{\xi^1_{\gamma_i}}\right| = \left|\widetilde{\sum_{l:x_l \in \gamma_i} \varphi^{1,s}_l}\right| \tag{21}$$

Proof.

1. Eq. (19), point 1). By eq. (17), ρ involves two functions H and T. H depends on γ_i, while T depends only on γ_i. Since they are summed in ρ, no interactions of γ_i with other parameter groups emerge. This happens for any change in γ_i as it varies across any scenario. Thus, $\xi^{k,s}_{\gamma_i,\gamma_{i_2},...,\gamma_{i_k}} = 0$ for all $k = 1, 2, ..., n$ and all $s = 1, 2, ..., p$. As a consequence, $\widetilde{\xi^{k,s}_{\gamma_i,\gamma_{i_2},...,\gamma_{i_k}}} = 0$. Eq. (19), point 2) is proven as follows. By definition $\xi^{1,s}_{\gamma_i} = T_s(\gamma^s_i) - T_s(\gamma^{s-1}_i)$. Since no interactions are present, then $\xi^{1,s}_{\gamma_i} = \xi^{T,s}_{\gamma_i}$, $\forall s$. By averaging these last two equalities, one obtains Eq. (19), point 2). The proof of Point 3) is similar, once observed that, by definition, $|\xi|^{1,s}_{\gamma_i} = \left|T_s(\gamma^s_i) - T_s(\gamma^{s-1}_i)\right|$.

2. By Theorem 2 in Borgonovo and Peccati (2009c), $\xi^{1(s)}_{\gamma_i} = \sum_{r=1}^{m_\gamma} \varphi_{i_1,i_2,...,i_r}$, where m_{γ_i} is the number of exogenous variables in γ_i, and $i_1, i_2, ..., i_r : x_{i_1}, x_{i_2}, ..., x_{i_r} \in \gamma_i$, $r = 1, 2, ..., m_{\gamma_i}$. That is, at each scenario change, the sensitivity index of γ_i is the sum of all the FCSI's of the parameters in group γ_i. By the additivity of T, then no interactions among the exogenous variables in γ_i emerge. As a consequence, $\xi^{1(s)}_{\gamma_i} = \sum_{r=1}^{m_\gamma} \varphi^{r,s}_{i_1,i_2,...,i_r} = \sum_{l:x_l \in \gamma_i} \varphi^{1,s}_l$. By definition,

$$\varphi^{1,s}_l = \Delta T^{s-1 \to s}_l = T_l(x^s_l) - T_s(x^{s-1}_l) \tag{22}$$

Consequently, one obtains:

$$\widetilde{\xi^1_{\gamma_i}} = \frac{\sum_{s=1}^{p} \xi^{1(s)}_{\gamma_i}}{p} = \frac{\sum_{s=1}^{p} \sum_{l:x_l \in \gamma_i} \varphi^{1,s}_l}{p} = \frac{\sum_{s=1}^{p} \sum_{l:x_l \in \gamma_i} \Delta T^{s-1 \to s}_l}{p} = \sum_{l:x_l \in \gamma_i} \frac{\sum_{s=1}^{p} \Delta T^{s-1 \to s}_l}{p} \tag{23}$$

Finally, one notes that $\dfrac{\sum_{s=1}^{p} \Delta T^{s-1 \to s}_l}{p}$ is the average change in T_l across the scenarios. This proves that $\widetilde{\xi^1_{\gamma_i}} = \sum_{l:x_l \in \gamma_i} \widetilde{\Delta T_l}$. Concerning $\left|\widetilde{\xi^1_{\gamma_i}}\right|$, one notes that $\left|\xi^1_{\gamma_i}\right| = \left|\sum_{l:x_l \in \gamma_i} \varphi^{1,s}_l\right|$. Hence, $\left|\widetilde{\xi^1_{\gamma_i}}\right| = \left|\widetilde{\sum_{l:x_l \in \gamma_i} \varphi^{1,s}_l}\right|$.

∎

Point 1 of theorem 1 states that, if Z is separable, the value of the coherent risk-measure at the optimum is influenced by γ_i, with an individual action. The effect of γ_i equals the average change in the portion of the loss function depending separately on γ_i. No interaction of γ_i with other groups is present. Point 2 adds to point 1 the following. If $T(\gamma_i)$ is additive, the effect of γ_i ($\widetilde{\xi^1_{\gamma_i}}$) equals the sum of the average individual effects of the parameters in group γ_i ($\widetilde{\xi^1_{\gamma_i}} = \sum_{l:x_l\in\gamma_i} \widetilde{\varphi^1_l}$). Since, at each scenario shift, $\varphi_l^{1,s} = \Delta^{s-1,s}T_l$ (see the proof), $\widetilde{\varphi^1_l} = \widetilde{\Delta T_l}$. Thus, if T is additive, the effect of group γ_i equals the sum of the average changes in T provoked parameters individually.

Theorems 1 and 1 allow us to obtain extended FCSI's directly. Eqs. (19) and (21) show that $\widetilde{\xi^1_{\gamma_i}}$ is computed directly from $T(\gamma_i)$. In principle, then, if the loss function is separable on a group of parameters, one obtains the sensitivity measures on that group without having to solve the optimization problem in correspondence of the variations of the parameters in the group. This turns into a computational advantage, since it contributes at reducing problem size.

In the next section, we set up the SA settings to gain managerial insight by extended FCSI's.

5 Result Interpretation and Managerial Insights

It is the purpose of this section to discuss the methodology for interpreting SA results and extracting managerial insights. We proceed as follows. We set up the SA settings and identify the corresponding sensitivity measures. We then analyze the endogenous variables in a multi-item risk-coherent problem.

According to Saltelli et al (2004), a setting is a clear statement of the SA question that allows the decision-maker to select the appropriate technique and obtain the consistent answer. In Borgonovo (2010) the following three settings have been envisioned in connection with FCSI's.

Setting 1 ("*Direction of Change*") Does a group of exogenous variables have a positive or negative impact on the endogenous variables?;

Setting 2 ("*Key-Drivers*") What is the most important factor? What factors have a minor influence?

Setting 3 ("*Model Structure*") Do exogenous variable act individually? How much of the endogenous variable response is attributable to interactions?

Answering Settings $1-3$ grants us with a full understanding of the model results. Let us see what are the appropriate sensitivity measures. We recall that setting 1 is related to Samuelson (1947)'s classical comparative statics quest: determining the slopes of the "*response of our system to changes in certain parameters* [Samuelson (1947)]*.*" The sensitivity measures for answering setting 1 are $\widetilde{\xi}^1_i, \widetilde{\xi}^k_{i_1,i_2,\ldots,i_k}$ and $\widetilde{\xi}^T_i$. In fact, they retain the sign and magnitude of the exogenous variable changes. We note that, by setting 2, one answers Eschenbach's question. By setting 3, one obtains a complete dissection of model behavior, answering Little's question. The sensitivity measures for settings 2 and 3, are $\widetilde{|\xi^1_i|}, \widetilde{|\xi^k_{i_1,i_2,\ldots,i_k}|}$ and $\widetilde{|\xi^T_i|}$.

We now address the selection of the endogenous variables. In a post-optimality analysis, a decision-maker is interested in the optimal policy and in the value of the objective function at the optimum [Monahan (2000)]. In a risk-coherent problem, these values are $\mathbf{y}^*$ and ρ^*. In a multi-item problem $\mathbf{y}^*$ is a vector of size n. Information consists of a $2^n \times I$ sensitivity matrix containing the extended FCSI's of each x_i, $i = 1, 2, .., n$ on each y_r, $r = 1, 2, ..., I$. As the problem size increases, result communication might become less straightforward.

A first way to reduce problem size is to consider, instead of the itemized list, the total number of ordered items, i.e., $\|\mathbf{y}^*\|$. $\|\mathbf{y}^*\|$ provides synthetic information on optimal order policies and has been adopted in Borgonovo and Peccati (2009a). However, by synthesizing $\mathbf{y}^*$ into its norm, one looses information on the detailed effect of exogenous variables. A second approach is to retain the vector $\mathbf{y}^*$ and integrate the analysis by Savage score correlation coefficients (SSCCs). SSCCs have been introduced by Iman and Conover (1987) to let decision-makers know whether exogenous variables influence different endogenous variables in the same way. In the context of multi-item inventories, we build them as follows. Denote by R_i^r the ranking of exogenous variable x_i, when the optimal order quantity of item r (y_r^*) is the endogenous variable (the complete ranking of the parameters is $\mathbf{R}^r = [R_1^r, R_2^r, ..., R_n^r]$). The corresponding vector of Savage scores is $\mathbf{SS}^r = [SS_1^r, SS_2^r, ..., SS_n^r]$, where

$$SS_i^r = \sum_{j=R_i^r}^{n} \frac{1}{j} \tag{24}$$

Similarly, let R_i^s be the ranking of x_i when item s is considered ($\mathbf{R}^s = [R_1^s, R_2^s, ..., R_n^s]$). Then, one computes the correlation coefficients $\rho_{\mathbf{R}^r,\mathbf{R}^s}$ and $\rho_{\mathbf{SS}^r,\mathbf{SS}^s}$. A high value of $\rho_{\mathbf{R}^r,\mathbf{R}^s}$ denotes an overall ranking agreement both when influential (top-ranked) and non-influential (low-ranked) parameters are considered. $\rho_{\mathbf{SS}^r,\mathbf{SS}^s}$ informs us of whether the ranking agreement is at the top-ranked factors. Comparing $\rho_{\mathbf{SS}^r,\mathbf{SS}^r}$ against $\rho_{\mathbf{R}^r,\mathbf{R}^s}$ one obtains information on whether the key-drivers of the problem are the same [Iman and Conover (1987), Campolongo and Saltelli (1997), Kleijnen and Helton (1999)].

We are then equipped with the tools necessary to examine the sensitivity of $\mathbf{y}^*$, $||\mathbf{y}^*||$ and ρ^* as endogenous variables of a risk-coherent problem. In the next section, we present results and managerial insights.

6 A Numerical Case Study

In this section, we illustrate the methodology developed in Sections 2, 3 and 5 by applying it to a risk-coherent case study. The case study is based on the risk neutral (Rockafellar and Uryasev (2002)) and conditional value at risk (CVaR) problems developed in Borgonovo and Peccati (2009a). Let I denote the number of inventoried items. $\boldsymbol{\omega} = [\omega_1, \omega_2, ..., \omega_I]$, $\boldsymbol{\omega} \in \Omega \subseteq \mathbb{R}^+$ the random demand for items in the inventory, and $(\Omega, \boldsymbol{\Omega}, F)$ the corresponding measure space. The exogenous variables are revenues per unit of inventoried goods ($\mathbf{r} = [r_1, r_2, ..., r_I]$), fixed order costs ($\mathbf{a} = [a_1, a_2, ..., a_I]$) and holding costs per unit of inventoried item ($\mathbf{c} = [c_1, c_2, ..., c_I]$). Hence, $\mathbf{x} = \{\mathbf{a}, \mathbf{c}, \mathbf{r}\}$. The system loss function is:

$$Z(\mathbf{y},\Omega,\mathbf{x}) = \mathbf{1a} + \sum_{i=1}^{I} \frac{\omega_i c_i y_i^2}{2} - \mathbf{ry} \tag{25}$$

Note that the loss function in eq. (25) is additively separable in fixed costs. The corresponding risk-neutral and risk-coherent problems are

$$\mathcal{P}_{risk-neutral} = \left\{ \min_{y \geq 0} \mathbf{1a} + \sum_{i=1}^{I} \frac{m_i c_i y_i^2}{2} - \mathbf{ry} \right. \tag{26}$$

where $m_i := \mathbb{E}_F \left[\frac{1}{\omega_i} \right]$, and

$$\mathcal{P}_{CVaR_\alpha} = \left\{ \min_{\mathbf{y},\zeta \in S \times R} \zeta + \frac{1}{1-\alpha} \left[(\mathbf{1a} - \mathbf{ry} - \zeta) F_\omega^{\zeta+} + \sum_{i=1}^{I} \frac{c_i y_i^2 m_i^{\zeta+}}{2} \right] \right\} \tag{27}$$

where $\Omega^{\zeta+} = \{\omega : Z > \zeta+\}$, $m_i^{\zeta+} := \int_{\Omega^{\zeta+}} \frac{1}{\omega_i}$ and $F_\omega^{\zeta+} = \int_{\Omega^{\zeta+}} \mathrm{d}F$ (See Borgonovo and Peccati (2009a) for further details.) The endogenous variables of interest are $\mathbf{y}^*_{risk-neutral}$, $||\mathbf{y}^*_{risk-neutral}||$, $\mathbb{E}[Z^*]$ for $\mathcal{P}_{risk-neutral}$, and $\mathbf{y}^*_{CVaR}$, $||\mathbf{y}^*_{CVaR}||$ and $CVaR^*$ for $\mathcal{P}_{CVaR_\alpha}$.

Theorems 3 and 1 allow us to state the following results concerning the sensitivity of the endogenous variables on fixed costs, before running both the optimization and sensitivity algorithms.

Proposition 1 *1.* $\widetilde{\xi}^1_{\mathbf{a}} = \widetilde{\xi}^1_{\mathbf{a},\mathbf{r}} = \widetilde{\xi}^1_{\mathbf{a},\mathbf{c}} = \widetilde{\xi}^1_{\mathbf{a},\mathbf{c},\mathbf{r}} = 0$ *for* $\mathbf{y}^*_{risk-neutral}$, $||\mathbf{y}^*_{risk-neutral}||$,$\mathbf{y}^*_{CVaR}$, $||\mathbf{y}^*_{CVaR}||$;

2. For both $CVaR^*_\alpha$ *and* $\mathbb{E}[Z^*]$*, then*

$$\begin{gathered} \widetilde{\xi}^1_{\mathbf{a},\mathbf{r}} = \widetilde{\xi}^1_{\mathbf{a},\mathbf{c}} = \widetilde{\xi}^1_{\mathbf{a},\mathbf{c},\mathbf{r}} = 0 \\ \text{and} \\ \widetilde{\xi}^1_{\mathbf{a}} = \sum_{i=1}^{I} \widetilde{\Delta a_i} = \widetilde{\xi}^T_{\mathbf{a}} \end{gathered} \tag{28}$$

where $\widetilde{\Delta a_i} = \sum_{s=1}^{p} \frac{a_i^s - a_i^{s-1}}{p}$ *is the average of the changes in the fixed cost of item* i*.*

Proof. The system loss function is additively separable in $\mathbf{a}$.

1. Point 1 follows by Theorem 1.
2. Point 2 follows by Theorem 1. In fact, item 2 in Theorem 1 states that $\widetilde{\xi}^1_{\mathbf{a}} = \sum_{l=1}^{I} \widetilde{\Delta T_l(a_l)}$. By eq. (25), $\Delta^{s-1\to s} T_l(a_l) = \Delta^{s-1\to s} a_l$, which, by averaging, leads to eq. (28). Also, by the separability of eq. (25) and Point 1 in Theorem 1, all interactions effects associated with $\mathbf{a}$ are null. Hence, $\widetilde{\xi}^1_{\mathbf{a}} = \sum_{i=1}^{I} \widetilde{\Delta a_i} = \widetilde{\xi}^T_{\mathbf{a}}$.

Table 2: Values of the exogenous variables in the central scenario.

Item	1	2	3	4	5	6	7	8	9	10
r	10	11	12.5	13	12	9.5	14	13.5	12.5	15
a	1	2	2.5	1.5	1.8	2.2	2.3	4.1	1.9	2.7
c	.55	.6	.65	.71	.53	.56	.68	.81	.92	.5

Table 3: Finite changes incurred by the endogenous variables as the exogenous variables vary across the 6 scenarios

$s \to s+1$	$0 \to 1$	$1 \to 2$	$2 \to 3$	$3 \to 4$	$4 \to 5$	$5 \to 6$
$\Delta^{s \to s+1} \|\|\mathbf{y}^*_{risk-neutral}\|\|$	-40	-36	-32	-29	-27	-24
$\Delta^{s \to s+1} \mathbb{E}[Z^*]$	-172	-181	-189	-195	-201	-207
$\Delta^{s \to s+1} \|\|y^*_{CVaR}\|\|$	-18	-16	-14	-13	-12	-13
$\Delta^{s \to s+1} CVaR^*_{\alpha}$	-74	-78	-81	-84	-86	-88

■

Proposition 1 states that: *i*) for the system at hand, the risk-neutral and CVaR optimal policies are insensitive on fixed costs; and *ii*) the fixed costs extended FCSI's [eq. (28)] are exactly equal to the sum of the average variations in fixed costs for both $CVaR^*_{\alpha}$ and $\mathbb{E}[Z^*]$.

We now examine whether numerical findings confirm these expectations. As in Borgonovo and Peccati (2009a), we consider a 10-item inventory ($I = 10$). The numerical values of $\mathbf{r}, \mathbf{a}, \mathbf{c}$ in the base-case scenario are reported in Table 2.

The decision-maker assesses the following exogenous variable ranges: $\mathbf{r} \in [.9\mathbf{r}, 1.1\mathbf{r}]$, $\mathbf{a} \in [0.85\mathbf{a}, 1.15\mathbf{a}]$ and $\mathbf{c} \in [.8\mathbf{c}, 1.2\mathbf{c}]$, respectively. Each range is split into $p = 7$ subintervals. Seven scenarios are then generated by the decision-maker, augmenting the parameters across the levels. One obtains 6 finite model-output changes [$\Delta^{s-1 \to s} g$, $s = 1, 2, .., p-1$; see eq. (12).] Table 3 displays the numerical values of the changes in $||\mathbf{y}^*_{risk-neutral}||$, $\mathbb{E}[Z^*]$, $||\mathbf{y}^*_{CVaR}||$ and $CVaR^*_{\alpha}$.

The second line in Table 3 shows that $||\mathbf{y}^*_{risk-neutral}||$ monotonically decreases across scenario ($\Delta^{s-1 \to s} ||\mathbf{y}^*_{risk-neutral}|| < 0$, $s = 1, 2, ..., 6$). The magnitude of the jumps decreases ($\left|\Delta^{s-1 \to s} ||\mathbf{y}^*_{risk-neutral}||\right| < \Delta^{s \to s+1} ||\mathbf{y}^*_{risk-neutral}||$, $\forall s$). The behavior of $||\mathbf{y}^*_{CVaR}||$ is similar (Line 4). Line 3 shows that the expected loss, $\mathbb{E}[Z^*]$ monotonically decreases across the scenarios. The magnitude of the changes, however, increases. The behavior of $CVaR^*_{\alpha}$ is similar to the behavior of $\mathbb{E}[Z^*]$.

Little's question, then, emerges: what has caused the outputs to came out as they did? To answer the question, we estimate the extended group FCSI's of the exogenous variables, by decomposing each of the finite changes in the objective function across the scenarios. Including the evaluation at $\mathbf{x}^0$, this is equivalent to exploring X at 43 points. Figure 2 displays the FCSI's for the first jump across scenarios $0 - 1$ of $||\mathbf{y}^*_{Risk-Neutral}||$, ($\Delta^{0 \to 1} \mathbf{y}^*_{Risk-Neutral} = -40$ units).

The values of the first order indices in Figure 2 are $\xi^1_{\mathbf{r}} = +72$, $\xi^1_{\mathbf{a}} = 0$, $\xi^1_{\mathbf{c}} = -108$, respectively. This indicates that the 3.3% increase in $\mathbf{r}$ produces an increase of 72 units in the order policy, which

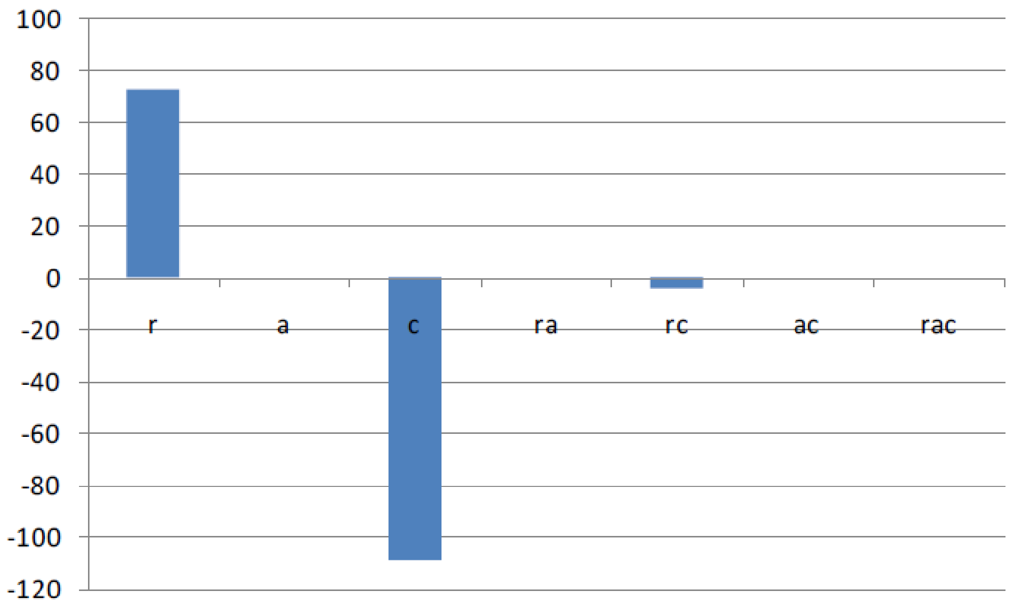

Figure 2: $\xi^1_{\mathbf{r}}, \xi^1_{\mathbf{a}}, \xi^1_{\mathbf{c}}, \xi^2_{\mathbf{r},\mathbf{a}}, \xi^2_{\mathbf{r},\mathbf{c}}, \xi^2_{\mathbf{c},\mathbf{a}}, \xi^3_{\mathbf{r},\mathbf{a},\mathbf{c}}$ for the change in $||\mathbf{y}^*_{risk-neutral}||$ (the total number of ordered items in the risk-neutral optimal policy) from scenario $\mathbf{x}^0$ to scenario $\mathbf{x}^1$. Note that the sum of the sensitivity indices equals the finite change: $\Delta^{0\to 1}\mathbf{y}^*_{Risk-Neutral} = \xi^1_{\mathbf{r}} + \xi^1_{\mathbf{a}} + \xi^1_{\mathbf{c}} + \xi^2_{\mathbf{r},\mathbf{a}} + \xi^2_{\mathbf{r},\mathbf{c}} + \xi^2_{\mathbf{c},\mathbf{a}} + \xi^3_{\mathbf{r},\mathbf{a},\mathbf{c}}$

Table 4: Extended FCSI's for the case study. The last three columns report the total order FCSI's

Output	$\widetilde{\xi}^1_{\mathbf{r}}$	$\widetilde{\xi}^1_{\mathbf{a}}$	$\widetilde{\xi}^1_{\mathbf{c}}$	$\widetilde{\xi}^2_{\mathbf{r},\mathbf{a}}$	$\widetilde{\xi}^2_{\mathbf{r},\mathbf{c}}$	$\widetilde{\xi}^2_{\mathbf{c},\mathbf{a}}$	$\widetilde{\xi}^3_{\mathbf{r},\mathbf{a},\mathbf{c}}$	$\widetilde{\xi}^T_{\mathbf{r}}$	$\widetilde{\xi}^T_{\mathbf{a}}$	$\widetilde{\xi}^T_{\mathbf{c}}$
$\|\|\mathbf{y}^*_{Risk-Neutral}\|\|$	64	0	-92	0	-3	0	0	61	0	-95
$\mathbb{E}[Z^*]$	-795	1.1	564	0	39	0	0	-756	1.1	$564-39$
$\|\|\mathbf{y}^*_{CVaR}\|\|$	28	0	-40	0	-1	0	0	27	0	39
$CVaR^*_{\alpha}$	-343	1.1	243	0	17	0	0	$-343+17$	1.1	$243+17$

is opposed by the decrease of 108 units provoked by the 3.3% change in $\mathbf{c}$. $\mathbf{a}$ have no effect, as expected. Overall, individual indices explain a decrease of 36 units, namely 90% of the change. The remaining 10% is not reducible to individual effects, but due to interactions. Of these, all interaction effects involving $\mathbf{a}$ are null, in accordance with Proposition 1. The additional decrease is, therefore, provoked by the residual interaction between unit revenues and costs. Indeed, $\xi^2_{\mathbf{r},\mathbf{c}} = -4$ (Figure 2). In terms of Setting 3, Figure 2 shows that results for the optimal policy $||y^*_{Risk-Neutral}||$ are driven by individual effects, with interaction effects playing a minor role.

Figure 2 explains the endogenous variable change across scenarios 0 and 1. Examination of the scenario-by-scenario results, shows that the behavior registered in shift across scenarios 0 and 1 is replicated at all jumps. Table 4 displays the extended FCSI's [eq. (13)].

The first line in Table 4 refers to $||\mathbf{y}^*_{Risk-Neutral}||$. It shows that changes in $||\mathbf{y}^*_{Risk-Neutral}||$ are mainly driven by the individual effects of $\mathbf{r}$ and $\mathbf{c}$, with interactions playing a minor role. $\mathbf{r}$

Table 5: Total order FCSI's for the parameters across the 7 scenarios. One notes their monotonically decreasing magnitude

Scenario	$0-1$	$1-2$	$2-3$	$3-4$	$4-5$	$5-6$
$\xi_{\mathbf{r}}^{T,s}$	68	65	62	59	56	54
$\xi_{\mathbf{c}}^{T,s}$	-113	-104	-97	-91	-85	-80
$\xi_{\mathbf{a}}^{T,s}$	0	0	0	0	0	0

is associated with an average increase of 64 units, $\mathbf{c}$ with an average decrease of 92 units, and their interaction effect is $\widetilde{\xi}^2_{\mathbf{r},\mathbf{c}} = -3$. Table 5 details the total order FCSI's across the scenarios for $||\mathbf{y}^*_{Risk-Neutral}||$.

Table 5 shows a monotonically decreasing magnitude of the sensitivity measures as s increases. This means that the exogenous variables impact $||\mathbf{y}^*_{Risk-Neutral}||$ with decreasing intensity. This explains the decreasing magnitude of the model output changes registered in Table 3.

The second line in Table 4 displays the extended FCSI's for the risk-neutral expected loss, $\mathbb{E}[Z^*]$. One notes that $\mathbf{r}$ has a negative effect on the expected loss. This is in accordance with intuition, since an increase in revenues decreases the expected loss. $\mathbf{c}$ on the other hand, has a positive effect on the loss. On $\mathbb{E}[Z^*]$ $\mathbf{a}$ has a non-null, albeit small effect ($\widetilde{\xi}^1_{\mathbf{a}} = 1.1$). By Proposition 1, one expects this value to be equal to the sum of the average changes in fixed costs. A numeric check confirms this expectation.

The fourth line in Table 4 displays the numerical values of the sensitivity measure for $||\mathbf{y}^*_{CVaR}||$. Results show that $||\mathbf{y}^*_{CVaR}||$ behaves in a similar way as $||\mathbf{y}^*_{Risk-Neutral}||$. The effect of $\mathbf{r}$ is an average increase in $||\mathbf{y}^*_{CVaR}||$ of 28 units. This is compensated by the effect of $\mathbf{c}$, that causes a decrease of 40 units in $||\mathbf{y}^*_{CVaR}||$. The action of $\mathbf{r}, \mathbf{c}$ is mainly individual, with a small and negative interaction effect of -1. All effects associated with $\mathbf{a}$ are null. The fifth line in Table 4 shows the sensitivity measures for $CVaR^*_\alpha$. $\mathbf{r}$ has a negative effect (its increase reduces the expected conditional loss). $\mathbf{c}$ has a positive effect (its increase augments $CVaR^*_\alpha$). There effect of $\mathbf{a}$ is equal to $\widetilde{\xi}^1_{\mathbf{a}} = \sum_{i=1}^{I} \widetilde{\Delta a_i} = 1.1$, in accordance with expectations (Proposition 1).

Let us now interpret the results in the light of the SA settings. We start with settings 1 and 3. The values of the sensitivity measures reveal that it is the individual effects of $\mathbf{r}$ and $\mathbf{c}$ that drive changes in order policy, with fixed costs playing no role and interaction effects playing a minor role. In this respect, one notes that interaction effects amplify the action of $\mathbf{c}$, while they oppose the action of $\mathbf{r}$. Thus, the total effect of $\mathbf{c}$ is higher then its individual effect; conversely, the total effect of $\mathbf{r}$ is smaller than its individual effect. The resulting total order sensitivity indices, $\widetilde{\xi}^T_{\mathbf{r}}, \widetilde{\xi}^T_{\mathbf{a}}, \widetilde{\xi}^T_{\mathbf{c}}$, are reported in the last three columns in Table 4. We recall that total order sensitivity indices measure the overall effect of an exogenous variable [Saltelli et al (2000), Saltelli and Tarantola (2002), Saltelli et al (2008)].

As far as Setting 2 is concerned, key-drivers are identified by the magnitudes of the total order sensitivity indices [eq. (14)]. Figure 3 displays $\widetilde{|\xi^T_{\mathbf{r}}|}$, $\widetilde{|\xi^T_{\mathbf{a}}|}$ and $\widetilde{|\xi^T_{\mathbf{c}}|}$ for $||y^*_{Risk-Neutral}||$, $\mathbb{E}[Z^*], ||y^*_{CVaR}||$ and $CVaR^*_\alpha$ are reported.

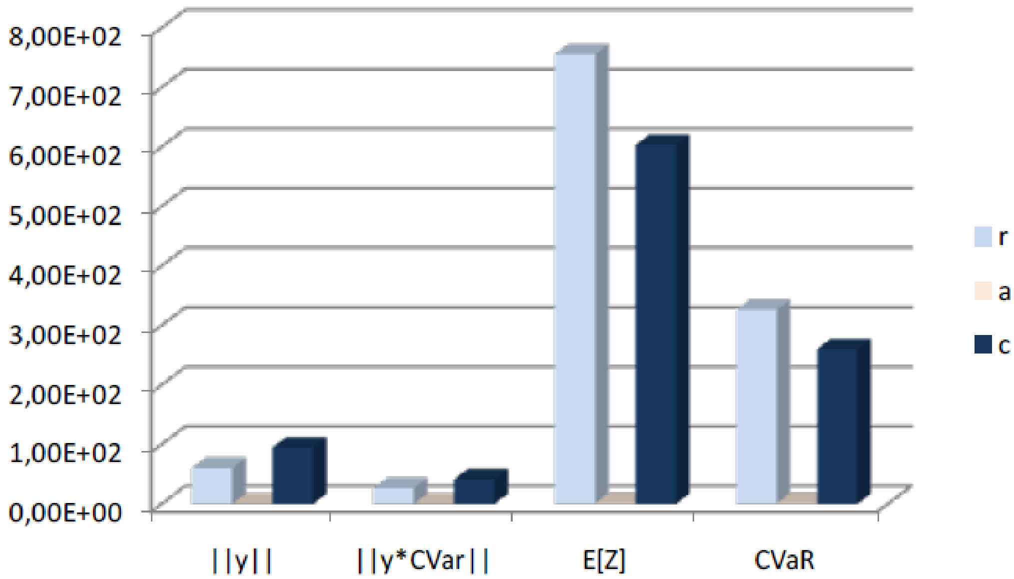

Figure 3: Key-drivers of $\mathcal{P}_{CVaR_\alpha}$ and $\mathcal{P}_{risk-neutral}$.

One notes that $\mathbf{c}$ is the main driver of the optimal order policies, both in $\mathcal{P}_{risk-neutral}$ and $\mathcal{P}_{CVaR_\alpha}$. It is closely followed by $\mathbf{r}$. Fixed costs play no role. Conversely, $\mathbf{r}$ is the main driver of the change in $\mathbb{E}[Z^*]$ and $CVaR^*_\alpha$ at the optimum. It is followed by $\mathbf{c}$, with $\mathbf{a}$ playing a minor role.

We now come to the item-by-item SA results. Figure 4 displays the FCSI's of all orders for all the items of $\mathbf{y}^*_{risk-neutral} = [\mathbf{y}^*_1, \mathbf{y}^*_2, ..., \mathbf{y}^*_{10}]$.

Figure 4 shows that $\mathbf{a}$ has a null effect on the optimal order policy of all items. For all items, increases in price increase the unit orders ($\widetilde{\xi}^1_{\mathbf{r}} > 0$ for all items); increases in holding costs impact $\mathbf{y}^*_i$ $(i = 1, 2., , , .10)$ in the opposite direction ($\widetilde{\xi}^1_{\mathbf{c}} < 0$). $\widetilde{\xi}^2_{\mathbf{r},\mathbf{c}}$ is the only interaction effects and is negative for all items. Furthermore, $\widetilde{\xi}^1_{\mathbf{c}} > \widetilde{\xi}^1_{\mathbf{r}} >> \widetilde{\xi}^2_{\mathbf{r},\mathbf{c}}$ for all items, signalling that individual effects prevail over interaction effects.

Thus, the item-by-item response maintains the structure of the response for $||\mathbf{y}^*_{risk-neutral}||$. The key-drivers are $\mathbf{c}$ and $\mathbf{r}$, with $\mathbf{a}$ being non-relevant.

In order to check whether the influence of $\mathbf{c}$, $\mathbf{r}$ and $\mathbf{a}$ is uniform across items, we compute the Savage score correlation coefficients (see Section 5). We have $\rho_{\mathbf{R}^r,\mathbf{R}^s} = 1$ and $\rho_{\mathbf{SS}^r,\mathbf{SS}^s} = 1$, $r, s = 1, 2, ..., 10$ $(r \neq s)$. This result indicates a uniform effect of the parameters on the item-by-item optimal order quantity. In other words, $\mathbf{c}$ influences the ordered quantity first any item $\mathbf{y}^*_i$ the most, $\mathbf{r}$ ranks second and $\mathbf{a}$ third, even if the order quantities of each item are different and also the changes across the scenarios.

Let us now examine the SA results for $\mathbf{y}^*_{CVCVar}$, Figure 5 reports the FCSI's of all orders for all items.

Figure 5 shows that $\mathbf{c}$ and $\mathbf{r}$ are the drivers of the change, individual effects play a major role in respect of interaction effects.

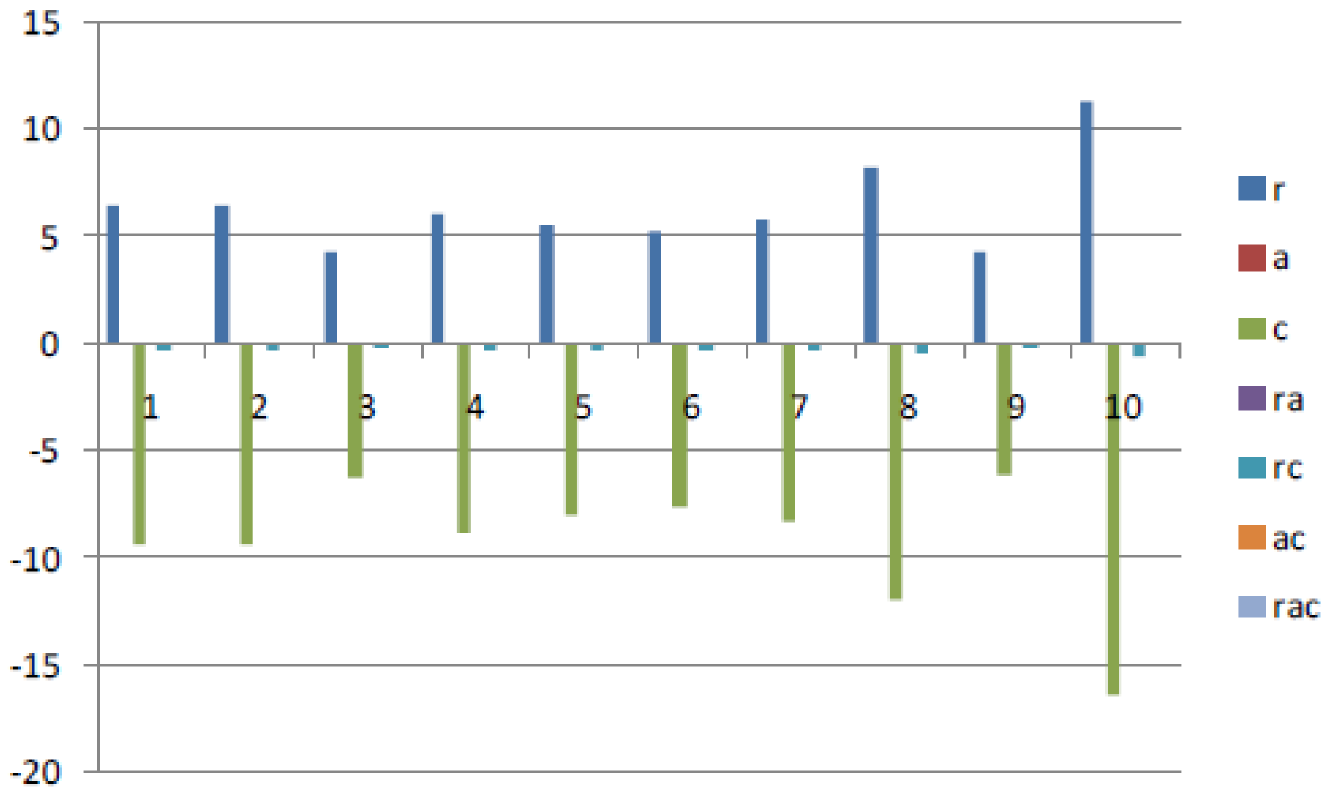

Figure 4: $\widetilde{\xi}^1_{\mathbf{r}}, \widetilde{\xi}^1_{\mathbf{a}}, \widetilde{\xi}^1_{\mathbf{c}}, \widetilde{\xi}^2_{\mathbf{r},\mathbf{a}}, \widetilde{\xi}^2_{\mathbf{r},\mathbf{c}}, \widetilde{\xi}^2_{\mathbf{c},\mathbf{a}}, \widetilde{\xi}^3_{\mathbf{r},\mathbf{a},\mathbf{c}}$ for the inventoried items of $\mathbf{y}^*_{risk-neutral}$. The horizontal axis displays the item number.

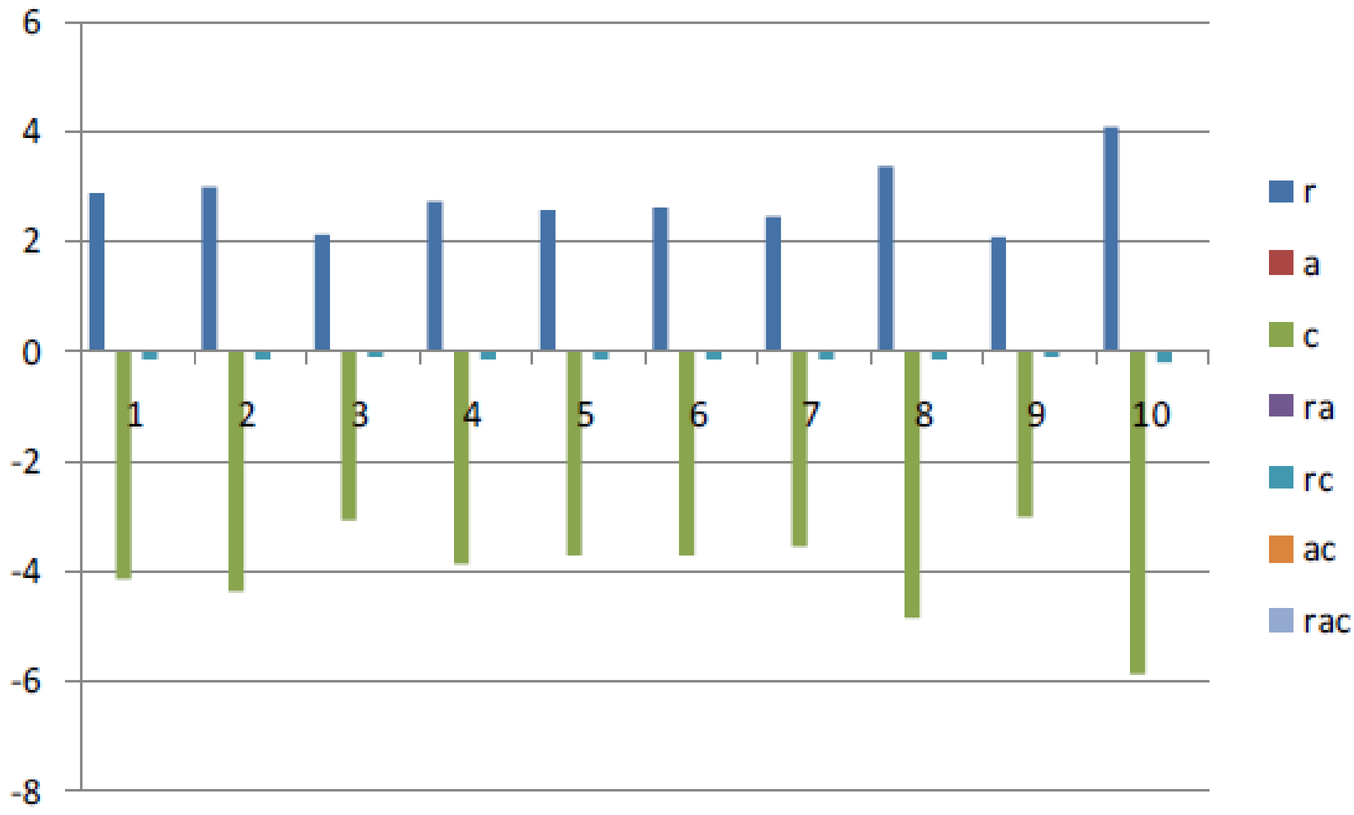

Figure 5: $\widetilde{\xi}^1_{\mathbf{r}}, \widetilde{\xi}^1_{\mathbf{a}}, \widetilde{\xi}^1_{\mathbf{c}}, \widetilde{\xi}^2_{\mathbf{r,a}}, \widetilde{\xi}^2_{\mathbf{r,c}}, \widetilde{\xi}^2_{\mathbf{c,a}}, \widetilde{\xi}^3_{\mathbf{r,a,c}}$ for the inventoried items of $\mathbf{y}^*_{CVaR}$.

The above results allow us to compare the sensitivities of a risk-neutral and of a CVaR decision-maker in respect of exogenous variable changes. Table 3 shows that $\Delta^{s\to s+1}||y^*_{CVaR}|| < \Delta^{s\to s+1}||y^*_{Risk-Neutral}||$ and $\Delta^{s\to s+1}CVaR^*_\alpha < \Delta^{s\to s+1}\mathbb{E}[Z^*]$, for all s. This indicates that changes in the risk-coherent optimal policy and in the corresponding value of the objective function are smaller than the corresponding risk-neutral ones. Table 4 shows that all sensitivity measures in $\mathcal{P}_{CVaR_\alpha}$ are smaller than in $\mathcal{P}_{risk-neutral}$ (compare lines 4 and 5 against lines 2 and 3). Thus, a $CVaR^*_\alpha$ decision-maker is less sensitive to exogenous variable changes than a risk-neutral one. These findings confirm the smoothening effect of risk-measures evidenced in previous literature. (See Borgonovo and Peccati (2009a), and Ahmed et al (2007).) For the optimal policies, one can then compare the results in Figures 4 and in Figure 5. The item-by-item dependence on the exogenous variables of the optimal policy in both $\mathcal{P}_{CVaR_\alpha}$ and $\mathcal{P}_{risk-neutral}$ is the same. However, the magnitude of the sensitivity different. In particular, the smoothening effect of CVaR is evidenced by comparing the absolute value of the sensitivity measures. Thus, a CVaR optimal policy is, item-by-item, less sensitive to exogenous variable changes than a risk-neutral one.

Let us now examine the interpretation of results from the perspective and retrospective operational viewpoints. If the scenarios were generated by a retrospective examination of order policies, then the endogenous variables changes are the consequence of actual changes in the exogenous variables. As parameter evolve through time, the optimal policy evolves. By the present methodology, one answers Little's question in full. In fact, one obtains an exact explanation of (paraphrasing Little's question) what has made the optimal order policy, expected loss (etc.) come out as they did. Conversely, assume that a perspective examination of future decisions is being performed (scenario analysis). In this case, the selected scenarios are plausible realizations of the future. Eschenbach's question then emerges. Through the extended FCSI's one then answers the question by unveiling in an exact and quantitative fashion the factors on which to focus during implementation. In particular, one knows that $\mathbf{r}$ and $\mathbf{c}$ are the drivers, with fixed costs $\mathbf{a}$ deserving much less attention.

We finally note that these findings provide guidance also on data and information collection. They highlight to the decision-maker what factors have priority when resources (time and costs) are limited — in our case, additional information on $\mathbf{a}$ would be much less influential than additional information on $\mathbf{c}$ or $\mathbf{r}$.—

7 Conclusions

In this work, we have introduced a comprehensive approach to the SA of risk coherent multi-item inventory problems.

We have addressed specific issues raised by: i) the piecewise-defined character of risk-coherent objective functions; ii) the presence of simultaneous and discrete changes in the parameters when shifted across multiple scenarios. We solved these problems by decomposing of the endogenous variable changes across all scenarios according to the finite change decomposition of Borgonovo (2010). We have formalized the notion of extended FCSI's. We have seen that they retain all properties of FCSI's granting full flexibility in performing SA with groups of exogenous variables.

We have obtained new results for the sensitivity analysis of risk-coherent optimization problems. We have seen that, if a loss function (not necessarily describing an inventory system) is separable in a group of exogenous variables, then the optimal policy is insensitive to that group. The risk-measure-at-the-optimum is sensitive, with an additive response. These results hold for any change and at all scenario jumps.

We have discussed the interpretation of results in the light three SA settings. We have seen that one is endowed with the exact explanation of what has made the outputs come out as they did (Little's question) and with information about the factors on which to focus managerial attention during implementation (Eschenbach's question).

To illustrate the methodology, we have applied it to an inventory management case study, involving a risk-neutral and CVaR problems. Numerical results confirm the theoretical findings. The extended FCSI's of fixed costs are null on the optimal policy, and equal to $\sum_{i=1}^{I} \widetilde{\Delta a_i}$ on both $\mathbb{E}[Z^*]$ and $CVaR_\alpha^*$, due to the separable structure of the loss function. The sensitivity measures allow us to identify the key drivers of the changes and unveil the detailed structure of the model response across scenarios — in the specific case, only the interaction between **c** and **r** is relevant. — The values of the sensitivity measures reveal that a CVaR decision-maker is less sensitive to exogenous variable changes than a risk neutral one.

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