Global Sensitivity Analysis in Inventory Management

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Abstract

This paper deals with the sensitivity analysis (SA) of inventory management models when uncertainty in the input parameters is given full consideration. We make use of Sobol' function and variance decomposition method for determining the most influential parameters on the model output. We first illustrate the method by means of an analytical example. We provide the expression of the global importance of demand, holding costs, order costs of the Harris EOQ formula. We then present the global SA of the inventory management model developed by Luciano and Peccati (1999) for the economic order quantity estimation in the context of the temporary sale problem. We show that by performing global SA in parallel to the modeling process an analyst derives insights not only on the EOQ structure when its expression is not analytically known, but also on the relevance of modeling choices, as the inclusion of financing policies and special orders.

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1 Introduction

Uncertainty in inventory policy making stems from a variety of factors. Just as a simple example, consider a firm that uses the Harris Economic Order Quantity (EOQ) formula as a support to its inventory policies ([18], [24], [30]). In order to come to a final decision on the EOQ, the firm must estimate demand, unit order costs and holding costs. Demand

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is seldom steady, and its value cannot be determined with certainty in most of the cases ([2], [35], [23], [47], [27], [15]). Costs can be a further source of uncertainty (see [30]): on the one hand the criteria of cost classification are not always be sharply set and, on the other hand, even once the criteria are set, variability characterizes the costs themselves [30]. Hence, rarely one can predict the behavior of an inventory system with the inputs fixed at a certain value; more likely, the decision maker will be able to assign parameters within ranges determined by the analysis ([30], [7], [8]). To cope with the corresponding uncertainty in model predictions, usually a sensitivity analysis (SA) exercise is performed. The more direct SA scheme is the testing of the change in model output that follows a change in the parameters when they are shifted within the limits of their variation ranges: this type of one-variable-at-a-time SA is performed in Ray and Sahu (1992) [34], Arcelus and Rowcroft (1993) [3], Ray and Chaudhuri (1997) [35], Powell (2000) [29]. Ganeshan et al study the sensitivity of supply chain performance to three inventory parameters [19]. Perturbation analysis has been developed and employed in the works of Glasserman, Bogataj et al ([20], [5], [16], [6], [9]). The previous approaches belong to the family of local SA approaches. Local SA techniques are the set of methods that study the behavior of a mathematical model around a point of the input parameter space for finite or small changes in the input parameters. To the family of local SA methods [49] belong the technique of comparative statics [39] and the differential importance measure ([10], [12].) An SA method is Global if it tests the sensitivity of the model in consideration of the uncertainty distribution reflecting the decision-maker state of knowledge in the parameters. Global SA techniques can be divided in the categories of non-parametric techniques [36], screening methods [28], response surface methodology [17], and variance-based methods ([40], [42], [44]). Target of variance-based global SA methods is the model output variance, which is decomposed in a series of summands of increasing dimensionality ([40], [44], [42]). In this work, we focus on variance decomposition through the Sobol' ([42], [44]) and the Extended FAST methods [40].

Our first application is the determination of the global importance of the parameters in the classical Harris inventory management model ([24], [18]). We derive analytically Sobol' function and variance decomposition of the Harris EOQ formula and provide the expression of the input parameter global importance (GI). By means of numerical results, we illustrate that parameters associated with the highest value of GI, are the most effective in reducing the variance of the EOQ.

We then apply the techniques to the inventory management model introduced by Luciano and Peccati (1999) in the context of the temporary sale problem [26]. Starting point of the model is the loss function corresponding to the consideration of the cost of capital which is obtained by making use of the adjusted present value (APV) technique ([21], [22], [31], [32], [48]). Luciano and Peccati (1999) then formulate the cost functions that include third party financing, and the presence of special orders. None of the cost functions allows for the analytical expression of the EOQ. We show that performing a step-by-step global SA an analyst gains insights on both the modeling aspects and the EOQ structure. As far as modeling aspects are concerned, the use of global SA allows to ascertain whether the inclusion/exclusion of a certain assumption has a significant or negligible impact on the EOQ. As far as the EOQ structure is concerned, one gains a quantitative indication on the type of dependency of the EOQ on the parameters. This information would not be gained without a global SA when one does not possess the analytical expression of the EOQ. Results show that, for the uncertainty ranges at hand, the cost of debt, followed by the cost of capital and the special order discount are the most relevant parameters. Not only, but their inclusion impacts the EOQ structure in a significant way, shifting EOQ dependence on the parameters from additive to non-additive .

Section 2 describes the principles and theorems at the basis of global SA. Section 3 illustrates the application and provides analytical results for the Harris EOQ formula. Section 4 presents the global SA of the Luciano-Peccati model, comparing the different cost functions and the corresponding SA results. Conclusions are offered in Section 5.

2 Variance-Based Global Sensitivity Analysis

The purposes of performing a global SA of model output are many, as the works of Saltelli (1999) [38] and Saltelli at al (2000) [42] discuss. Two are the characteristics of global SA that we are going to exploit and discuss in this work:

the ability to enable the understanding of the type of model structural dependence on the input parameters when the explicit dependence is not available;

the ability to assess the influence of parameters, thus providing guidance in data collection and input estimation (see also Borgonovo and Peccati (2005) [14]).

We start with introducing the mathematical definitions and theorems at the basis of global SA based on variance decomposition [42].

Let $X \subseteq \mathbb{R}^n$ be the input parameter space, $\mathcal{B}(X)$ a Borel algebra on X, μ a probability measure on $\mathcal{B}(X)$.

Then, let

$$Y = f(\mathbf{x}), \qquad f : X \subseteq \mathbb{R}^n \to \mathbb{R} \tag{1}$$

denote the generic model output, where $\mathbf{x} = (x_1, x_2, ..., x_n) \in X$ is the "set of input parameters."

Target of the analysis is the model output variance, $V[Y]$, for which it is sought a decomposition of the type ([25], [44]):

$$V[Y] = \sum_{i=1}^{n} V_i + \sum_{i<j} V_{i,j} + \sum_{i<j<m} V_{i,j,m} + ... + V_{1,2,...n} \tag{2}$$

In eq. (2), V_i represents the contribution to the output variance provided for by parameter x_i individually. $V_{i,j}$ is the contribution to the variance of the interaction[1] between parameter x_i and x_j. $V_{i,j,m}$ is the contribution to the output variance of the interaction between parameters x_i, x_j and x_m. Finally, $V_{1,2,...n}$ is the residual portion of the variance that can only be explained as effect of the interaction among all the parameters. There follows that $\sum_{i=1}^{n} V_i$ represents the portion of the model variance explained by individual parameter contributions. $\sum_{i<j} V_{i,j}$ is the portion of $V[Y]$ explained by terms containing parameter pairs and so on. As highlighted in Sobol' (2001) and Rabitz and Alis (1999), eq. (2) generalizes ANOVA ([44], [45], [43], [33]).

The variance decomposition terms in eq. (2) can be computed from the function decomposition proven by Sobol' [44].

Assumption 1 $d\mu = \prod_{i=1}^{n} d\mu_i$, where μ_i is the measure of each parameter [Rabitz and Alis (1999) [33]].

Assumption 2 f is measurable [Sobol' (1993) [44]].[2]

[1] Rabitz and Alis (1999) also use the term "cooperation" as a synonym for "interaction."

[2] Assumption 2 is of a technical nature, insofar it is required to perform the integrations. Assumption 1, however, shares two interpretations. In Rabitz and Alis (1999) it is broadly underlined that the decomposition of a measurable function can be performed without adding to the integration operation a state-of-knowledge meaning. That is, suppose one is given a multivariate measurable function $f(\mathbf{x})$ and that $x_1, x_2, ..x_n$ vary in ranges $[-\infty \leq a_i, b_i \leq \infty]$ one can compute its decomposition by mechanically performing nested integrations as stated in the theorem. With this respect Sobol'/Rabitz function decomposition would not differ conceptually from a decomposition in Taylor polynomials. However, if the measure μ is selected to reflect the uncertainty of a decision-maker on the input parameters, then Assumption 1 is equivalent to state that μ is such that the parameters are independent. In the case parameters are not independent and correlations emerge, Bedford (1998) shows that the variance decomposition in eq. (2) depends on the lexicographic ordering of the parameters (see also Bedford (1998) [13] and Borgonovo (2005) [13] for a further discussion.)

Theorem 1 *(Sobol' (1990) [44] and also Rabitz and Alis (1999) [33]). Under the above assumptions, the following decomposition of $f(\mathbf{x})$ is unique*

$$f(\mathbf{x}) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{i,j}(x_i, x_j) + ... + f_{1,2,...n}(x_1, x_2, ..., x_n) \quad (3)$$

where

$$\begin{array}{c} f_0 = E_x[Y] = \int \cdots \int f(\mathbf{x}) d\mu \\ f_0 + f_i(x_i) = \int \cdots \int f(\mathbf{x}) \prod_{k \neq i} d\mu_k \\ f_0 + f_i(x_i) + f_j(x_j) + f_{i,j}(x_i, x_j) = \int \cdots \int f(\mathbf{x}) \prod_{k \neq i,j} d\mu_k \\ ... \end{array} \quad (4)$$

We briefly discuss eq. (3). First, let us rewrite it as follows:

$$f(\mathbf{x}) - f_0 = \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{i,j}(x_i, x_j) + ... + f_{1,2,...n}(x_1, x_2, ..., x_n) \quad (5)$$

and then compare eq. (5) to eq. (2). The right hand sides of the two equations contain the same number of terms. Furthermore, since $V[Y] = E\left[(f(\mathbf{x}) - f_0)^2\right]$, this means that the right hand side of eq. (2) is the expected of the square of the left hand side of eq.(5), *i.e.*

$$V[Y] = E\left[\left\{\sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{i,j}(x_i, x_j) + ... + f_{1,2,...n}(x_1, x_2, ..., x_n)\right\}^2\right] \quad (6)$$

This has been proven in Sobol' (1993) and is possible due to the following property of the summands in eq. (5) [see also eqs. (4)]:

$$\int \cdots \int f_{i_1, i_2 ..., i_k} \cdot f_{j_1, j_2 ..., j_m} d\mu^3 = 0 \begin{cases} if & k \neq m \\ if \ (i_1, i_2 ..., i_k) \neq (j_1, j_2 ..., j_k) \end{cases} \quad (7)$$

Eq. (7) states that the integral of the product of any two terms in the function decomposition of eq. (5) is zero when either the two terms contain a different number of parameters — $k \neq m$ — or their argument differs in even only one parameter [33].

As a result of this property, the partial variances of Y can be written as ([44], [45]):

$$V_{i,j...,m} = \int \cdots \int [f_{i,j...,m}(x_i, x_j, ..., x_m)]^2 \prod_{k=i,j,...,m} d\mu_k \quad (8)$$

[3] We have dropped the dependency of the integrands on $x_{i_1}, x_{i_2}, ..., x_{i_k}$ and $x_{j_1}, x_{j_2}, ..., x_{j_m}$ for synthetic notation purposes.

It is easy to verify that if a model is of the form

$$f(\mathbf{x}) = \sum_i g_i(x_i) \tag{9}$$

then only individual terms appear in the decompositions of eqs. (3) and (2) [11]. Eq.(9) is a sufficient condition for uncertainty to be explained by first order effects only. In the SA jargon, one says that the model is additive over the uncertainty range. However — and we show this in our first example — the converse can happen: a model can respond "additively" over a certain input range without being of the form of eq.(9). This would mean that the higher order terms in the function decomposition [eq. (3)] had a negligible influence for the uncertainty range at hand.

In order to quantify the effect of parameter groups, the global sensitivity indices (GSI) of order r, where r is the number of parameters in the group, are introduced as the ratios of the terms in eq. (2) and $V[Y]$ ([45], [40]):

$$S_{i_1,i_2\ldots,i_r} = \frac{V_{i_1,i_2\ldots,i_r}}{V[Y]} \tag{10}$$

The total order GSI of parameter x_l,(ST_l), is the sum of all the GSI related to x_l:

$$ST_l = \sum_{r=1}^{n} \sum_{i_1,\ldots,i_r} S_{i_1,\ldots,i_r} \ (i_1 = l) \tag{11}$$

i.e. ST_l is the ratio of all individual and interaction terms involving x_l and $V[Y]$.

By global importance of x_l [GI_l] one means the relevance of x_i on Y when the entire distributions of $\mathbf{x}$ and Y are considered. Then ([40]):

$$GI_l \equiv \sum_{r=1}^{n} \sum_{i_1,\ldots,i_r} S_{i_1,\ldots,i_r} \ [(i_1 = l)] = ST_l \tag{12}$$

A note on the computation of eqs. (3), (2) and (12). The estimation of the GSI can be achieved either analytically or numerically. An analytical approach is obviously feasible only if the expression of Y is known explicitly as a function of $\mathbf{x}$. The convenience of an analytical approach deteriorates rapidly with the number of parameters: the number of integrals to be computed for the sole decomposition of $f(\mathbf{x})$ is $2^n - 1$. Thus, a numerical approach based on Monte Carlo integration is adopted in most of the cases ([33], [1], [45]). The computational cost of a numerical calculation is defined in terms of number of model runs necessary to estimate the sensitivity measure. The cost (M) for computing all terms in the variance decomposition of eq. (2) is given by [33]:

$$M = N \times \sum_{i=0}^{n} \frac{n!}{(n-i)!i!} \tag{13}$$

where N is the sample size and n the number of parameters. Supposing that a model contains $n = 10$ parameters, one finds: $M = N \times 1024$. Since sample sizes $N < 1000$ produce appreciable results in few cases, it is not difficult to realize that one is dealing with a substantially high number of model runs. This problem has been tackled by Saltelli et. al (1999) [40] and Alis and Rabitz (2001) [1]. In particular, utilizing the Extended FAST approach proposed by Saltelli et al (1999) [40], one would be able to compute the first order and the total order indices at a cost of order N, with a gain of $\sum_{i=0}^{n} \frac{n!}{(n-i)!i!}$ model runs.

3 Analytical Example: Application to the Harris EOQ Formula

The purpose of this Section is to illustrate the global SA of an inventory management model when it is possible to perform the function decomposition of eq. (3) analytically. We consider a firm that uses the classical Harris EOQ formula ([30]). The model is deterministic, considers a fixed order cost x_1, constant demand x_2 and fixed holding cost per unit time x_3, no time lag between order and delivery. Under these assumptions, the EOQ is expressed as:

$$Q^* = f(\mathbf{x}) = \sqrt{2} \prod_{i=1}^{3} x_i^{p_i} \tag{14}$$

where $\mathbf{p} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. If costs and demand were known with certainty, Q^* would assume a certain value. For example with $\mathbf{x} = (30, 8000, 1)$, $Q^* = 693$.

Suppose now, that the firm wants to consider the impact on Q^* of uncertainty in the value of the parameters (see Section 1). Formally, we suppose that the firm reflects such uncertainty by means of measure $d\mu = \prod_{i=1}^{3} h_i(x_i)dx_i$ where $h_i(x_i)$ are the density functions of $x_i, i = 1..3$. Q^* becomes an uncertain quantity, being a function of random variables.

To compute GI_l, we are allowed to apply Sobol' decomposition method analytically [eq. (14)], since $f(\mathbf{x})$ is analytically known. We describe in detail the application of eq. (4) to eq. (14). For f_0, One gets:

$$f_0 = \int\int\int \sqrt{2}\prod_{i=1}^{3} x_i^{p_i} h_i(x) dx_i = \sqrt{2}\prod_{i=1}^{3}\int x_i^{p_i} h_i(x_i) dx_i = \sqrt{2}\prod_{i=1}^{3} E\left[x_i^{p_i}\right] = E[Q^*] \tag{15}$$

For the first order terms one gets ($l = 1, 2, 3$):

$$f_l(x_l) = \int\int \sqrt{2}\prod_{i=1}^{3} x_i^{p_i} \prod_{s=1, s\neq l}^{3} h_s(x) dx_s - f_0 = \sqrt{2} x_l^{p_l} \prod_{s\neq l, s=1}^{3} \int x_s^{p_s} h_s(x_s) dx_s - f_0$$

$$= \sqrt{2} x_l^{p_l} \prod_{s\neq l, s=1}^{3} E\left[x_s^{p_s}\right] - \sqrt{2}\prod_{i=1}^{3} E\left[x_i^{p_i}\right] = \sqrt{2}\left(x_l^{p_l} - E[x_l^{p_l}]\right) \prod_{s\neq l, s=1}^{3} E\left[x_s^{p_s}\right] \tag{16}$$

For the second order terms, the application of eqs. (4) to eq. (14) is as follows:

$$f_{l,m}(x_l, x_m) = \int \left(\sqrt{2}\prod_{i=1}^{3} x_i^{p_i}\right) h_s(x) dx_{s\ (s\neq l,m)} - f_l(x_l) - f_m(x_m) - f_0 =$$

$$= \sqrt{2} x_l^{p_l} x_m^{p_m} \int x_s^{p_s} h_s(x_s) dx_{s\ (s\neq l,m)} - f_l(x_l) - f_m(x_m) - f_0 =$$

$$= \sqrt{2} x_l^{p_l} x_m^{p_m} E\left[x_s^{p_s}\right] - \sqrt{2} x_l^{p_l} \prod_{s\neq l, s=1}^{3} E\left[x_s^{p_s}\right] - \sqrt{2} x_m^{p_m} \prod_{s\neq m, s=1}^{3} E\left[x_s^{p_s}\right] + \sqrt{2}\prod_{i=1}^{3} E\left[x_i^{p_i}\right] =$$

$$= \sqrt{2}\left(x_l^{p_l} x_m^{p_m} - x_l^{p_l} E[x_m^{p_m}] - E[x_l^{p_l}] x_m^{p_m} - E[x_l^{p_l}] E[x_m^{p_m}]\right) E[x_s^{p_s}] \tag{17}$$

Finally:

$$f_{1,2,3}(x_1, x_2, x_3) = f(\mathbf{x}) - \sum_{i=1}^{3} f_i(x_i) - \sum_{i<j} f_{i,j}(x_i, x_j) - f_0 \tag{18}$$

The expression of the terms in Sobol' variance decomposition [eq. (2)] is obtained by squaring and integrating the terms of Sobol' function decomposition in eqs. (15-18). Thus:

$$V_l = 2\left(\prod_{j\neq l} E[x_j^{p_j}]\right)^2 V[x_l^{p_l}] \tag{19}$$

$$V_{l,m} = 2E[x_s^{p_s}]^2 \left\{E[x_m^{2p_m}] E[x_l^{2p_l}] - E[x_m^{p_m}]^2 E[x_l^{2p_l}] - E[x_l^{p_l}]^2 E[x_m^{2p_m}] + E[x_m^{p_m}]^2 E[x_l^{p_l}]^2\right\} \tag{20}$$

Order (r)	Term	$V_{i_1,i_2...,i_r}$	$S_{i_1,i_2...,i_r}$
1	V_1	2554	0.329
1	V_2	2554	0.329
1	V_3	2621	0.337
2	$V_{1,2}$	13.51	0.0017
2	$V_{1,3}$	13.87	0.0018
2	$V_{2,3}$	13.87	0.0018
3	$V_{1,2,3}$	0	0

Table 1: V[Q*] decomposition results for the example

The residual term $V_{1,2,3}$ is obtained from:

$$V_{1,2,3} = V[Q^*] - \sum_{s=1}^{3} V_s - V_{1,2} - V_{1,3} - V_{2,3} \tag{21}$$

where

$$V\left[Q^*\right] = 2\left[\prod_{i=1}^{3} E\left[x_i^{2p_i}\right] - \left(\prod_{i=1}^{3} E\left[x_i^{p_i}\right]\right)^2\right] \tag{22}$$

GI_l is then found as:

$$GI_l = \frac{V_l + \sum_{k=1,k\neq l}^{3} V_{l,k} + V_{1,2,3}}{V\left[Q^*\right]}, \quad l = 1, 2, 3 \tag{23}$$

Eqs. (19)-(23) suggest that the global importance of the parameters is a function of the moments of $x_l^{p_l}$. In appendix A we illustrate the analytical calculations of such moments.

We now detail some numerical examples, considering two cases. For the analysis, suppose that the decision-maker assignes a $\pm 25\%$ variation range for each of the input parameters. More formally, he utilizes a uniform distribution for the parameters over the uncertainty range ($x_1 \sim u[22.5, 37.5]$, $x_2 \sim u[6000, 10000]$, $x_3 \sim u[.75, 1.25]$). Utilizing eqs.(15) and (22), leads to: $E\left[Q^*\right] = 695$ and $V\left[Q^*\right] = 7730$. In order to understand how uncertainty is apportioned by individual and group contributions, we utilize eqs. (19-20). Substituting for the numbers, we get the results in Table 1.

Table 1 shows that the model variance is explained by first order terms, since $\sum_{l=1}^{3} S1_l = 99\%$. This means that the Harris EOQ behaves

Parameter	GI_l
x_1	0.33
x_2	0.33
x_3	0.34

Table 2: Parameter global importance for the example

Term	$S_{i,j,..m}$
S_1	0.13
S_2	0.13
S_3	0.56
$S_{1,2}$	0.017
$S_{1,3}$	0.071
$S_{2,3}$	0.071
$S_{1,2,3}$	0

Table 3: V[Q*] decomposition results for the example with increased uncertainty

additively over the parameter uncertainty distributions considered. GI_l [eq. (23)] is displayed in Table 2.

Table 2 shows that x_1, x_2 and x_3 have practically the same importance, *i.e.* they contribute to the EOQ variance in the same way.

Let us now consider the same example, but supposing that the decision-maker assigns larger ($\pm 100\%$) uncertainty range to the parameters (we shall refer to this case as "case 2"). Utilizing still uniform distributions, we have: $E[Q^*] = 811$, and $V[Q^*] = 6.14 \times 10^5$. We note two effects of the increased uncertainty: $E[Q^*]$ differs now significantly from the deterministic Q^* and $V[Q^*]$ is one order of magnitude bigger than in the previous case. The global sensitivity indices [eq.(10)] are detailed in Table 3.

As mentioned, $S_{i,j,..m}$ represents the normalized variance contribution of parameter groups. Thus, Table 3 shows that the model is less additive over the new input uncertainty. In fact, individual terms cover now a lower portion of the model variance ($\sum_{l=1}^{3} S1_l = 83\%$), against almost 100% of the previous case. The corresponding parameter importance is shown in Table 4.

In this case x_3 — holding cost — is the most important parameter. We recall that in the previous case x_1, x_2 and x_3 had the same importance.

A practical implication of the above results in terms of uncertainty

Parameter	GI_l
x_1	0.23
x_2	0.23
x_3	0.66

Table 4: Parameter Global Importance for the case of increased uncertainty

Order (r)	**Term**	$V_{i_1,i_2\ldots,i_r}$(case 1)	$S_{i_1,i_2\ldots,i_r}$(case 1)	$V_{i_1,i_2\ldots,i_r}$(case 2)	$S_{i_1,i_2\ldots,i_r}$(case 2)
1	V_1	2520	0.329	$4.661E+4$	0.257
1	V_2	2520	0.329	$4.661E+4$	0.257
1	V_3	2574	0.336	$7.101E+4$	0.392
2	$V_{1,2}$	13.2	$1.72E-3$	$4.032E+3$	0.022
2	$V_{1,3}$	13.4	$1.756E-3$	$6.142E+3$	0.0034
2	$V_{2,3}$	13.5	$1.756E-3$	$6.142E+3$	0.0034
3	$V_{1,2,3}$	0	0	$5.31E+2$	0

Table 5: Global SA rsults when the shape of the distribution is changed from uniform to gamma

management is the following: *ceteris paribus*, collecting information on x_3 would be the most effective way of reducing the variance of the EOQ. We illustrate this concept by means of a numerical exercise. Suppose that the decision maker orients resources to eliminate his uncertainty on demand (x_2), and becomes sure on the value $x_2 = 8000$. In so doing the model variance decreases from 6.2×10^5 to 5.3×10^5, with a reduction of around 13%. Suppose now that, instead, the same resources were spent in collecting information on x_3, (say that it is fixed at $x_3 = 1$). In this case, the model experiences a much more significant reduction, from 6.2×10^5 to 1.00×10^5, which corresponds to an 84% decrease.

We then tested the robustness of the above results following a change in the form of the distribution. We utilized gamma distributions $\gamma_i(x_i; \alpha_i, \beta_i)$ (see Appendix A) and considered two cases with the values of α_i, β_i reflecting the expected value and variance of x_i in the 25% and in the 100% case respectively. The results are reported in Tables 5 and 6.

Parameter	GI_l (Case 1)	GI_l (Case 2)
x_1	0.33	0.32
x_2	0.33	0.32
x_3	0.34	0.45

Table 6: Parameter global importance with gamma distributions

The comparison of Tables 5 and 6 and Tables 1, 2, 3 and 4 shows that the results are robust w.r.t. the choice of the uncertainty distributions made in this exercise. In particular the following remarks hold: *i)* the response of the EOQ is additive in the first uncertainty case for both the uniform and the gamma distribution choices (Tables 2 and 5) and non-additive over an increased uncertainty range (Tables 3 and 5); *ii)* in case 1 ; *iii)* the fact that $V_1 = V_2$ and $GI_1 = GI_2$ in all cases reflects the symmetry of the Harris EOQ formula' dependence on x_1 and x_2.

4 Application: the Evaluation of Financing Policies and the Temporary sale Problem

The purpose of this Section is to present the insights in terms of EOQ structure and modelling choices that are gained from a variance-based global SA when an analytical expression of the EOQ is not available.

We do so by studying the model proposed by Luciano and Peccati (1999), which allows the evaluation of inventory management policies with explicit consideration of financing choices [26]. Utilizing the Adjusted Present Value (APV) approach ([21], [22], [31], [32]), the model offers, in a logic cascade, the cost functions that enable the EOQ calculation in the presence of, respectively, the cost of capital, third party financing and special orders — we refer the reader to [26] for the complete illustration of the model. —

The first case treated by the model is the inclusion in the EOQ of the cost of capital [26]. The following cost function is introduced [26]:

$$\mathcal{L}_1(Q) = \frac{\left(c + \frac{\alpha}{2}\right) Q + \gamma}{1 - e^{-\rho Q/D}} \tag{24}$$

$$\text{where} \begin{cases} c = \text{ price of the good in inventory} \\ \alpha = \text{holding cost percentage} \\ D = \text{demand} \\ \gamma = \text{unit order cost} \\ \rho = \text{ cost of capital} \end{cases}$$

The EOQ, Q_1^*, that cannot be determined analytically, depends implicitly on the following 5 parameters:

$$Q_1^* = q_1(c, \alpha, D, \gamma, \rho) \tag{25}$$

With the following numerical assumptions: $c = 10$, $D = 8000$, $\alpha = 0$, $\gamma = 30$, $\rho = 30\%$, the optimal order quantity is estimated at $Q_1^* = 399$ [26].

Parameter	**Distribution**	$S1_l$	GI_l	% of interaction	Rank
c	$\beta(5,15,2,2)$	0.33	0.37	9	2
α	$\beta(0,0.3,2,4)$	0	0.02	99	5
γ	$\beta(20,40,2,4)$	0.10	0.12	17	3
D	$\beta(7500,8000,2,4)$	0.003	0.02	85	4
ρ	$\beta(0.1,0.4,2,2)$	0.53	0.55	4	1

Table 7: Uncertainty distributions of the input parameters for Q*1

Let us now consider the effect of uncertainty in the five input parameters. They are assigned the densities[4] displayed in Table 7.

Since an analytical expression for Q_1^* is not available, one has to resort to a numerical approach ([45], [25], [40]), computing the terms of eqs. (3) and (2) via the appropriate random generation technique ([40], [41], [25]). In correspondence to each input generation, Q_1^* is re-evaluated. The sampled values of Q_1^* are then utilized to estimate the global sensitivity indices. We have employed the FAST method, thus obtaining GI_l and $S1_l$ at a computational cost equal to the sample size (see the end of Section 2). For the inputs of Table 7, we determined the sample size by iteratively increasing the number of model runs until the estimation of GI_l differed by less than 1% in two next calculations. For the case at hand, convergence was obtained with a sample size of $N = 5000$.

The result for the global importance of the parameters are reported in Table 7. Q_1^* is almost additive, since $\sum_{i=1}^{5} S1_l = 96\%$. Note that $S1_\rho + S1_c = 86\%$, i.e. ρ,and c, individually explain around 86% of the model variance with the other parameters playing a relatively minor role. α is the least influential parameter, with practically no individual influence and with all its (low) importance due to interactions with the other parameters (the column "% of interaction" in Table 7 represents the percentage of the parameter importance due to interaction with the other parameters). We note that the ranking does not change if one considers the first order or the total order indices, as a consequence of the high additivity of Q_1^* in this case.

In the case inventories are funded by external debt, one needs to model the presence of third party financing. The cost function becomes [26]:

$$\mathcal{L}_2(Q) = -\frac{\left(c + \frac{\alpha}{2}\right) Q + \gamma}{1 - e^{-\rho Q/D}} + \left[\left(c + \frac{\alpha}{2}\right) Q + \gamma\right] \cdot k \qquad (26)$$

[4] For notation clarity: $\beta(a,b,r,q) = \frac{1}{\beta(r,q)} \frac{(x-a)^{r-1}(b-x)^{q-1}}{(b-a)^{r+q-1}}$

Parameter	$S1_l$	GI_l	% of interaction	Rank
c	0.005	0.051	91	3
α	0.002	0.048	96	6
γ	0.003	0.049	94	5
D	0.003	0.050	94	4
ρ	0.27	0.60	54	2
δ	0.41	0.72	44	1

Table 8: Importance of the parameters in the presence of financing

with $k = 1 - \int_0^\infty \beta(s)e^{-\rho s}ds$ is the NPV of the cash flows[5] generated by a debt increase of one unit [26]. Under the assumption of a constant repayment policy, with installments coinciding with interest repayments at maturities $s = 1, 2...n$, then $k = 1 - \frac{\delta}{e^\rho - 1} = g(\delta, \rho)$, where $\delta =$ interest rate for one period [26]. In the case of eq. (26), the EOQ turns out to be a function of 6 parameters, with δ adding to the previous 5:

$$Q_2^* = q_2(c, \alpha, D, \gamma, \rho, \delta) \tag{27}$$

Assuming $\delta = 20\%$, the point estimate of Q_2^* produces $Q_2^* = 1013$. Hence, the choice of modeling third party financing with a point value of 20% leads to an increase in the point estimate of the EOQ of around 600 units, from $Q_1^* = 399$ to $Q_2^* = 1013$. Let us now utilize variance-based SA to determine whether in correspondence of the modeling of the new aspect, *i.e.* the presence of financing, a relevant or non-relevant parameter has been introduced and whether the structure of the EOQ has changed. Utilizing $\delta \sim \beta(0, 0.4, 2, 2)$, one finds the results reported in Table 8.

Table 8 shows that δ is the most influential parameter, when both its individual and group importance are considered. ρ, that resulted as the most influential parameter on Q_1^*, ranks now second. GI_δ and GI_ρ are much higher than the GI_l of the other parameters. c, α, γ and D have a similar impact on Q_2^*, and their GI is mainly attributable to interactions with the other parameters. With respect to the global SA of Q_1^*, a high relevance of interaction terms is registered (see Column 4 of Table 8). Note also that the ranking differs if one utilizes first order terms rather than GI_l. Thus, the effect of modeling external debt is felt both on the parameter ranking, with δ being the most important parameter, and on the EOQ structure, with a reduction of additivity.

Luciano and Peccati (1999) consider also the case of a special order

[5] $\beta(s)$ is the cash flow generated at s by a debt increase of 1 unit.

Parameter	$S1_l$	GI_l	% of interaction	Rank
c	0.0005	0.129	99	4
α	0.0001	0.114	99	7
γ	0.0002	0.128	99	5
D	0.0003	0.125	99	6
ρ	0.02	0.180	90	2
δ	0.76	0.957	20	1
d	0.0014	0.177	99	3

Table 9: Importance of the parameters in the presence of financing and of a special order

to be placed at $t_0 = 0$ [26]. The cost function turns out to be [26]:

$$\mathcal{L}_3(Q) = -\frac{\left(c + \frac{\alpha}{2}\right) Q + \gamma}{1 - e^{-\rho Q/D}} + \left[\left(c - d + \frac{\alpha}{2}\right) Q + \gamma\right] \cdot k + d \cdot Q \qquad (28)$$

where d = reduction of c in the special order. In this case, the EOQ is a function of the following seven parameters:

$$Q_3^* = q_3(c, \alpha, D, \gamma, \rho, \delta, d) \qquad (29)$$

Assuming $d = 1$, and maintaining all the previous numerical assumptions, $Q_3^* = 1769$ [26]. Again, the modeling of a special order leads to a significant change in the point value of the EOQ: from $Q_2^* = 1013$ to $Q_3^* = 1769$.

Let us now perform the global SA of the model letting $d \sim \beta(0.5, 2, 2, 2)$. One finds the results shown in Table 9, with the sample size increased to $N = 12000$ to assure convergence.

Table 9 shows that δ is still the most influential parameter, with increased influence. The newly introduced parameter, d, turns out to be a relevant one, ranking 3^{rd} over 7, and having an influence similar to ρ. The remaining parameters, c, α, γ, and D have a much lower influence on Q_3^*.

In terms of model structure, one notes that the addition of d causes the EOQ to further deviate from the additive result obtained in the absence of δ and d (Table 9, Column 4).

5 Conclusions

This work has concerned the application of global Sensitivity Analysis (SA) to inventory management models. Global SA techniques are the most appropriate for performing the SA of model output when full consideration is given to the uncertainty in the parameters.

We have presented the mathematical bases of global SA and the relevant theorems and introduced the definition of global importance of a parameter (GI_l). We have first dealt with the analytical application of Sobol' function and variance decomposition method to the classical Wilson-Harris inventory model. We have discussed the derivation of the analytical expressions for the global importance of demand, holding and order costs. We have shown how the method can be used to analyse the model dependence on the input parameters with two set of assumptions, reflecting different states of knowledge of the decision maker.

We have then turned to the global SA of the Luciano-Peccati model. The model presents the cost functions that determine the EOQ's in consideration of the cost of capital, the presence of financing and the presence of a special order, respectively. We have seen that in the absence of financing and special orders, the EOQ depends on five parameters. The most relevant one is the cost of equity, ρ. In the presence of third party financing, the EOQ becomes a function of six parameters, the previous five plus the cost of debt, δ. δ turns out to be the most relevant input, followed by ρ. With the evaluation of a special order, the EOQ becomes a function of seven parameters. The parameter added to model the special order, d, results third, after ρ; δ ranks still first. In none of the above cases the analytical expression of the EOQ was feasible. However, the analysis has shown that the application of variance based global SA enables one to gain:

Modeling insights. The results provide an analyst with direct and quantitative information on whether the inclusion/exclusion of certain aspects in/from the model would correspond to consider/neglect relevant aspects. In the Luciano-Peccati model, for example, not to model third party financing (δ), the cost of equity (ρ) or the presence of a special order would have resulted in neglecting aspects with a significant impact on the EOQ.

EOQ structure insights. We have seen that the EOQ dependence on the parameters changed from additive to non-additive as third party financing and the presence of a special order were included in the model.

Uncertainty management insights. By definition, parameters associated with high values of GI_l are the ones that are capable of reducing variance in the output the most. In the presence of limited time and resources, there can result a precious information the knowledge of the parameters on which to orient resource and data collection to reduce the variance of model output the most.

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6 Appendix A: Moment Calculations

As mentioned in Section 3, the variance and function decomposition for the Harris EOQ formula is analytically feasible and is a function of the moments of the three parameters. This Appendix details the calculation of the relevant moments that appera in (15) - (23).

Let us start with $E\left[x_i^{p_i}\right]$ $i = 1, 2, 3$. We have:

$$\begin{cases} E[x_1^{1/2}] = \int_{a_1}^{b_1} \frac{z^{1/2}}{b_1 - a_1} dz = \frac{1}{b_1 - a_1}\left(\frac{2}{3} b_1^{\frac{3}{2}} - \frac{2}{3} a_1^{\frac{3}{2}}\right) \\ E[x_2^{1/2}] = \int_{a_2}^{b_2} \frac{z^{1/2}}{b_2 - a_2} dz = \frac{1}{b_2 - a_2}\left(\frac{2}{3} b_2^{\frac{3}{2}} - \frac{2}{3} a_2^{\frac{3}{2}}\right) \\ E[x_3^{-1/2}] = \int_{a_3}^{b_3} \frac{z^{-1/2}}{b_3 - a_3} dz = \frac{1}{b_3 - a_3}\left(2\sqrt{b_3} - 2\sqrt{a_3}\right) \end{cases} \tag{30}$$

The above three moments can then be inserted in eqs. (15) - (18).

In order to compute the numerical values of the variance decomposition in eqs. (19) - (22), one must also compute the following moments: $V[x_i^{p_i}]$ and $E[x_i^{2p_i}]$. Noting first that:

$$V[x_i^{p_i}] = E[x_i^{2p_i}] - E[x_i^{p_i}]^2 \tag{31}$$

and that $E[x_i^{p_i}]$ have been provided in eq. (30), then one needs only to estimate $E[x_i^{2p_i}]$, $i = 1, 2, 3$. We have:

$$\begin{cases} E[x_1^{2p_1}] = E[x_1] = \frac{a_1+b_1}{2} \\ E[x_2^{2p_2}] = E[x_2] = \frac{a_2+b_2}{2} \\ E[x_3^{2p_3}] = E[x_3^{-1}] = \frac{\ln b_3 - \ln a_3}{b_3 - a_3} \end{cases} \tag{32}$$

For the case of $u[22.5, 37.5]$, $x_2 \sim u[6000, 10000]$, $x_3 \sim u[.75, 1.25]$ we have:

$$\begin{cases} a_1 = 22.5 \\ b_1 = 37.5 \\ a_2 = 6000 \\ b_2 = 10000 \\ a_3 = 0.75 \\ b_3 = 1.25 \end{cases} \tag{33}$$

The numerical values of the moments become:

$$\begin{cases} E[x_1^{1/2}] = 5.46 \\ E[x_2^{1/2}] = 89.21 \\ E[x_3^{-1/2}] = 1.01 \\ E[x_1^{2p_1}] = E[x_1] = 30 \\ E[x_2^{2p_2}] = E[x_2] = 8000 \\ E[x_3^{2p_3}] = E[x_3^{-1}] = 1.02 \end{cases} \tag{34}$$

One can then substitute back in eqs. (19) - (21) and finds the results reported in Section 3.