Financial Management in Inventory Problems: Risk Averse vs Risk Neutral Policies

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Abstract

In this work, we discuss the effect of risk measure selection in the determination of inventory policies. We consider an inventory system characterized by the loss function of Luciano et al (2003.) We derive the optimization problems faced by risk neutral, quadratic utility, mean-absolute and CVaR decision makers. Results show that while the global nature of the optimal policy is assured for risk coherent and risk neutral decision makers, the convexity of the quadratic utility problem depends on the stochastic properties of demand. We investigate the economic and stochastic determinants of the different policies. This allows us to establish the conditions under which each type of decision maker is indifferent to imprecision in the distribution families. Finally, we discuss the numerical impact of the choice of the risk measure by means of a multi-item inventory. The introduction of an approach based on Savage Scores allows us to offer a quantitative measurement of the similarity/discrepancy of policies reflecting different risk attitudes.

Keywords: *Inventory Management, Coherent Risk Measures, Optimization with Coherent Risk Measures, Random Demand Modelling.*

1 Introduction

The purpose of this work is to investigate the quantitative implications of the risk measure choice on optimal inventory policies. We introduce a structured approach to allow the determination of the extent of the discrepancies in the policies selected by decision makers with different risk attitudes — in particular, we compare risk neutral policies to the policies of decision makers selecting variance, absolute deviation (MAD) and conditional value at risk (CVaR) as risk measures[1]. —

Relevant literature in the last 20 years has evidenced the importance of financial and decision theoretical aspects in inventory management. The works of Grubbström and Thorstenson (1986), Thorstenson (1988), Luciano (1998), Luciano and Peccati (1999) and Luciano

[1] We refer to Ogryczak and Ruszczynski (1999) for the relationship between stochastic dominance and mean-risk models, to Artzner *et al* (1999) for a definition of coherent risk measures and to Rockafellar and Uryasev, 2002 for the definition of conditional value at risk. A definition of mean-absolute deviation can be found in Ruszczynski and Shapiro, 2005, who also discuss in detail the theory of risk measure optimization.

et al (2003), Koltay (2006) focus on applications of the discounted cash flow methodology to inventory policies. Bogataj and Hvalica (2003) propose to utilize, besides an expected value criterion, the maximin approach. The maximin approach to the newsvendor problem is discussed in Gallego and Moon (1993) and Gallego *et al* (2001.) Earlier, alternative optimization criteria for the newsvendor problem have been studied in Lau (1980.) This lines of thought can be seen as leading to the recent formulation of inventory management problems in terms of coherent risk measures [Chen *et al* (2005,) Ahmed *et al* (2006,) Gotoh and Takano (2006).] Chen *et al* (2005) analyze risk aversion in inventory problems comparing risk measures and expected utility optimization. Ahmed *et al* (2006) derive the structure of the solution of coherent risk measure optimization for the newsvendor loss function. Gotoh and Takano (2006) discuss the solution of the newsvendor problem with conditional value at risk (CVaR.) Common feature in these works is the utilization of the newsvendor loss function. Distinctive features are, in Gotoh and Takano (2006,) CVaR minimization with the extension of the loss function to a multi-item single-period problem, and, in Ahmed *et al* (2006), the treatment of the single-item multi-period (finite horizon) problem.

In this work, we consider a multi-item inventory system whose financial characteristics are described by the profit function of Luciano *et al* (2003) — "LCP model" from now on. In order to assess the effect of alternative risk aversion representations, we are faced with formulating and studying the optimization problems of the four decision makers in the presence of the LCP model. Results show that while a risk neutral and a coherent risk averse optimal policy is always a global one, the conditions under which a mean-variance decision maker's optimal policy is globally optimal are determined by the stochastic properties of demand.

We then investigate the determinants of the optimal policies. By deriving the analytical expressions of the optimization problems, we identify and discuss the stochastic properties that are needed by the four types of inventory managers to identify the optimal policies. This allows us to determine the conditions under which the decision makers are insensitive to imprecision in the demand distributions. As far as economic aspects are concerned, findings show that while a risk neutral policy can be determined based on the sole knowledge of revenues and variable costs, risk averse decision makers need the additional knowledge of the system fixed costs.

We then carry out a numerical discussion aimed at highlighting the quantitative differences among the optimal policies selected by the different decision makers. To compare the policy structures we introduce a methodology based on Savage's score correlation coefficients (Iman and Conover, 1987.) The numerical impact of imprecision in the demand distribution is assessed by confronting numerical results obtained with finite support distributions (Beta) to the results obtained via an infinite support distribution (Gamma.)

The remainder of the paper is organized as follows. Section 2 illustrates the problem settings in the presence of a generic loss function. Section 3 presents the problem settings for the LPC profit function. In particular, Section 3.1 discusses the optimization problem for a risk neutral decision maker. Section 3.2 derives the optimization problem for a quadratic utility risk averse decision maker. Section 3.3 discusses the problem for an inventory manager utilizing MAD. Section 3.4 presents the optimization problem for a CVaR decision maker.

Section 4 compares the different problems, discusses numerical results and evaluates the effect of imprecision in the demand distributions. Section 5 offers conclusions.

2 Problems Settings for Generic Inventories

In this Section, we present a brief overview of risk measure optimization, introducing results that are relevant in the remainder of the paper.

We start with considering a real valued random variable, $Z = f(\mathbf{x}, \omega)$ that depends on decision vector ($\mathbf{x} \in \mathbb{R}^N$) and $\omega \in \Omega$, where $(\Omega, \mathcal{B}(\Omega), P)$ is a measure space [Ruszczynski and Shapiro (2005).] If Z represents a loss or a disutility (Ruszczynski and Shapiro, 2005) the optimal risk-neutral choice solves the stochastic programming problem:

$$\min_{\mathbf{x} \in S} \mathbb{E}_P[f(\mathbf{x}, \omega)] \tag{1}$$

where S is the feasible set.

Many authors, have questioned expected value optimization, as the resulting policy is optimal "*on average* [Ruszczynski and Shapiro (2005)]." Indeed, the most general formulation of an optimization problem is in terms of expected utility maximization. The problem is stated as

$$\max_{\mathbf{x} \in S} \mathbb{E}_P[u(f(\mathbf{x}, \omega))] \tag{2}$$

The utility function, $u(\cdot)$, captures the decision maker preferences, giving full consideration to His/Her risk aversion/proneness. However, the form of $u(\cdot)$ can be "*extremely difficult to elicit* [Ruszczynski and Shapiro (2005)]." A first way to go around such a difficulty is to pre-determine the shape of the utility function [see Chen *et al* (2005),] or to approximate it through a series expansion. When the expansion is arrested at the second order one obtains the quadratic assumption which is at the basis of Markowitz's (1952) portfolio selection model. A decision maker possessing a quadratic utility function, ought to select $\mathbf{x}$ such that:

$$\min_{\mathbf{x} \in S} \mathbb{V}_P[f(\mathbf{x}, \omega)] \tag{3}$$

As second way to take risk aversion into consideration, which has been successfully proposed in the finance literature in the seminal work of Artzner *et al* (1999,) consists of making use of coherent measures of risk. Let $\rho(Z)$ denote a function satisfying the following axioms of Artzner *et al*,1999:

1) Translational Invariance: $\rho[Z + a] = \rho[Z] + a$
2) Subadditivity: $\rho[Z_1 + Z_2] \leq \rho[Z_1] + \rho[Z_2]$
3) Positive Homogeneity: $\forall \alpha > 0 \quad \rho[\alpha X] = \alpha \rho[X]$
4) Monotonicity: Given Z_1, Z_2 such that $Z_1(\omega) \geq Z_2(\omega)$ $\forall \omega \in \Omega$ then $\rho[Z_2] \leq \rho[Z_1]$
5) $\forall Z \neq 0$, $\rho[Z] > 0$

Then, $\rho(Z)$ is a coherent measure of risk and a decision maker modeling risk aversion through $\rho(Z)$ solves the following problem [Artzner *et al* (1999), Ruszczynski and Shapiro (2005):]

$$\min_{\mathbf{x} \in S} \rho[f(\mathbf{x}, \omega)] \tag{4}$$

For a complete commentary on the meaning of the Axioms, we refer to Artzner *et al* (1999.) We, however, place emphasis on the following results that provide the background for the findings presented in the next sections.

Remark 1 *1. The axioms of Subadditivity and Positive Homogeneity lead to the convexity of $\rho[f(\mathbf{x},\omega)]$;*

2. Monotonicity assures that $\rho[f(\mathbf{x},\omega)]$ is convex in $\mathbf{x}$ if $f(\mathbf{x},\omega)$ is convex[2];

3. If the set S is convex, then the optimization Problem becomes a convex program [see also Ruszczynski and Shapiro (2005).]

The implementation of Problem (4) in inventory management is accomplished if one lets $f(\mathbf{x},\omega)$ coincide with the loss function of the inventory model under consideration. In particular, the observations in Remark 1 assure that:

Remark 2 *if the loss function of the inventory model is convex, then the optimization Problem (4) is a convex program.*

Variance can be taken as a measure of dispersion, and originates the so called mean-variance risk function [Ogryczak and Ruszczynski (1999,) Ruszczynski and Shapiro (2005).] However, Ogryczak and Ruszczynski (1999) show that the mean-variance risk measure is not consistent with second order stochastic dominance in the presence of asymmetric distributions, and Ruszczynski and Shapiro (2005) show that it does not satisfy the monotonicity axiom, *i.e.*, the mean-variance risk function is not a coherent measure of risk. We then contrast the result of a quadratic decision maker to those obtained by decision makers utilizing the following two coherent risk measures: mean-absolute deviation measure (MAD) and CVaR. The MAD measure is defined [Ruszczynski and Shapiro (2005) and Ahmed *et al* (2006)] as:

$$\rho(Z) = \mathbb{E}_P[Z] + \gamma\left\{\mathbb{E}_P[|Z - \mathbb{E}_P[Z]|]\right\} \tag{5}$$

with $0 < \gamma \leq 0.5$.

The CVaR measure (denoted as $\phi_\alpha(\mathbf{x})$, with $\alpha \in [0,1]$, see also Table 1) has been introduced in Rockafellar and Uryasev (2002):

$$\phi_\alpha(\mathbf{x}) = \text{ mean of the } \alpha - \textit{tail} \text{ distribution of } Z \tag{6}$$

CVaR is a generalization of Value at Risk (VaR.) CVaR enables to overcome the lack of subadditivity associated with VaR and satisfies the axioms of Artzner *et al* (1999) [Rockafellar and Uryasev (2002,) Ruszczynski and Shapiro (2005), Gotoh and Takano (2006) and Ahmed *et al* (2006).]

In the next sections, we derive the Problems faced by a risk neutral, a mean-variance, a MAD and a CVaR decision maker in the case the inventory system is described by the LPC profit function.

[2] See Ruszczynski and Shapiro (2005), Section 6 and Theorem 5.1 in Rockafellar (1970.)

3 Problem Settings for the Multi-Item LPC Inventory Model

The cost function at the basis of our analysis is the one proposed in Luciano *et al* (2003). The model foresees a single item inventory characterized by stochastic demand, D, with one-period profit function given by [Luciano *et al* (2003)]:

$$\pi(x, D) = \left(px - a - \frac{hx^2}{2D}\right) \tag{7}$$

where x is the order quantity, $p > 0$ the revenue per unit of inventoried good, $a > 0$ the fixed order cost component, and $h > 0$ the unit holding cost [Luciano *et al* (2003)]. We let $D \in \Theta_D \subseteq \mathbb{R}^+ - \{0\}$, and $(\Theta_D, \mathcal{B}(\Theta_D), P)$ is the corresponding measure space (we refer to Table 1 for notation and symbols.) As far as the convexity of eq. (7) is concerned, it holds that:

Remark 3 *Since $p > 0$, $a > 0$, $h > 0$ and $D > 0$, $\pi(x, D)$ is (strictly) concave $\forall D$, while the corresponding loss function*

$$\mathcal{L}(x, D) = -\pi(x, D) \tag{8}$$

is (strictly) convex $\forall D$.

Proof. By differentiation, $\pi''(x, D) = -\frac{h}{D} < 0$, and $\mathcal{L}''(x, D) > 0$ $\forall D$. ■

Then, a risk-averse inventory manager faces the Problem:

$$\min_{x \in S} \rho[\mathcal{L}(x, D)] \tag{9}$$

We note that combining Remarks 2 and 3 one obtains the following:

Remark 4 *Given the LPC loss function in eq. (8), (9) is a convex program for any coherent risk measure*

On the other hand, a risk neutral inventory manager would face the Problem:

$$\max_{x \geq 0} \mathbb{E}_P[\pi(x, D)] \tag{10}$$

whose unique solution is:

$$x^* = \frac{p}{hm} \tag{11}$$

where $m = \mathbb{E}_P[1/D]$.

The settings in eqs. (10)-(5) apply to an inventory system which is represented as composed of one item. When the inventory is disaggregated into its various items, let $i = 1, 2, ..., N$ denote the types of goods (items). Then, p, a, h, D become N-dimensional vectors ($\mathbf{p}$, $\mathbf{a}$, $\mathbf{h}$, $\mathbf{D} \in (\mathbb{R}^+)^N$.) There follows that

$$\Theta_D = \Theta_{D_1} \times \Theta_{D_2} \times ... \times \Theta_{D_N} \subseteq (\mathbb{R}^+)^N \tag{12}$$

and one needs to characterize an N-variate random demand[3]. We let $F = P(\mathbf{D} < \mathbf{d})$ the corresponding joint probability function. As far as cost (profit) aggregation is concerned, maintaining the same hypothesis as in Assumption 2.2 of Gotoh and Takano (2006)[4], *i.e.* value additivity, the total inventory cost or profit is the sum of the costs or profits deriving from the individual products in the inventory [eq. (7)]. In the case of the LPC model one gets the total profit function:

$$\pi(\mathbf{x}, \mathbf{D}) = \sum_{i=1}^{N} \left(p_i x_i - a_i - \frac{h_i x_i^2}{2D_i} \right) \tag{13}$$

As far as the concavity of eq. (13) is concerned, by direct differentiation or by noting that eq. (13) is the sum of convex functions, one obtains that:

Remark 5 *The profit function of eq. (13) is concave* $\forall \mathbf{D}$ *and the corresponding loss function*

$$\mathcal{L}(\mathbf{x}, \mathbf{D}) = -\pi(\mathbf{x}, \mathbf{D}) \tag{14}$$

is convex $\forall \mathbf{D}$.

We have now all the tools needed to discuss the optimization problems that are faced by risk-neutral, quadratic utility, mean-absolute and CVaR risk averse decision makers for multi-item inventories with loss function given by the extended LPC model [eq. (13).]

3.1 Risk Neutral Decision Maker

We start with a risk neutral decision maker. One can state the following:

Proposition 1 *Given the LPC profit function [eq. (13)], a risk neutral decision maker solves the following Problem:*

$$\mathcal{P}_{risk-neutral} = \left\{ \max_{\mathbf{x} \geqq \mathbf{0}} \sum_{i=1}^{N} \left(p_i x_i - a_i - \frac{m_i h_i x_i^2}{2} \right) \right. \tag{15}$$

where

$$m_i := \mathbb{E}_F \left[\frac{1}{D_i} \right] \tag{16}$$

Proof. A risk neutral decision maker maximizes the expected value of profit [Problem (10)]. We have:

$$\begin{aligned} \mathbb{E}_F \left[\pi(\mathbf{x}, \mathbf{D}) \right] = \mathbb{E}_F \left[\textstyle\sum_{i=1}^{N} \left(p_i x_i - a_i - \frac{h_i x_i^2}{2D_i} \right) \right] = \left[\textstyle\sum_{i=1}^{N} \mathbb{E}_F \left(p_i x_i - a_i - \frac{h_i x_i^2}{2D_i} \right) \right] = \\ = \textstyle\sum_{i=1}^{N} \left(p_i x_i - a_i - \frac{h_i x_i^2}{2} \mathbb{E}_F \left[\frac{1}{D_i} \right] \right) \end{aligned} \tag{17}$$

[3]Gotoh and Takano (2006) discuss the solution of a multi-item newsvendor problem with N goods and a discrete joint mass function.

[4]The assumption of value additivity, albeit standard in literature, is hiding the possibility of interactions (discounts, sinergies) among inventory items. It is utilized in this work as the construction of a non-addivite multi item function is out of the scope of the present paper and it is the subject of future work of the authors.

■

We note that, by the linearity of the expectation operator or by direct differentiation, $\mathbb{E}_F[\pi(\mathbf{x}, D)]$ is strictly concave. This implies that a unique maximum (minimum for $\mathcal{L}(\mathbf{x}, D)$) exists [Takayama (1993), Ch. 2.], which is given by:

$$\mathbf{x}^* = \left[\frac{1}{h_1 m_1} p_1, \frac{1}{h_2 m_2} p_2, ..., \frac{1}{h_N m_N} p_N\right] \tag{18}$$

One can note that $\mathbf{x}^*$ is determined by holding costs ($\mathbf{h}$), revenue per unit of inventoried items $\mathbf{p}$, and by $\mathbf{m}$, which is a stochastic property of demand and, in general, depends on the choice of the demand distribution. In this respect, the following holds.

Remark 6 *The optimal risk-neutral inventory policy is invariant if two distributions F_1 and F_2 are such that:*

$$\mathbb{E}_{F_1}[\pi(\mathbf{x}, \mathbf{D})] = \mathbb{E}_{F_2}[\pi(\mathbf{x}, \mathbf{D})] \tag{19}$$

i.e., *risk neutral decision makers are indifferent among demand distributions that lead to the same* $\mathbf{m}$*.*

As far as the computation of $\mathbf{m}$ is concerned, we note that a simplification is possible if stochastic independence is assumed. In fact, under the independence assumption it holds that $F = \prod_{i=1}^{N} F_i$ and $dF = \prod_{i=1}^{N} dF_i$. Then the m_i's can be computed for each item separately[5] and $m_i = \int_{\Theta_{D_i}} \frac{1}{D_i} dF_i$.

We now turn to the problem of determining the optimal policy for a risk averse decision maker selecting variance as a risk measure.

3.2 Risk Averse Decision Maker: Variance as a Risk Measure

For a risk averse decision maker with a quadratic utility function, the dispersion measure is variance [in this respect, see Ogryczak and Ruszczynski (1999).] The Problem of the optimal inventory policy selection becomes:

$$\mathcal{P}'_{\mathbb{V}} = \left\{ \min_{\mathbf{x} \in S} \mathbb{V}_F[\pi(\mathbf{x}, \mathbf{D})] \right. \tag{20}$$

In order to discuss the conditions under which Problem $\mathcal{P}'_{\mathbb{V}}$ is a convex one, we derive the explicit expression of the objective function for a quadratic utility inventory manager when the cost function is the multi-item LPC one [eq. (14).]

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[5] In fact: $m_i = \int_{\Theta_D} \frac{1}{D_i} dF = \int_{\Theta_{D_1} \times \Theta_{D_2} \times \ldots \times \Theta_{D_N}} \frac{1}{D_i} \prod_{l=1}^{N} dF_l = \int_{\Theta_{D_i}} \left(\frac{1}{D_i} dF_i\right) \left(\prod_{l=1, l \neq i}^{N} \int_{\Theta_{D_l}} dF_l \right)$
$= \int_{\Theta_{D_i}} \frac{1}{D_i} dF_i$, since $\left(\prod_{l=1, l \neq i}^{N} \int_{\Theta_{D_l}} dF_l \right) = 1$.

Proposition 2 *Let*

$$M = [m_{i,j} : m_{i,j} = \mathbb{C}ov_F \left(D_i^{-1}, D_j^{-1} \right) \tag{21}$$

be the covariance matrix of inverse item demands. Given the LPC profit function [eq. (13),] a quadratic risk averse the decision maker solves the following Problem ($\mathcal{P}_{\mathbb{V}}$):

$$\mathcal{P}_{\mathbb{V}} = \left\{ \min_{\mathbf{x} \in S} \tfrac{1}{4} \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} h_i h_j x_i^2 x_j^2 m_{i,j} \right. \tag{22}$$

Proof. See Appendix A. ■

Concerning matrix M [eq. (21),] the following properties hold:

Remark 7 1. *M is symmetric.*

2. *If stochastic independence holds, then M is diagonal and*

$$m_{i,i} = \int_{\Theta_{D_i}} D_i^{-2} dF_i - m_i^2 \tag{23}$$

From a technical point of view, clearly $\mathbb{V}_F[\pi(\mathbf{x}, \mathbf{D})] \in C^{\infty}(\mathbb{R}^N)$, and is increasing in $\mathbf{x}$. As far as its convexity is concerned, we note that letting $x_i^2 = y_i$, one can write $\mathbb{V}_F[\pi(\mathbf{y}, \mathbf{D})] = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{h_i h_j y_j y_i}{4} \mathbb{C}ov[D_i^{-1}, D_j^{-1}]$, *i.e.* $\mathbb{V}_F[\pi(\mathbf{y}, \mathbf{D})]$ is a quadratic form in $\mathbf{y}$. Since h_i,$h_j > 0$ the concavity (convexity) of $\mathbb{V}_F[\pi(\mathbf{y})]$ is determined by the covariance matrix, M. If $\mathbb{C}ov[D_i^{-1}, D_j^{-1}]$ is such that $\mathbb{V}_F[\pi(\mathbf{x}, \mathbf{D})]$ is convex, Problem $\mathcal{P}_{\mathbb{V}}$ becomes a convex program[6]. Some significant simplification is achieved in the case of stochastic independence, as the above Remark 7 implies that the definiteness of M is determined only by the sign of the diagonal elements and so is the convexity of $\mathbb{V}_F[\pi(\mathbf{x}, \mathbf{D})]$.

In terms of imprecision about the demand distribution, the following observation holds:

Remark 8 *A quadratic risk averse decision maker is indifferent among demand distributions that lead to the same M,* i.e., *the optimal quadratic risk-averse inventory policy is invariant if two distributions F_1 and F_2 are such that:*

$$\mathbb{C}ov_{F_1} \left(D_i^{-1}, D_j^{-1} \right) = \mathbb{C}ov_{F_2} \left(D_i^{-1}, D_j^{-1} \right) \tag{24}$$

Remark 8 can be compared to Remark 6. Remark 8 implies that quadratic risk averse decision makers are interested in the covariance matrix of the inverse demand distributions, while risk neutral are interested only in their expected values.

We now deal with the Problem solved by a mean-absolute risk averse decision maker.

[6] From an analytical point of view, one can adopt a Kuhn-Tucker formulation, as the non-negativity constraint $\mathbf{x} \geq \mathbf{0}$ holds due to economic considerations [Takayama (1993).]

3.3 Risk Coherent Decision Maker: the case of Mean-Absolute Deviation

We start this Section with a result that assures the convexity of the optimization problem for a decision maker selecting any coherent risk measure in the presence of the LCP loss function. Let $\rho[Z]$ denote the generic coherent risk measure. Then, a risk-coherent decision maker solves the Problem:

$$\mathcal{P}_\rho = \left\{ \min_{\mathbf{x}\in S} \rho[\mathcal{L}(\mathbf{x},\mathbf{D})] \right. \tag{25}$$

The following result holds.

Remark 9 *Given $\mathcal{L}(\mathbf{x},\mathbf{D})$ of eq. (14), $\mathcal{P}_\rho$ [25] is a convex program for any coherent risk measure.*

We now focus on a decision maker selecting the MAD measure [eq.(5)]. Problem (25) is written as:

$$\mathcal{P}'_{MAD} = \left\{ \min_{\mathbf{x}\in S} \mathbb{E}_F[\mathcal{L}(\mathbf{x},\mathbf{D})] + \gamma \left\{ \mathbb{E}_F[|\mathcal{L}(\mathbf{x},\mathbf{D}) - \mathbb{E}_F[\mathcal{L}(\mathbf{x},\mathbf{D})]|] \right\} \right. \tag{26}$$

We derive the explicit expression of $\mathcal{P}'_{MAD}$ in the following Proposition.

Proposition 3 *A decision maker with mean-absolute risk aversion and LPC profit function [eq.(13)] solves the following Problem:*

$$\mathcal{P}_{MAD} = \left\{ \min_{\mathbf{x}\in S} -\mathbf{px} + \mathbf{1a} + \sum_{i=1}^N \frac{h_i x_i^2}{2} \left[m_i + \gamma(m_i^+ + m_i^-) \right] \right. \tag{27}$$

where

$$\begin{aligned} m_i^+ &= \int_{\Theta_D^+} (\mathbb{E}_F[\tfrac{1}{D_i}] - \tfrac{1}{D_i}) dF \\ m_i^- &= \int_{\Theta_D^-} (\tfrac{1}{D_i} - \mathbb{E}_F[\tfrac{1}{D_i}]) dF \end{aligned} \tag{28}$$

Proof. See Appendix B. ■

About the calculation of $m_i^+ + m_i^-$, we note that:

Remark 10

$$m_i^+ + m_i^- = m_i[2F(\Theta_D^+) - 1] + \int_{\Theta_D^-} \frac{1}{D_i} dF - \int_{\Theta_D^+} \frac{1}{D_i} dF \tag{29}$$

where $F(\Theta_D^+)$ is the measure of Θ_D^+, i.e., *the probability that $\mathbf{D} \in \Theta_D^+$.*
Proof. *Proof:*

$$\begin{aligned} &\int_{\Theta_D^+} (\mathbb{E}_F[\tfrac{1}{D_i}] - \tfrac{1}{D_i}) dF + \int_{\Theta_D^-} (\tfrac{1}{D_i} - \mathbb{E}_F[\tfrac{1}{D_i}]) dF = \\ &\int_{\Theta_D^+} (\mathbb{E}_F[\tfrac{1}{D_i}] - \tfrac{1}{D_i}) \prod_{i=1}^N dF_i + \int_{\Theta_D^-} (\tfrac{1}{D_i} - \mathbb{E}_F[\tfrac{1}{D_i}]) \prod_{i=1}^N dF_i = \\ &= m_i \left(\int_{\Theta_D^+} dF \right) - \int_{\Theta_D^+} \tfrac{1}{D_i} dF + \int_{\Theta_D^-} \tfrac{1}{D_i} dF - m_i \int_{\Theta_D^-} dF = \\ &m_i F(\Theta_D^+) - m_i F(\Theta_D^-) + \int_{\Theta_D^-} \tfrac{1}{D_i} dF - \int_{\Theta_D^+} \tfrac{1}{D_i} dF = \\ &= m_i [F(\Theta_D^+) - F(\Theta_D^-)] + \int_{\Theta_D^-} \tfrac{1}{D_i} dF - \int_{\Theta_D^+} \tfrac{1}{D_i} dF \end{aligned} \tag{30}$$

Now, it holds that: $F(\Theta_D^+) + F(\Theta_D^-) = 1$and substituting $F(\Theta_D^+) = 1 - F(\Theta_D^-)$ completes the proof. ■

As far as the effect of the demand distribution is concerned, the following holds:

Remark 11 *Mean-absolute risk averse policies are invariant for distributions leading to the same* $\mathbf{m}$ *and* $\mathbf{m}^+ + \mathbf{m}^-$.

The above Remark can be seen also as stating that, to find the optimal policy, a mean-absolute risk averse decision maker needs to determine the quantities $\mathbf{m}, \mathbf{m}^+, \mathbf{m}^-$ – besides the economic parameters, of course.

3.4 Risk Coherent Decision Maker: the case of CVaR

In this Section, we formulate the optimization problem for a decision-maker adopting CVaR as a coherent measure of risk (see Section 2.) Following Rockafellar and Uryasev (2002) [see also Gotoh and Takano (2006),] we introduce the auxiliary function (see also Table 1 for notation):

$$H_\alpha(\mathbf{x}, \zeta) = \zeta + \frac{1}{1-\alpha}\mathbb{E}_F\left\{[\mathcal{L}(\mathbf{x}, \mathbf{D}) - \zeta]^+\right\} \tag{31}$$

where

$$[\mathcal{L}(\mathbf{x}, \mathbf{D}) - \zeta]^+ = \max[\mathcal{L}(\mathbf{x}, \mathbf{D}) - \zeta, 0] \tag{32}$$

Theorem 10 in Rockafellar and Uryasev (2002) then assures that the following holds:

$$\zeta_\alpha(\mathbf{x}) = \arg\min_\zeta H_\alpha(\mathbf{x}, \zeta) \ \text{ and } \ \phi_\alpha(\mathbf{x}) = H_\alpha(\mathbf{x}, \zeta_\alpha(\mathbf{x})) \tag{33}$$

One can write the optimization problem in one step (see also Rockafellar and Uryasev (2002), Gotoh and Takano (2006)) as:

$$\mathcal{P}'_{CVaR_\alpha} = \left\{\min_{\mathbf{x},\zeta \in S\times\mathbb{R}} H_\alpha(\mathbf{x}, \zeta)\right\} \tag{34}$$

In Appendix C, we show that the following holds.

Proposition 4 *A decision maker utilizing the LPC loss function and adopting CVaR as a coherent risk measure solves the problem:*

$$\mathcal{P}_{CVaR_\alpha} = \left\{\min_{\mathbf{x},\zeta\in S\times\mathbb{R}} \zeta + \frac{1}{1-\alpha}\left[(\mathbf{1a} - \mathbf{px} - \zeta)\, F(\Theta_D^{\zeta+}) + \sum_{i=1}^{N} \frac{h_i x_i^2 m_i^{\zeta+}}{2}\right]\right\} \tag{35}$$

where

$$\begin{aligned} F(\Theta_D^{\zeta+}) &= \textstyle\int_{\Theta_D^{\zeta+}} dF \\ m_i^{\zeta+} &:= \textstyle\int_{\Theta_D^{\zeta+}} \frac{1}{D_i} dF \end{aligned} \tag{36}$$

Note that $F(\Theta_D^{\zeta+})$ being the measure of $\Theta_D^{\zeta+}$ is the probability that $D \in \Theta_D^{\zeta+}$, *i.e.*, the probability of incurring a loss greater than ζ.

As far as imprecision in the demand distribution is concerned, the following holds:

Remark 12 *A CVaR decision maker is indifferent among demand distributions that lead to the* $F(\Theta_D^{\zeta+})$ *and* $m_i^{\zeta+}$ *[eq. (36).]*

The next Section compares the numerical results that are obtained by decision makers that solve the four problems presented in this Section.

4 Numerical Discussion and the Effect of Demand Distribution Imprecision

In this section, we analyze and compare structure, numerical results and sensitivity to the choice of the demand distribution of the four problems discussed in Sections 3.1, 3.2, 3.3 and 3.4.

We utilize as a constraint the fact that the decision maker wishes to minimize risk while maintaining a certain expected profit, *i.e.,* in problems $\mathcal{P}_{\mathbb{V}}$, $\mathcal{P}_{MAD}$ and $\mathcal{P}_{CVaR}$, we set S:

$$S = \{\mathbf{x} : \mathbb{E}_F[\pi(\mathbf{x}, \mathbf{D})] \geq \overline{\pi} \text{ and } \mathbf{x} \geq \mathbf{0}\} \tag{37}$$

We have seen in the above section that $\mathbb{E}_F[\pi(\mathbf{x}, \mathbf{D})]$ is concave. This assures that S is convex [Theorem 1.8 in Takayama (1993).] Thus, the global nature of the solution to $\mathcal{P}_{\mathbb{V}}$, $\mathcal{P}_{MAD}$ and $\mathcal{P}_{CVaR}$ is preserved.

The first question we are going to answer by means of our numerical analyses is the sensitivity of the optimal policy to the choice of the risk measure[7]. For this discussion, we suppose that the decision maker assumes that each item's demand can vary between its lowest value $D_i^{\inf}$ and its highest value, $D_i^{\sup}$ (see Table 1 for the notation and Table 2 for the numerical values) according to a Beta density, *i.e.*, She/He lets:

$$\begin{cases} \Theta_D^{Beta} = (D_1^{\inf}, D_1^{\sup}) \times (D_2^{\inf}, D_2^{\sup}) \times ... \times (D_N^{\inf}, D_N^{\sup}) \\ f_{Beta}(\mathbf{D}; \mathbf{r}, \mathbf{q}, \mathbf{D}^{\inf}, \mathbf{D}^{\sup}) = \prod_{i=1}^{N} \frac{1}{\int_0^1 y_i^{r-1}(1-y_i)^{q-1} dy_i} \cdot \frac{(D_i - D_i^{\inf})^{r-1}(D_i^{\sup} - D_i)^{q-1}}{(D_i^{\sup} - D_i^{\inf})^{r+q+1}} \end{cases} \tag{38}$$

where $\mathbf{r}$ and $\mathbf{q}$ are shape factors (Table 1.) Given this density, P^{Beta} is such that the demand among items are independent. Thus, the covariance matrix M is diagonal (Remark 7.)

We utilize an inventory made of $N = 10$ items and the demand distribution numerical assumptions reported in Table 2.

[*** Table 2 is about here ****]

The Economic parameters of the inventory Problems are reported in Table 3.

[*** Table 3 is about here ****]

The optimal policy of the four decision makers are reported in Table 4.

[*** Table 4 is about here ****]

[7] We utilize $\overline{\pi} = 5000$ in this illustration.

The unconstrained expected profit equals $1.1E{+}4$ and the unconstrained variance $2.7E{+}7$. The total number of items ordered is $\sum_{i=1}^{10} x_i^{*RiskNeutral} \simeq 1850$. A quadratic utility risk-averse inventory manager, would adhere to the optimal policy reported in Table 4, third row. The value of the minimized variance is $\mathbb{V}^*_{Beta}[\pi(\mathbf{x},\mathbf{D})] = 9.0E+4$, with a sensible reduction w.r.t. the risk neutral case. The total number of purchased items is $\sum_{i=1}^{10} x_i^{*Quadratic} = 464$. The same reduction effect can be seen in Figure 1, which shows the sensitivity of the optimal inventory policy for a mean-absolute risk averse decision maker with γ increasing from 0.2 to 0.5: the number of purchased items decreases linearly with γ for each type of good.

[*** Figure 1 is about here ****]

The optimal policy obtained with $\gamma = 0.5$ is reported in the fourth row of Table 4. The total number of ordered items is $\sum_{i=1}^{10} x_i^{*MAD} = 1429$. A CVaR decision maker would adopt the policy in the fifth row of Table 4 with $\sum_{i=1}^{10} x_i^{*CVAR} = 804$. These results lead to the following first observation: risk aversion causes the decision maker to reduce the total number of ordered items in order to incur in lower losses [8]. More precisely, we have obtained that $\sum_{i=1}^{10} x_i^{*Quadratic} < \sum_{i=1}^{10} x_i^{*CVAR} < \sum_{i=1}^{10} x_i^{*MAD} < \sum_{i=1}^{10} x_i^{*RiskNeutral}$. Such an ordering does not concern only the total number of ordered items, but Table 4 shows that it is maintained for individual items, *i.e.*, $x_i^{*Quadratic} < x_i^{*CVAR} < x_i^{*MAD} < x_i^{*RiskNeutral}$, $i = 1, 2, ..., 10$. This leads us to analyze the similarity/discrepancies in the structure of the policies. To do so, we make use of an approach based on the technique of Savage Scores[9]. The method requires to first transform the raw figures of the ordered items ($\mathbf{x}$) into their ranks and then convert the ranks into Savage Scores ($\boldsymbol{\xi}$ - Table 1.) The resemblance among the policies is then synthesized by the correlation coefficient on the scores. In our case, results are as follows. The correlation coefficient between $\boldsymbol{\xi}^{*CVaR}$ and $\boldsymbol{\xi}^{*Quadratic}$is unity, as indeed the two policies have indeed the same structure. Similarly, the correlation coefficient between $\boldsymbol{\xi}^{*MAD}$ and $\boldsymbol{\xi}^{*RiskNeutral}$ is 1. The correlation coefficient between $\boldsymbol{\xi}^{*MAD}/\boldsymbol{\xi}^{*RiskNeutral}$ and $\boldsymbol{\xi}^{*Quadratic}/\boldsymbol{\xi}^{*CVaR}$ is equal to 0.98, signalling a very high structural agreement among the policies. More in detail, item 10 is the most ordered across all types of risk measure choices, followed by items 8, 2, 1, 4. Items 5, 9 and 3 are the least ordered for all policies. The only shift happens between item 6 and 7, which rank 6^{th} and 8^{th} respectively for variance and CVaR decision makers, while they rank eight and sixth for the risk neutral and MAD decision makers.

Finally, we analyze the effect of imprecision in the demand distribution. We suppose that the decision maker wishes to shift from a finite (Beta) to an infinite support distribution

[8] Technically, this is consistent with the form of the loss function as $\mathcal{L}(\mathbf{x},\mathbf{D})$ is decreasing in $\mathbf{D}$ (see eq. (14).)

[9] Savage Scores have been introduced in Statistcs as a measure of agreement among rankings by Iman and Conover, 1987. For their mathematical definition and the illustration of their application we also refer to Campolongo and Saltelli (1997.) Borgonovo and Peccati (2005) and Borgonovo (2006.)

(Gamma):

$$\begin{cases} \Theta_D^{Gamma} = (D_1^{\inf}, \infty) \times (D_2^{\inf}, \infty) \times ... \times (D_N^{\inf}, \infty) \\ f_{Gamma}(\mathbf{D}; \mathbf{D}^{\inf}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{i=1}^{N} \frac{\beta_i^{\alpha_i}(D_i - D_i^{\inf})^{\alpha_i - 1} e^{-\beta_i (D_i - D_i^{\inf})}}{\Gamma(\alpha_i)} \end{cases} \quad (39)$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are the parameters of the Gamma densities (Table 1.) Before proceeding, we recall Remarks 6, 8, 11, and 12 that identify the conditions under which an optimal policy is insensitive to distribution changes. To implement the findings of these Remarks, we select the values of $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ so that the Gamma densities [eq. (39)] lead to the same $\mathbf{m}$ and $\mathbf{m}^+ + \mathbf{m}^-$ obtained with the Beta densities (eq. (38 and Table 2.) The matching equations are solved numerically and produce the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ reported in Table 2. It can be seen that, although such a choice does not preserve the expected value and variance of the individual item demands, the optimal policies of a risk neutral and a MAD decision maker do not change. However, the policies of a quadratic and a CVaR decision makers are affected by the change, as the choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ does not preserve the moments in Remarks 11, and 12 (sixth and seventh rows of Table 4.)

5 Conclusions

In this work, we proposed an approach to quantify the differences among inventory policies determined by alternative risk aversion attitudes. We have dealt with the determination of inventory policies in the presence of alternative risk measures for a multi item inventory system described by the (extended) LCP loss function. We have considered the policies of risk neutral decision makers, and of three risk averse inventory managers selecting variance, MAD and CVaR as risk measures.

The optimization problem for each decision maker has been derived. This has enabled us to observe that decision makers choosing any coherent risk measure always solve a convex program, thus being assured of the global nature of the optimal policy. The same happens to a risk neutral decision maker. However, the nature of the optimization problem for decision maker selecting variance depends on the stochastic properties of demand.

We have then studied the economic and stochastic determinants of the inventory policies to identify what information is relevant to the different types of managers in order to come to an inventory management decision. Findings show that per unit revenues and variable costs concur in the determination of all four policies. However, while a risk neutral decision makers would not need to measure fixed costs to identify the optimal policy, such knowledge is required to all risk averse decision makers. We have identified the stochastic properties needed to come to an optimal policy vary across the different risk measure selections. This has also enabled us to state conclusions about the sensitivity of the policies to imprecision in the demand distribution. More in detail: : *i)* risk neutral policies require the knowledge of $\mathbf{m}$ (Table 1) and do not change in the presence of distributions leading to the same $\mathbf{m}$ (Table 1); *ii)* quadratic risk averse policies require the knowledge of M (Table 1) and are unaltered by distributions leading to the same M (Table 1;) *iii)* MAD policies require the knowledge of $\mathbf{m}$, $\mathbf{m}^+ + \mathbf{m}^-$ and are invariant for distributions leading to the same $\mathbf{m}$, $\mathbf{m}^+ + \mathbf{m}^-$(Table 1;) and finally *iv*) CVaR policies require the knowledge of $F(\Theta_D^{\zeta+})$ and $\mathbf{m}^{\zeta+}$ (Table 1) and are unaffected if imprecision in the distributions leads to the same $F(\Theta_D^{\zeta+})$ and $\mathbf{m}^{\zeta+}$.

We have finally addressed the quantitative comparison of optimal policies induced on the same system by different risk measures. The approach to compare the structure of inventory policies has been based on the statistical technique of Savage Scores (Iman and Conover, 1987.) Results for the 10 item inventory system analyzed in this exercise are as follows. Risk aversion leads to a reduction in the optimal order quantity, with MAD decision makers ordering less than risk neutral ones but more than CVaR, and quadratic utility decision makers ordering the lowest number of items. However, risk aversion has not altered the structure of the policies in a significant way. Namely, items that were ordered the most by a risk neutral decision maker have remained the most ordered also across the three risk averse policies. The same has happened to the least ordered items.

This work opens future research directions by the authors. The first direction is the study of optimal inventory policies in the presence of a non-additive loss function, so as to evidence the effects of synergies and discounts. The second direction is the implementation of the methodology in the reverse direction, to infer the risk measure which the decision maker is selecting from the actually chosen policy in the context of a case study.

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6 Appendix A: Proof of Proposition 2

Proof. From eq. (13), one gets:

$$\mathbb{V}_F[\pi(\mathbf{x},\mathbf{D})] = \mathbb{E}_F\left[\left\{\sum_{i=1}^{N}\left(p_i x_i - a_i - \frac{h_i x_i^2}{2D_i}\right) - \sum_{i=1}^{N}\left(p_i x_i - a_i - \frac{m_i h_i x_i^2}{2}\right)\right\}^2\right] \tag{40}$$

Now, we have that:

$$\sum_{i=1}^{N}\left(p_i x_i - a_i - \frac{h_i x_i^2}{2D_i}\right) - \sum_{i=1}^{N} p_i x_i - a_i - \frac{m_i h_i x_i^2}{2} = \sum_{i=1}^{N}\left(-\frac{h_i x_i^2}{2D_i} + \frac{m_i h_i x_i^2}{2}\right) = \sum_{i=1}^{N}\frac{x_i^2 h_i}{2}\left(m_i - \frac{1}{D_i}\right) \tag{41}$$

and therefore

$$\mathbb{V}_F[\pi(\mathbf{x},\mathbf{D})] = \mathbb{E}_F\left[\left\{\sum_{i=1}^{N}\frac{x_i^2 h_i}{2}\left(m_i - \frac{1}{D_i}\right)\right\}^2\right] \tag{42}$$

Then, since it is true that:

$$\left\{\sum_{i=1}^{N}\frac{x_i^2 h_i}{2}\left(m_i - \frac{1}{D_i}\right)\right\}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{h_i h_j x_j^2 x_i^2}{4}\left(m_i - \frac{1}{D_i}\right)\left(m_j - \frac{1}{D_j}\right) \tag{43}$$

by the linearity of the expectation operator, one derives:

$$\mathbb{V}_F[\pi(\mathbf{x},\mathbf{D})] = \mathbb{E}_F\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{h_i h_j x_j^2 x_i^2}{4}\left(m_i - \frac{1}{D_i}\right)\left(m_j - \frac{1}{D_j}\right)\right] \tag{44}$$

which equals

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{h_i h_j x_j^2 x_i^2}{4}\mathbb{E}_F\left[\left(\frac{1}{D_i} - \mathbb{E}_F\left[\frac{1}{D_i}\right]\right)\left(\frac{1}{D_j} - \mathbb{E}_F\left[\frac{1}{D_j}\right]\right)\right] \tag{45}$$

But $\mathbb{E}_F\left[\left(\frac{1}{D_i} - \frac{1}{\mathbb{E}_F[D_i]}\right)\left(\frac{1}{D_j} - \frac{1}{\mathbb{E}_F[D_j]}\right)\right]$ is nothing but the covariance between the random variables D_i^{-1} and D_j^{-1}. Thus:

$$\mathbb{V}_F[\pi(\mathbf{x},\mathbf{D})] = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{h_i h_j x_j^2 x_i^2}{4}\mathbb{C}ov[D_i^{-1}, D_j^{-1}]$$

q.e.d.. ■

7 Appendix B: Proof of Proposition 3

Proof. We have:

$$\mathbb{E}_F[\mathcal{L}(\mathbf{x},\mathbf{D})]+\gamma\left\{\mathbb{E}_F[|\mathcal{L}(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\mathcal{L}(\mathbf{x},\mathbf{D})]|]\right\}=-\left[\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]+\gamma\left\{\mathbb{E}_F[|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|]\right\}\right]$$

$$=-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]-\gamma\left\{\mathbb{E}_F[|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|]\right\}$$

Now, $|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|$ is equivalent to:

$$|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|=\begin{cases}\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]\,if\;\pi(\mathbf{x},\mathbf{D})>\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]\\ \mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]-\pi(\mathbf{x},\mathbf{D})\,if\;\pi(\mathbf{x},\mathbf{D})\le\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]\end{cases}\tag{46}$$

The condition $\pi(\mathbf{x},\mathbf{D})>\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]$ partitions the space Θ_D into two regions, $\Theta_D^+=\{D:\pi(\mathbf{x},\mathbf{D})>\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]\}$ and $\Theta_D^-=\Theta_D\backslash\Theta_D^+$ or $\Theta_D^-=\{D:\pi(\mathbf{x},\mathbf{D})<\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]\}$. Hence, one can write:

$$\mathbb{E}_F[|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|]=\int_{\Theta_D}|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|\,dF=\tag{47}$$

$$=\int_{\Theta_D^+}|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|\,dF+\int_{\Theta_D^-}|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|\,dF=\tag{48}$$

As shown in Appendix A, it holds that

$$\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]=\sum_{i=1}^{N}\frac{h_ix_i^2}{2}(\mathbb{E}_F[\frac{1}{D_i}]-\frac{1}{D_i})\tag{49}$$

Therefore:

$$\begin{gathered}\mathbb{E}_F[|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|]=\\ =\int_{\Theta_D^+}\left\{\sum_{i=1}^{N}\frac{h_ix_i^2}{2}(\mathbb{E}_F[\frac{1}{D_i}]-\frac{1}{D_i})\right\}dF+\int_{\Theta_D^-}\left\{\sum_{i=1}^{N}\frac{h_ix_i^2}{2}(\frac{1}{D_i}-\mathbb{E}_F[\frac{1}{D_i}])\right\}dF=\\ =\left\{\sum_{i=1}^{N}\frac{h_ix_i^2}{2}\int_{\Theta_D^+}(\mathbb{E}_F[\frac{1}{D_i}]-\frac{1}{D_i})dF\right\}+\left\{\sum_{i=1}^{N}\frac{h_ix_i^2}{2}\int_{\Theta_D^-}(\frac{1}{D_i}-\mathbb{E}_F[\frac{1}{D_i}])dF\right\}=\end{gathered}\tag{50}$$

Letting $\int_{\Theta_D^+}(\mathbb{E}_F[\frac{1}{D_i}]-\frac{1}{D_i})dF=m_i^+$ and $\int_{\Theta_D^-}(\frac{1}{D_i}-\mathbb{E}_F[\frac{1}{D_i}])dF=m_i^-$, one can write:

$$\mathbb{E}_F[|\pi(\mathbf{x},\mathbf{D})-\mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]|]=\sum_{i=1}^{N}\frac{h_ix_i^2(m_i^++m_i^-)}{2}\tag{51}$$

Substituting eqs. (51) and (17) into eq. (26) completes the proof. ■

8 Appendix C: proof of Proposition 4

Proof. In Problem 34, let us study

$$\mathbb{E}_F\left\{[\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta]^+\right\}=\mathbb{E}_F\left[\max(\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta,0)\right] \tag{52}$$

We have:

$$\max[\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta,0]=\begin{cases}\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta & if\; \mathcal{L}(\mathbf{x},\mathbf{D})-\zeta\geq 0\\ 0 & if\; \mathcal{L}(\mathbf{x},\mathbf{D})-\zeta<0\end{cases} \tag{53}$$

Thus, similarly to the MAD case, one can see Θ_D partitioned into two regions which, now, depend on ζ. We call the regions $\Theta_D^{\zeta+}=\{D:\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta\geq 0\}$ and $\Theta_D^{\zeta-}=\Theta_D\backslash\Theta_D^{\zeta+}$ or $\Theta_D^{\zeta-}=\{D:\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta<0\}$. We then write:

$$\mathbb{E}_F[\max[\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta,0]]=\int_{\Theta_D^{\zeta+}}[\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta]\,dF+\int_{\Theta_D^{\zeta-}}0dF=\int_{\Theta_D^{\zeta+}}\mathcal{L}(\mathbf{x},\mathbf{D})dF-\int_{\Theta_D^{\zeta+}}\zeta dF \tag{54}$$

Let us study the first integral

$$\int_{\Theta_D^{\zeta+}}\mathcal{L}(\mathbf{x},\mathbf{D})dF=\int_{\Theta_D^{\zeta+}}\sum_{i=1}^{N}\left(a_i+\frac{h_ix_i^2}{2D_i}-p_ix_i\right)dF=\int_{\Theta_D^{\zeta+}}\sum_{i=1}^{N}a_i-\sum_{i=1}^{N}p_ix_i+\sum_{i=1}^{N}\frac{h_ix_i^2}{2D_i}dF= \tag{55}$$

$$=\left(\sum_{i=1}^{N}a_i-\sum_{i=1}^{N}p_ix_i\right)F(\Theta_D^{\zeta+})+\sum_{i=1}^{N}\frac{h_ix_i^2m_\zeta^+}{2} \tag{56}$$

where $F(\Theta_D^{\zeta+})=\int_{\Theta_D^{\zeta+}}dF$ is the measure of $\Theta_D^{\zeta+}$, *i.e.*, the probability that $D\in\Theta_D^{\zeta+}$ and $m_i^{\zeta+}:=\int_{\Theta_D^{\zeta+}}\frac{1}{D_i}dF$. The second integral is:

$$\int_{\Theta_D^{\zeta+}}\zeta dF=\zeta F(\Theta_D^{\zeta+}) \tag{57}$$

Summarizing:

$$\mathbb{E}_F[\max[\mathcal{L}(\mathbf{x},\mathbf{D})-\zeta,0]]=\left(\sum_{i=1}^{N}a_i-\sum_{i=1}^{N}p_ix_i-\zeta\right)F(\Theta_D^{\zeta+})+\sum_{i=1}^{N}\frac{h_ix_i^2m_i^{\zeta+}}{2} \tag{58}$$

Inserting eq. (58) into eq.(34) completes the proof. ■

Table 1: List of Symbols.

Symbol	Definition	Meaning
$(\Theta_D,\mathcal{B}(\Theta_D),P)$		Demand Measure Space
$\mathbf{D}$	$\{D_1, D_2, ..., D_N\}$	Random Demand
$\mathbf{x}$	$\{x_1, x_2, ..., x_N\}$	LCP profit function
$\pi(\mathbf{x},\mathbf{D})$	eq. (13)	LCP profit function
$\mathcal{L}(\mathbf{x},\mathbf{D})$	eq. (14)	LCP loss function
$\mathbf{a}$	$\{a_1, a_2, ..., a_N\}$	Unit fixed costs per inventoried item
$\mathbf{p}$	$\{p_1, p_2, ..., p_N\}$	Unit revenues per inventoried item
$\mathbf{h}$	$\{h_1, h_2, ..., h_N\}$	Unit variable costs per inventoried item
$\mathbf{m}$	eq. (16)	Expected values of of $1/D_i$
M	eq. (21)	Covariance matrix of $1/D_i$
$\mathbf{m}^+$	eq. (28)	Coefficient of x_i^2 in MAD
$\mathbf{m}^-$	eq. (28)	Coefficient of x_i^2 in MAD
γ	eq. (28)	MAD parameter
Θ_D^+	eq. (29)	$\Theta_D^+ = \{D : \pi(\mathbf{x},\mathbf{D}) > \mathbb{E}_F[\pi(\mathbf{x},\mathbf{D})]\}$
$\zeta_\alpha(\mathbf{x})$	eq. (6)	α-VaR at $\mathbf{x}$
$\phi_\alpha(\mathbf{x})$	eq. (6)	α-CVaR at $\mathbf{x}$
$H_\alpha(\mathbf{x},\zeta)$	eq. (31)	Auxiliary Function for CVaR minimization
$m_i^{\zeta+}$	eq. (36)	Coefficient of x_i^2 in CVaR
$F(\Theta_D^{\zeta+})$	eq. (36)	Probability that $D \in \Theta_D^{\zeta+}$
$\Theta_D^{\zeta+}$	eq. (36)	$\Theta_D^{\zeta+} = \{D : \mathcal{L}(\mathbf{x},\mathbf{D}) - \zeta \geq 0\}$
$\overline{\pi}$	eq. (37)	Minimum Profit in Profit Constraint
$\mathbf{r}, \mathbf{q}, \mathbf{D}^{\text{inf}}, \mathbf{D}^{\text{sup}}$	eq. (38)	Parameters of the Beta Distributions
$\mathbf{D}^{\text{inf}}, \boldsymbol{\alpha}, \boldsymbol{\beta}$	eq. (39)	Parameters of the Gamma Distributions
$\boldsymbol{\xi}$	Section 4	Symbol for Savage Scores

Table 2: Stochastic inputs and properties for the case of the Beta distribution.

Item	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{D}^{\text{inf}}$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$\mathbf{D}^{\text{sup}}$	40	30	25	28	27	26	31	42	35	32
$\mathbf{r}$	2	2	2	2	2	2	2	2	2	2
$\mathbf{q}$	3	2	3	2	3	2	3	2	3	2
$\boldsymbol{\alpha}$	2.5	2.7	2.5	2.7	2.5	2.7	2.5	2.7	2.5	2.7
$\boldsymbol{\beta}$	0.15	0.17	0.24	0.18	0.22	0.20	0.19	0.12	0.16	0.16

Table 3: Economics inputs for the optimization problems.

Item	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
p	10	11	12.5	13	12	9.5	14	13.5	12.5	15
a	1	2	2.5	1.5	1.8	2.2	2.3	4.1	1.9	2.7
h	.55	.6	.65	.71	.53	.56	.68	.81	.92	.5

Table 4: Optimal policies.

Optimal Policy | **Item**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{x}_{Beta}^{*RiskNeutral} = \mathbf{x}_{Gamma}^{*RiskNeutral}$	188	189	126	176	160	152	166	239	123	329
$\mathbf{x}_{Beta}^{*Quadratic}$	48	49	36	45	43	44	42	53	35	66
$\mathbf{x}_{Beta}^{*MAD(\gamma=0.5)} = \mathbf{x}_{Gamma}^{*MAD(\gamma=0.5)}$	144	147	97	137	123	118	127	185	95	256
$\mathbf{x}_{Beta}^{*CVaR}$	83	87	61	78	74	75	71	97	60	118
$\mathbf{x}_{Gamma}^{*Quadratic}$	48	51	35	46	42	46	40	55	34	67
$\mathbf{x}_{Gamma}^{*CVaR}$	97	98	70	90	85	87	79	112	68	140

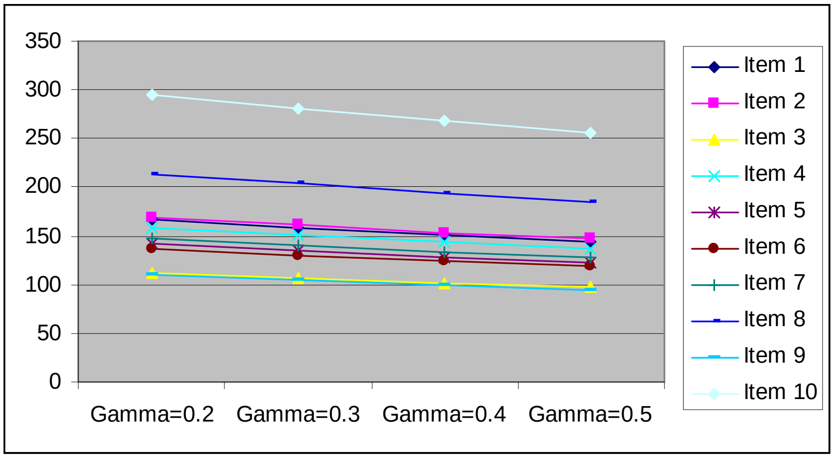

Figure 1: Sensitivity of the optimal order policy to the risk-aversion constant γ.