Differential Importance and Comparative Statics: an Application to Inventory Management

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Abstract

In this work, we deal with the problem of establishing which parameters impact an inventory policy the most. A new sensitivity measure (Γ) is introduced by relating the differential importance (D) and the comparative statics (CS) techniques. We discuss the properties of the new indicator, and show that it shares the additivity property. We provide the expression of Γ for optimization models. Numerical results are offered for the sensitivity analysis of a recently introduced inventory management model.

Keywords: Comparative Statics, Sensitivity Analysis, Inventory Management, Importance Measures, Economic Order Quantity.

1 Introduction

In the recent past, the field of sensitivity analysis (SA) of model output has been steadily growing and has assisted to the development of several new SA methods capable of studying *how the variation in the output of a model can be apportioned to variations in the input*[1] [Tarantola (2000), [31]]. These methods are traditionally called importance measures ([5], [23], [24].)

Saltelli *et al* (2000) [24] demonstrates that the application of these SA methods benefits both the modelling process and the utilization of model results. However, the use of importance measures in inventory management (IM) has not been fully explored yet. Applications of "what if" SA schemes can be found in the works of Ray and Sahu (1992), in Ray and Chaudhuri (1997), in Arcelus and Rowcroft (1993) ([21], [22], [1]). In these works, the sensitivity of the model results is tested for individual changes in the parameters. Ganeshan *et al* (2001) study the sensitivity of supply chain performance to three inventory parameters [11]. Perturbation analysis has been developed and employed in the works of Glasserman and Tayur (1995) [12], Bogataj and Cibej (1994) [3], and Bogataj and Bogataj (2004) [4]. As far as importance measures are concerned, a first application in IM is the one in Borgonovo and Peccati (2006) [8] that introduces global SA methods to deal with uncertainty in parameters of IM models.

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[1] In some sensitivity analysis applications the term input is used as a synonym for parameter.

The purpose of this paper is to establish which of the parameters influences an inventory policy the most given the IM model at hand. This task presents several new features. In fact, all importance measures ([23], [5]), have been defined for models of the explicit form:

$$x = h(\alpha) \quad h : A \subseteq \mathbb{R}^n \longrightarrow X \subseteq \mathbb{R}^m \tag{1}$$

where $x = (x_1, x_2, ..., x_m)$ denote the set of choice variables and α=$(\alpha_1, \alpha_2, ..., \alpha_n) \in A$ the parameters. In many IM models, the choice variable x is the result of an optimization process and the model takes on the more general form:

$$f(x, \alpha) = 0 \quad f : X \times A \to Y \subseteq \mathbb{R}^m \text{ with } X \subseteq \mathbb{R}^m,\ A \subseteq \mathbb{R}^n \tag{2}$$

where $f(x; \alpha) = [f^1(x; \alpha), f^2(x; \alpha), ..., f^m(x; \alpha)]^T$ is the set of model equations.

From eq. (2) is not always possible to extract the analytical expression of $x = h(\alpha)$[2].

Thus, the first part of this paper deals with the extension of the definition of parameter importance from explicit [eq. (1)] to implicit models [eq. (2)]. We show that to come to this definition it is necessary to link the comparative statics (CS) technique and the differential importance measure (D). Result of the analysis is the introduction of a generalized importance indicator, denoted by Γ.

We illustrate the mathematical properties of Γ, and show that it shares the additivity property[3].

We next discuss the application of Γ to IM models. As inventory policies are often determined as the solution to an optimization problem (minimization of a loss function or maximization of a utility function) we derive the expression of Γ for generic optimization models.

We then apply CS and Γ to an IM model proposed by Luciano and Peccati (1999) [18] (LP model from now on). The model output is the modified EOQ that takes into account financing policies. CS results show that an increase in demand and order costs lead to an increase in the modified EOQ, while an increase in the cost of capital, and in the unit price of goods in inventories decreases the modified EOQ. However, we illustrate that no indication can be inferred on the importance of parameters from CS results. In fact, since parameters have different units, partial derivatives cannot be compared. Thus, making use of CS results to infer parameter relevance, one would draw misleading conclusions. Instead, the application of Γ enables one to estimate the importance of the parameters and to identify the relevant and non-relevant ones. Numerical results show that demand, cost of capital and order costs have the same influence on the modified EOQ, while holding costs have a negligible influence. Next, the additivity property of Γ allows to assess the joint effect of changes in the parameters in a straightforward way. Results indicate that a contemporary proportional increase in demand and cost of capital hedge each other away, leaving the modified EOQ unchanged.

In Section 2 the relevant SA background on CS and D is discussed. Section 3 introduces the definition of Γ, and proves its main properties. Section 4 illustrates the derivation of importance measures for optimization problems. Section 5 provides an application and discusses numerical results. Conclusions are offered in Section 6.

[2] As an example, in the Arrow-Harris-Marshak [2] models, the modified EOQ of the static model is an explicit function of the parameters, while the (s, S) policy of the dynamic model is defined by an equation which, in general, does not give rise to an explicit solution.

[3] By additivity we refer to the ability of a sensitivity technique to obtain the importance of groups of parameters as sum of the results of the one-parameter-at-a-time SA. This property enables one to obtain joint sensitivity results without having to perform further model runs and, thus, it is desirable in many applications (Borgonovo and Apostolakis, 2001 [5]).

2 Sensitivity Analysis background

2.1 The Methodology of Comparative Statics: a brief Literature Survey and Introduction to the Mathematical Formulation

The methodology of Comparative Statics (CS) has been fully developed by Samuelson in his seminal work of 1947 [25]. CS has since then provided as the basic methodology for the sensitivity analysis in Economics and its importance in the economic literature cannot be overemphasized. A comprehensive literature review is out of the scope of this paper. We limit ourselves to a brief summary, starting with the applications in Microeconomics by Samuelson Himself that leads to the Duality Principle ([26]), and by Silberberg in his works on the theory of the firm ([27], [28]). Following Samuelson's statement: "*it is hoped to formulate qualitative restrictions on slopes, curvatures etc. of our equilibrium, so as to be able to derive definite qualitative restrictions upon the response of our system to changes in certain parameters*" ([25] p. 20) the most recent developments in CS have focused on the use of CS for predicting the direction of changes in equilibria provoked by parameter changes. The works of Quirk, Yamada, Jeffries and other have established the qualitative CS methodology (see [20] for a comprehensive literature review). The works of Milgrom and Shannon (1994) [19] and Topkis (1995) [32] have made monotone CS a cornerstone in modern Economics.

For the purpose of this work, *i.e.*, to quantify the importance of parameters in IM models, we deal with CS in the form presented in [9], and [30], in the line of the classical approach of Samuelson's Foundations ([25]). Starting point of the problem is the set of implicit equations of the form of eq. (2). Then, we let

$$J_x(x,\alpha)=\left[f^j_{x_i}(x,\alpha)\right] \;\; i,j=1,2,...,m \tag{3}$$

denote the (Jacobian) matrix of the partial derivatives of $f(x,\alpha)$ w.r.t. x (the output) and

$$J_\alpha(x,\alpha)=\left[f^j_{\alpha_s}(x,\alpha)\right] \;\; j=1,2,...,m, \; s=1,2,...,n \tag{4}$$

denote the matrix of the partial derivatives of $f(x;\alpha)$ w.r.t. α (the parameters.) We also let

$$J^s_x(x,\alpha) \text{ and } J^s_\alpha(x,\alpha) \tag{5}$$

denote the s^{th} column vector of $J_x(x,\alpha)$ and $J_\alpha(x,\alpha)$ respectively. The following vectors are also going to be used:

$$d_x J(x,\alpha) = J_x(x,\alpha)dx \tag{6}$$

$$d_\alpha J(x,\alpha) = J_\alpha(x,\alpha)d\alpha \tag{7}$$

The usual regularity assumptions on $f(x;\alpha)$ and the condition $|J_x(x^*,\alpha^*)| \neq 0$ — where x^*,α^* satisfy eq. (2) — assure that the implicit function theorem holds and that the system of eqs. (2) defines the set of continuously differentiable functions $x^*=h(\alpha^*)$ on an interval around (x^*,α^*) ([30]). CS aims at quantifying the rate of change in the output x^* that is caused by a change in the parameters α^*. The change is obtained differentiating both sides of eq.(2) ([9], [30]):

$$d_x J(x^*,\alpha^*) + d_\alpha J(x^*,\alpha^*)= \mathbf{0} \tag{8}$$

Eq. (8) can be solved for matrix

$$\frac{dx}{d\alpha}\Big|_{(x^*,\alpha^*)} = -J_x(x^*,\alpha^*)^{-1}J_\alpha(x^*,\alpha^*) \tag{9}$$

whose components are the partial derivative of the output w.r.t. the parameters. Eqs. (2), (8) and (9) set forth the mathematical framework for CS.

2.2 The Differential Importance Measure

As mentioned in the Introduction, in order to measure the importance of parameters in IM models it is necessary to extend the definition of importance measures from explicit to implicit models. We discuss such extension in Section 3. In the remainder of this Section, we briefly review the definition and properties of the differential importance measure (D), since its link to the CS technique enables us to derive the importance of parameters in implicit models.

Given a model in the form of eq. (1), the differential importance (D) for parameters α_s is given by [5]:

$$D_s(\alpha^0, d\alpha) = \frac{h_s(\alpha^0)d\alpha_s}{\sum_{j=1}^n h_j(\alpha^0)d\alpha_j} = \frac{d_s h(\alpha^0)}{dh(\alpha^0)} \tag{10}$$

The requirements under which D_s is defined are that $h(\alpha)$ is differentiable at α^0 and $d\alpha = [d\alpha_1, d\alpha_2, ..., d\alpha_n]$ is not orthogonal to the gradient of h at α^0.

It is worth recalling that:

- D shares the additivity property with respect to the various parameters: the D of some set of parameters is given by the sum of the individual D of the parameters in that set ([5], [6], [7]). As a consequence the sum of all the individual parameter D ($i = 1...n$) is always equal to unity ([5], [6], [7]).

- D accounts for the way parameters are varied through the dependence on $d\alpha$ ([5], [6], [7]). In particular:

  - in the hypothesis of uniform parameter changes (H1):

  $$H1 : d\alpha_s = d\alpha_l \quad \forall s, l \tag{11}$$

  one finds:

  $$D1_s(\alpha^0, d\alpha) = \frac{h_s(\alpha^0)}{\sum_{j=1}^n h_j(\alpha^0)} \tag{12}$$

  Eq. (12) implies that $D1_s(\alpha^0, d\alpha) \propto h_s(\alpha^0)$, *i.e.*, partial derivatives assign the same relevance to parameters as D under the assumption that all parameters variations are equal [eq.(11)];

  - in the hypothesis of proportional changes (H2),

  $$H2 : \frac{d\alpha_s}{\alpha_s^0} = \frac{1}{\omega} \quad \forall s \tag{13}$$

  one finds:

  $$D2_s(\alpha^0) = \frac{h_s(\alpha^0) \cdot \alpha_s^0}{\sum_{j=1}^n h_j(\alpha^0) \cdot \alpha_j^0} \tag{14}$$

  Recalling the definition of Elasticity (E), eq. (14) shows that $D2_s(\alpha^0, d\alpha) \propto E_s(\alpha^0)$, *i.e.*, E assigns the same relevance to parameters as D under the assumption of proportional parameter changes.

2.3 A Remark

We can summarize the discussion in Sections 2.1 and 2.2 as follows. The framework of CS is more general than the one set forth in Section 2.2 for D, in the sense that CS deals with implicit models, that contain explicit models as a particular case. D and other importance measures have been, up to now, defined for explicit models.

The purpose of CS and D is also different. In fact, through CS one mainly aims at assessing whether a change in a parameter (α_s) provokes an increase or a decrease in the output (x^*) — direction of change, Section 2.1. — Through D one quantifies the importance of the parameters (α) and ranks them based on their influence on the output (x^*). D then indicates what parameters matter the most w.r.t. the final choice.

Consider the case of a model whose parameters carry different units of measure. CS allows the evaluation of the partial derivatives, but their comparison does not lead information on the parameter importance. In fact, partial derivatives corresponding to parameters with different units have different units and cannot be compared. Thus, one cannot utilize CS results, in general, to determine the importance of parameters. Technically, this is due to the fact that the uniform changes assumption [eq. (11)] cannot hold. As mentioned, in Section 2.2, not only the rate of change is to be taken into consideration, but also the relative way in which parameters are varied ([6]). Thus, one needs to integrate the results of CS with D to acquire information on the importance of parameters. The link between CS and D is discussed in the next Section.

3 Parameter Importance in Implicit Models

In this Section, we define the importance of parameters in implicit models by making use of the relationship between CS and D.

We start with introducing the following Definition:

Definition 1 *Let*

$$\Gamma(x^*, \alpha^*) = \left[\gamma_{j,s}\right] \;\; j = 1...m, \; s = 1...n \tag{15}$$

be the matrix whose components represent the differential importance of parameter α_s $(s = 1...n)$ w.r.t. output x_j $(j = 1...m)$.

We then prove the following proposition.

Proposition 2

$$\Gamma(x^*, \alpha^*) = \left[\gamma_{j,s} : \gamma_{j,s} = \frac{|\Phi^{js}|}{|\Phi_j|}\right] \tag{16}$$

where

$$\begin{array}{c} \Phi^{js} = \left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \, J_\alpha^s d\alpha_s \, J_x^{j+1} \, ... \, J_x^m\right] \\ \Phi_j = \left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \, dJ_\alpha \, J_x^{j+1} \, ... \, J_x^m\right] \end{array} \tag{17}$$

In $\gamma_{j,s}$ [eq. (16)] the denominator, $|\Phi^j|$, is the total differential of output x_j w.r.t. the parameters and, as we show in the following proof, it is obtained by substituting vector dJ_α [eq. (7)] for the j^{th} column of matrix J_x [eq. (3)]. The numerator of $\gamma_{j,s}$ [eq. (16)], $|\Phi^{js}|$, represents the portion of the differential relating to the sole parameter α_s. Note that in eq. (16) the condition$|\Phi_j(x^*, \alpha^*)| \neq 0$ substitutes the CS condition $|J_x(x^*, \alpha^*)| \neq 0$ discussed in Section 2.1.

The proof of the previous proposition can expressed in two alternative ways. We present here the proof based on the differential [4].

Proof. According to Definition 1, applying eq. (10), we have:

$$\gamma_{j,s} = \frac{d_s x^j}{dx^j} \tag{18}$$

We need to compute dx^j and dx_s^1. Solving eq.(8) for dx^j leads to (Cramer's rule):

$$dx^j = -\frac{|\Phi_j|}{|J_x(x^*, \alpha^*)|} \tag{19}$$

dx^j represents the change in dx^j due to a change in all the parameters. The sensitivity of x^1 on parameter α_s alone is given by:

$$dx_s^j = -\frac{|\Phi^{sj}|}{|J_x(x^*, \alpha^*)|} \tag{20}$$

Substituting eqs. (20) and (19) into 18 one obtains eq. (16), q.e.d. ■

We now show that $\Gamma(x^*, \alpha^*)$ shares the additivity property w.r.t. the parameters.

Proposition 3 *Let $S = \{s_1, s_2, ..., s_k\}$, $k \leq n$, be the indices of a subset of the parameters. Then:*

$$\gamma_{j,S} = \gamma_{j,s_1} + \gamma_{j,s_2} + ... + \gamma_{j,s_k} \tag{21}$$

Proof. The effect of a change in S is given by:

$$dx_S^j = \frac{|\Phi^{jS}|}{|J_x(x^*, \alpha^*)|} \tag{22}$$

where now

$$\Phi^{jS} = \left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \sum_{i \in S} J_\alpha^i d\alpha_i \, J_x^{j+1} \, ... \, J_x^m \right] \tag{23}$$

Hence:

$$\gamma_{j,S} = \frac{|\Phi^{jS}|}{|\Phi_j|} = \frac{\left| \left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \sum_{s \in S} J_\alpha^s d\alpha_s \, J_x^{j+1} \, ... \, J_x^m \right] \right|}{\left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \, dJ_\alpha \, J_x^{j+1} \, ... \, J_x^m \right]} \tag{24}$$

But thanks to the properties of determinants (property nr. 26.6, p.729 in [29]), one gets:

$$\left| \left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \sum_{s \in S} J_\alpha^s d\alpha_s \, J_x^{j+1} \, ... \, J_x^m \right] \right| = \sum_{s \in S} \left| \left[J_x^1 \, J_x^2 \, ... \, J_x^{j-1} \, J_\alpha^s d\alpha_s \, J_x^{j+1} \, ... \, J_x^m \right] \right| \tag{25}$$

Hence:

$$\gamma_{j,S} = \frac{|\Phi^{jS}|}{|\Phi_j|} = \frac{\sum_{s \in S} |\Phi^{js}|}{|\Phi_j|} = \sum_{s \in S} \frac{|\Phi^{js}|}{|\Phi_j|} = \sum_{s \in S} \gamma_{j,s} \tag{26}$$

q.e.d ■

Note that the sum of the rows of matrix $\Gamma(x^*, \alpha^*)$ is always equal to unity.

We now discuss the effect of relative parameter changes. To do so we prove the following Proposition.

[4] A second way of proving the results is based on computing the partial derivatives first.

Proposition 4 *If the parameters undergo a uniform change (case H1), then*

$$\Gamma(x^*, \alpha^*) = \Gamma 1(x^*, \alpha^*) \tag{27}$$

with

$$\Gamma 1(x^*, \alpha^*) = \left[\gamma 1_{j,s} : \gamma 1_{j,s} = \frac{\left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^s \, J_x^{j+1} \ldots J_x^m \right] \right|}{\sum_{i=1}^n \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^i \, J_x^{j+1} \ldots J_x^m \right] \right|} \right] \tag{28}$$

If proportional parameter changes are considered, then, (case H2):

$$\Gamma(x^*, \alpha^*) = \Gamma 2(x^*, \alpha^*) \tag{29}$$

with

$$\Gamma 2(x^*, \alpha^*) = \left[\gamma 2_{j,s} : \gamma 2_{j,s} = \frac{\left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^s \, J_x^{j+1} \ldots J_x^m \right] \right| \alpha_s^*}{\sum_{i=1}^n \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^i \, J_x^{j+1} \ldots J_x^m \right] \right| \alpha_i^*} \right] \tag{30}$$

Proof. Using the properties of determinants, we can write:

$$\gamma_{j,s} = \frac{|\Phi^{js}|}{|\Phi_j|} = \frac{|\Phi^{js}|}{\sum_{i=1}^n |\Phi^{ji}|} \tag{31}$$

Thanks to the determinant property relating to column multiplication by scalars (fact 26.4, p. 728 in [29],) one can write:

$$\left|\Phi^{js}\right| = \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^s \, J_x^{j+1} \ldots J_x^m \right] \right| d\alpha_s \tag{32}$$

i.e., it is possible to "extract" the differential $d\alpha_s$ from the j^{th} column. Utilizing properties nr. 26.6 and nr. 26. 4 in [29] one can write:

$$|\Phi_j| = \sum_{i=1}^n \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^i d\alpha_i \, J_x^{j+1} \ldots J_x^m \right] \right| d\alpha_i \tag{33}$$

and $\gamma_{j,s}$ becomes:

$$\gamma_{j,s} = \frac{|\Phi^{js}|}{|\Phi_j|} = \frac{\left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^s \, J_x^{j+1} \ldots J_x^m \right] \right| d\alpha_s}{\sum_{i=1}^n \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^i \, J_x^{j+1} \ldots J_x^m \right] \right| d\alpha_i} \tag{34}$$

In the case of uniform parameter changes eq. (11) holds and one finds:

$$\gamma 1_{j,s} = \frac{\left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^s \, J_x^{j+1} \ldots J_x^m \right] \right|}{\sum_{i=1}^n \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^i \, J_x^{j+1} \ldots J_x^m \right] \right|} \tag{35}$$

For proportional parameter changes, eq. (13) holds and one finds:

$$\gamma 2_{j,s} = \frac{\left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^s \, J_x^{j+1} \ldots J_x^m \right] \right| \alpha_s^*}{\sum_{i=1}^n \left| \left[J_x^1 \, J_x^2 \ldots J_x^{j-1} \, J_\alpha^i \, J_x^{j+1} \ldots J_x^m \right] \right| \alpha_i^*} \tag{36}$$

q.e.d. ■

Some observations:

Eqs. (16), (28) and (30) display that the way in which parameters are varied influences their impact on the output. For uniform and proportional changes, eqs. (28) and (30) apply. If the change under investigation is generic, then eq. (16) applies;

Each element in $\Gamma 1(x^*, \alpha^*)$ [$\gamma 1_{j,s}$, eq. (28)] is the partial derivative of output x_j w.r.t. parameter α_s divided by the sum of all the partial derivatives. Hence, ranking parameters utilizing $\Gamma 1(x^*, \alpha^*)$ is equivalent to rank them according to their partial derivatives. This statement can also be read in the opposite direction: ranking parameters based on their partial derivatives is equivalent to stating the assumption that the parameters undergo a uniform change (see also [5], [6], [7].) Since partial derivatives are an output of CS, one can see that CS results can be utilized to infer parameter importance only if one assumes a uniform perturbation of the parameters;

Each element in $\Gamma 2(x^*, \alpha^*)$ [$\gamma 2_{j,s}$, eq. (30)] can be regarded, after a simple manipulation, as the elasticity of output x_j w.r.t. parameter α_s normalized by the sum of all elasticities. As we mentioned, ranking parameters based on their elasticities is equivalent to stating the assumption that the parameters variations are proportional to their values [eq. (13)].

4 Parameter Importance ($\Gamma(x^*, \alpha^*)$) in Optimization Models

The purpose of this Section is to specialize the definition of Γ to optimization models. The reason is that most often the choice variables determining optimal inventory policies — EOQ, or one of its modified forms, (S, s) or others depending on the IM model — are the result of the maximization of a utility function or minimization of an expected loss or cost function ([2], [17], [22], [18], [33]).

We denote such Loss/Utility function as $\mathcal{L}(x, \alpha)$ and consider an optimization problem of the form:

$$\begin{cases} \max\limits_{x} \mathcal{L}(x, \alpha) \\ \quad s.t. \\ g(x, \alpha) \geq 0 \end{cases} \tag{37}$$

with $g(x; \alpha) = (g^1(x; \alpha), g^2(x; \alpha), ..., g^c(x; \alpha))^{\mathrm{T}}$, where c is the number of constraints, with $c < m$. Letting

$$\Lambda(x, \lambda, \alpha) = \mathcal{L}(x, \alpha) + \lambda g(x, \alpha) \tag{38}$$

denote the Lagrangian function of the problem, and considering $x > 0$ and $\lambda > 0$, the FOCs for this problem [eq. (37)] are [30]:

$$\begin{cases} \Lambda_x(x, \lambda, \alpha) = 0 \\ \Lambda_\lambda(x, \lambda, \alpha) = 0 \end{cases} \tag{39}$$

where $\Lambda_x(x, \lambda, \alpha) = \mathcal{L}_x(x, \alpha) - \lambda g_x(x, \alpha)$, and $\Lambda_\lambda(x, \lambda, \alpha) = g(x, \alpha)$. The implicit differentiation of eq. (39) leads to:

$$\begin{cases} \Lambda_{x\lambda} d\lambda + \Lambda_{xx} dx + \Lambda_{x\alpha} d\alpha = 0 \\ \qquad \Lambda_{\lambda x} dx + \Lambda_{\lambda\alpha} d\alpha = 0 \end{cases} \tag{40}$$

From eq. (40), one finds the *fundamental equation of comparative statics* ([30], p. 128):

$$\begin{bmatrix} \lambda_{\alpha_s}(x^*, \alpha^*) \\ x_{\alpha_s}(x^*, \alpha^*) \end{bmatrix} = - \begin{bmatrix} 0 & g_x(x^*, \alpha^*) \\ g_x(x^*, \alpha^*)^T & \Lambda_{xx} \end{bmatrix}^{-1} \begin{bmatrix} \Lambda_{\lambda\alpha_s} \\ \Lambda_{x\alpha_s} \end{bmatrix} \tag{41}$$

where $\lambda_{\alpha_s}(x^*, \alpha^*)$ is the matrix of the derivatives of the Lagrangian multipliers w.r.t. parameter α_s and x_{α_s} is the matrix of the partial derivatives of the output w.r.t. α_s. Matrix

$\begin{bmatrix} 0 & g_x(x^*,\alpha^*) \\ g_x(x^*,\alpha^*)^T & \Lambda_{xx} \end{bmatrix}$ is the bordered Hessian (BH) of $\Lambda(x,\alpha)$. The non-singularity of BH is required for performing CS [30].

For the computation of the parameter importance, it is convenient to collect all the dependent variables in vector $z = (\lambda_1, \lambda_{2,...}, \lambda_c, x_1, x_{2,...}, x_m)$. Eq. (40) becomes:

$$\Lambda_{zz} dz + \Lambda_{z\alpha} d\alpha = 0 \tag{42}$$

where

$$\Lambda_{zz} = \begin{bmatrix} \Lambda_{x\lambda} \, \Lambda_{xx} \\ 0 \;\; \Lambda_{\lambda x} \end{bmatrix} \text{ and } \Lambda_{z\alpha} = \begin{bmatrix} \Lambda_{x\alpha} \\ \Lambda_{\lambda\alpha} \end{bmatrix} \tag{43}$$

Note that Λ_{zz} is an alternative way of writing the BH [30].

In order to find the importance of α w.r.t. x, *i.e.*, $\Gamma(x^*,\alpha^*)$, we apply Proposition 2 to the first order conditions of problem (37) in eq. (42). We get:

$$\Gamma(z^*,\alpha^*) = \left[\gamma_{j,s} : \gamma_{j,s} = \frac{\left| \Lambda^1_{zz} \, \Lambda^2_{zz} \dots \Lambda^{j-1}_{zz} \, \Lambda^s_{z\alpha} \, \Lambda^{j+1}_{zz} \dots \Lambda^{c+m}_{zz} \right| d\alpha_s}{\sum_{r=1}^n \left| \Lambda^1_{zz} \, \Lambda^2_{zz} \dots \Lambda^{j-1}_{zz} \, \Lambda^r_{z\alpha} \, \Lambda^{j+1}_{zz} \dots \Lambda^{c+m}_{zz} \right| d\alpha_r} \right] \tag{44}$$

with $s = 1, 2, ...n$ and $j = 1, 2, ...c + m$.

Note that the elements of the first c rows of $\Gamma(z^*,\alpha^*)$ in eq. (44) represent the importance of the parameters w.r.t. the Lagrangian multipliers. Elements $\gamma_{j,s}$ from row $c+1$ to $c+m$ represent the sought importance of the parameters w.r.t. the choice variables x.

Finally, in the determination of the parameter importance in unconstrained optimization, the Hessian of the loss function $\mathcal{L}_{xx}$ replaces the Bordered Hessian of the Lagrangian function (Λ_{zz}).

5 An application

With the purpose of illustrating the previous framework, we present the SA of the IM model proposed by Luciano and Peccati (1999) ("LP model") [18].

To cope with the economic, financial and managerial aspects of Inventory Management (IM) the Operations Research and Management Science literature has assisted to the development of several models and methods. Models have evolved from the early Harris work on the Economic Order Quantity of 1913 ([17], [10]), through the classical Arrow model [2], to the dynamic optimization approach by Vienott [33], up to the use of system dynamics and the analysis of the stochastic and stability properties of inventory systems ([15], [16].)

The purpose of the LP model is to allow the evaluation of IM policies while explicitly revealing the impact of financing choices [18], through an Adjusted Present Value (APV) approach ([13], [14].) We refer the reader to [18] for the complete illustration of the model.

The model estimates the modified EOQ as the quantity Q^* that minimizes the following cost (loss) function [18]:

$$\mathcal{L}(Q,\alpha) = \frac{\left(u + \frac{a}{2}\right) Q + \gamma}{1 - e^{-\rho Q/R}} \tag{45}$$

In the framework of eq. (2), the choice variable is Q and the vector of the parameters is

$$\alpha = \begin{cases} u = \text{unitary price of the good in inventory} \\ a = \text{unitary holding cost} \\ R = \text{constant consumption intensity} \\ \gamma = \text{the cost of placing one order} \\ \rho = \text{cost of capital} \end{cases}$$

Table 1: CS results for the case study

Parameter	CS	Expression	Sign	Magnitude	Unit
u	Q_u	$-\frac{R(e^{\frac{Q}{R}\rho}-1)-Q\rho}{\rho(e^{\frac{Q}{R}\rho}-1)(\frac{1}{2}a+u)}$	$-$	38	$\frac{\#\text{ items}}{\text{cost per item}}$
a	Q_a	$-\frac{\frac{1}{2}R(e^{\frac{Q}{R}\rho}-1)-\frac{1}{2}Q\rho}{\rho(e^{\frac{Q}{R}\rho}-1)(\frac{1}{2}a+u)}$	$-$	19	$\frac{\#\text{ items}}{\text{cost per item}}$
R	Q_R	$-\frac{\frac{1}{R}(\frac{1}{2}a+u)\left(R(e^{\frac{Q}{R}\rho}-1)-Q\rho e^{\frac{Q}{R}\rho}\right)}{\rho(e^{\frac{Q}{R}\rho}-1)(\frac{1}{2}a+u)}$	$+$	0.05	$\frac{\#\text{ items}}{\#\text{ items}}$
ρ	Q_ρ	$-\frac{Q(\frac{1}{2}a+u)(e^{\frac{Q}{R}\rho}-1)-\gamma}{\rho(e^{\frac{Q}{R}\rho}-1)(\frac{1}{2}a+u)}$	$-$	5770	# items
γ	Q_γ	$-\frac{-\rho}{\rho(e^{\frac{Q}{R}\rho}-1)(\frac{1}{2}a+u)}$	$+$	13	$\frac{\#\text{ items}}{\text{cost}}$

In the remainder, we utilize the following numerical assumptions: $u = 10$ [\$ per item], $a = 1$ [\$ per Item], $R = 8000$ [Item], $\gamma = 30$ [\$], $\rho = 7\%$.

Q^* is found form the FOC:

$$\mathcal{L}_Q(Q^*, \alpha) = \left(\frac{1}{2}a + u\right)\left(e^{\frac{Q^*}{R}\rho} - 1\right) R - \rho\left(\gamma + Q^*\left(\frac{1}{2}a + u\right)\right) = 0 \tag{46}$$

Eq. (46) cannot be solved to provide the analytical expression of $Q^*(\alpha)$. The optimal order quantity is then estimated numerically to be $Q^* \cong 807$.

Let us now analyze what information can be derived by the SA of the model performed through the joint application of CS [eq. (46)] and Γ [eq. (44)]. CS results are reported in Table 1.

We note that:

- an increase in u, a and ρ leads to a decrease in the EOQ;
- an increase in R and γ leads to an increase in the EOQ;
- ρ is the parameter associated with the highest rate of change in the EOQ (Table 1). From the results in Table 1, one could be induced to infer that ρ is the most relevant parameter — practically the only relevant one. However, note that the partial derivatives have different units and cannot be compared. Indeed, when parameters have different dimensions, the assumption of uniform changes (H1) cannot hold. More precisely, CS results cannot be utilized even in the case parameters have the same dimensions, but the direction of change is not the uniform one (see Section 2.2).

To evaluate the importance of the parameters w.r.t. the EOQ one needs to account for the relative parameter changes and utilize $\Gamma(x, \alpha)$ [see Definition 1, eq. (16)]. For the LP model in eq.(45), we have 1 choice variable, namely the modified EOQ, and 5 parameters. Hence, in this case:

$$\Gamma(Q^*, \alpha) = [\gamma_{j,s}] \; j = 1, \; s = 1, 2, ..., 5 \tag{47}$$

In the LP model, Q^* is determined through unconstrained optimization. Hence eq. (44) holds with the Bordered Hessian of the Lagrangian function coinciding with the Hessian of the loss function. Applying eq. (44), under H2 [eq. (13),] one finds the results reported

Table 2: Sensitivity Analysis results for Parameter Importance

Index	Parameter	Importance	Expression	Sign	Magnitude	Rank
1	u	$\Gamma 2_u(Q^*,\alpha^*)$	$\frac{\left[R(e^{\frac{Q}{R}\rho}-1)-Q\rho\right]u}{\sum_{j=1}^5 Q_j^*\alpha_j}$	$-$	1.10×10^3	4
2	a	$\Gamma 2_a(Q^*,\alpha^*)$	$\frac{\left[\frac{1}{2}R(e^{\frac{Q}{R}\rho}-1)-\frac{1}{2}Q\rho\right]a}{\sum_{j=1}^5 Q_j^*\alpha_j}$	$-$	5.6×10^1	5
3	R	$\Gamma 2_R(Q^*,\alpha^*)$	$\frac{(\frac{1}{2}a+u)\left(R(e^{\frac{Q}{R}\rho}-1)-Q\rho e^{\frac{Q}{R}\rho}\right)}{\sum_{j=1}^5 Q_j^*\alpha_j}$	$+$	1.183×10^3	1
4	ρ	$\Gamma 2_\rho(Q^*,\alpha^*)$	$\frac{\left[Q(\frac{1}{2}a+u)(e^{\frac{Q}{R}\rho}-1)-\gamma\right]\rho}{\sum_{j=1}^5 Q_j^*\alpha_j}$	$-$	1.182×10^3	2
5	γ	$\Gamma 2_\gamma(Q^*,\alpha^*)$	$\frac{-\rho\gamma}{\sum_{j=1}^5 Q_j^*\alpha_j}$	$+$	1.180×10^3	3

in Table 2. Columns 5 and 6 of Table 2 display the importance of the parameters and their rank according to $\Gamma 2$, respectively.

Note that $|\Gamma 2_R(Q^*,\alpha^*)| \simeq |\Gamma 2_\rho(Q^*,\alpha^*)| \simeq |\Gamma 2_\gamma(Q^*,\alpha^*)|$. This means that an increase in R, ρ or γ proportional to their values would impact the modified EOQ in (practically) the same way[5]. Figure **??** also shows that a is by far the less relevant parameter, while u is slightly less significant than R, ρ, γ. The information an analyst would infer from these results is that a change in the modified EOQ provoked by a change in a is negligible w.r.t. the change in the modified EOQ that is provoked by a change in each one of R, ρ, or γ.

The utilization of the additivity property of Γ also leads to the following result: a contemporary proportional increase in ρ and R, or in γ and R would leave the modified EOQ unchanged. In fact, let us apply Proposition 3. Let $S=(\rho,R)$. Then, we have:

$$\Gamma 2_S(Q^*,\alpha^*) = \Gamma 2_\rho(Q^*,\alpha^*) + \Gamma 2_R(Q^*,\alpha^*) \simeq 0 \tag{48}$$

signalling that a change in ρ neutralizes a change in R w.r.t. the modified EOQ.

Similarly, for $S=(\gamma,R)$

$$\Gamma 2_S(Q^*,\alpha^*) = \Gamma 2_\gamma(Q^*,\alpha^*) + \Gamma 2_R(Q^*,\alpha^*) \tag{49}$$

signalling that if order costs and demand grow by the same proportion, the modified EOQ would not change. Borrowing from Finance terminology, proportional changes in ρ and/or γ are locally hedged by proportional changes in R.

6 Conclusions

The purpose of this work has been the determination of which of the parameters influences an IM model result the most.

To achieve this goal, we have linked the differential importance measure (D) to the technique of Comparative Statics (CS). The link has been necessary since: *i)* D and importance measures utilized in the literature have not been defined for implicit models, and, *ii)* the CS technique is not designed to infer the importance of parameters.

The analysis has lead to the introduction of a generalized importance indicator (Γ) for implicit models. We have discussed the mathematical framework of the new indicator, proven its properties, with particular reference to additivity.

[5]Note that ρ is no more the only infuential parameter.

We have obtained the expression of Γ for optimization problems, as inventory policies are often determined as a result of an optimization process (maximization of a utility/profit function or minimization of a utility/loss function.)

We have illustrated the application of Γ and CS in the SA of an IM model proposed by Luciano and Peccati (1999). The application of CS has enabled to understand the rate of change of the modified EOQ w.r.t. changes in the parameters. However, since the model parameters have different units, CS results could not be utilized to determine the importance of the parameters. This task has been made possible by the application of the new importance indicator, Γ, that has allowed the identification of the influential and non-influential variables. Furthermore, we have assessed the joint effect of contemporary changes in the parameters by making use of the additivity property of Γ. The analysis has evidenced that changes in the modified EOQ provoked by changes in demand can be neutralized by contemporary proportional changes in order costs or in the cost of capital, leading to a natural hedging/management strategy for a decision maker willing to maintain a constant modified EOQ.

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