A Study of Interactions in the Risk Assessment of Complex Engineering Systems: an Application to Space PSA

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Abstract

Risk-managers are often confronted with the evaluation of operational policies in which two or more system components are simultaneously affected by a change. In these instances, the decision-making process should be informed by the relevance of interactions. However, because of system and model complexity, a rigorous study for determining whether and how interactions quantitatively impact operational choices has not been developed yet. In the light of the central role played by the multilinearity of the decision-support models, we investigate of the presence of interactions in multilinear functions first. We identify interactions that can be a-priori excluded from the analysis. We introduce sensitivity measures that apportion the model output change to factors individual and interaction contributions in an exact fashion. The sensitivity measures are linked to graphical representation methods as tornado diagrams and Pareto charts, and a systematic way of inferring managerial insights is presented. We then specialize the findings to reliability and probabilistic safety assessment (PSA) problems. We set forth a procedure for determining the magnitude of changes that make interactions relevant in the analysis. Quantitative results are discussed by application to a PSA model developed at NASA to support decision-making in space mission planning and design. Numerical findings show that sub-optimal decisions concerning the components on which to focus managerial attention can be made, if the decision-making process is not informed by the relevance of interactions.

Keywords: *Risk Analysis; Sensitivity Analysis; Multilinearity; Interactions; System Risk; Safety; Probabilistic Safety Assessment; Importance Measures; Operational Decision-Making.*

To appear in Operations Research. Accepted on February 27, 2011.

1 Introduction

Quantitative models play a crucial role in supporting decision-makers in several operations research (O.R.) applications. In engineering risk assessment problems, managers "*face a challenge when deciding how to allocate scarce resources to minimize the risks of failure. As resource constraints become tighter, balancing these failure risks is more critical, less intuitive and can benefit from the power of quantitative analysis* [Dillon et al (2003); p. 354]." In his seminal work about the use of quantitative models, Little (1970) highlights the need to know "*what it was about the input* *that made the output come out as they did* [Little (1970); p. B469]." Little's statement warns one about the black box effect, namely leaving a model's informational content unexploited. In this respect, sensitivity analysis techniques "*appear to be the key ingredient needed to draw out the maximum capabilities of mathematical modelling* [Rabitz (1989)]." Nonetheless, a recent survey [Saltelli and Annoni (2010)] shows that the most widely utilized sensitivity methods are based on one-factor-at-a-time (OFAT) variations. To this group belong techniques as tornado diagrams [Howard (1988)], spiderplots [Eschenbach (1992)] and Pareto charts [Hart and Hart (1989)], as well as reliability sensitivity indices as the Birnbaum, Differential, Criticality, Risk Achievement Worth (RAW) and Risk Reduction Worth (RRW) importance measures [Birnbaum (1969), Cheok et al (1998), Borgonovo and Apostolakis (2001).]

Calculation simplicity and easiness of interpretation make OFAT techniques attractive. However, their limitations have been highlighted in design of experiment [Myers and Montgomery (1995)], sensitivity analysis [Saltelli and Annoni (2010), Saltelli and D'Hombres (2010)] and O.R. [Wagner (1995)]. The main drawback of OFAT methods is their inability of delivering information about interactions — among others, see the geometric argument recently offered by Saltelli and Annoni (2010). — If a model is not an additive function of the factors, the superimposition of individual effects does not explain model results [Myers and Montgomery (1995)].

This issue can critically affect the allocation of resources in operational decisions when one or more system components are simultaneously affected by a change (change in inspection and maintenance policies, ageing etc.). The problem is specifically unveiled by Vesely et al (1990), where sensitivity measures for the presence of interactions in system ageing are introduced. Recently, several studies have redirected attention to the problem [Zio and Podofillini (2006), Lu and Jiang (2007), Gao et al (2007), Borgonovo (2010b)]. However, a methodology for determining how and when interactions matter in the evaluation of operational policies has not been proposed yet. Indeed, because the industrial systems under investigation are made of several tens of components, the corresponding decision-support models contain hundreds of factors. In an n-variate model, $2^n - 1$ potential interactions are present. When n is greater than 20, the number of interactions is greater than 1 million. Such complexity hinders the analysis of interactions — general reflections on model complexity and interactions are offered by Herman (1992). —

This work introduces a systematic approach to the quantification of interactions in risk-informed operational decision models. Because several of these models are multilinear (probabilistic safety assessment models, reliability functions, belief-networks, etc.), we start with analyzing interactions in generic multilinear functions (f). The equivalence of the Taylor and integral representations of a finite change in f allows us to introduce sensitivity measures that identify the exact fraction of Δf associated with a given factor. Furthermore, a triplet of sensitivity measures that dissect a factor's impact in its individual, interaction and total contributions is introduced. We show that they can be estimated at a cost of $2n+2$ model runs. This result grants applicability of the method to full-fledged operational models.

Two aspects emerging in practical applications are addressed next. First, because multilinear functions are infinitely many times differentiable, interactions might not be numerically significant if changes are small. A procedure for investigating the magnitude of changes that makes interactions relevant in the analysis (interaction threshold) is introduced. Knowledge of the threshold has the following practical implication: if the policy under investigation involves changes that are below the threshold, the decision-making process does not need to be informed by the relevance of interactions. Second, in several instances, managers are interested in an analysis at the group rather than individual factor level. However, the relationship between individual and group sensitivity measures is not straightforward [Cheok et al (1998)]. By exploiting the mathematical link between the differential and integral decompositions, we obtain sensitivity measures for quantifying interactions among factor groups in multilinear functions. The extension preserves both the properties and the computational advantages that hold for individual factors, allowing decision-makers to select the level of aggregation.

These findings are then specialized to operational decision problems supported by reliability and probabilistic safety assessment (PSA) models. We prove that the PSA risk metric is a multilinear function of the basic event probabilities and initiating event frequencies. Then, the results obtained for generic multilinear functions allow us to: *i*) identify the interactions that can be a-priori excluded; and *ii*) extend the notion of total order reliability importance [Borgonovo (2010b)] to PSA models.

We present a systematic way of obtaining managerial insights via three sensitivity analysis settings concerning model structure, direction of change and key-driver identification [for the concept of sensitivity setting, see Saltelli and Tarantola (2002)] and illustrate the graphical representation of results in the form of Pareto charts and extended tornado diagrams. A sample belief-network example is used to demonstrate the procedure [Park and Darwiche (2004)].

We address quantitatively the presence of interactions in a complex operational decision-making problem: the design of a multi-phased lunar space mission. The decision-support tool is a probabilistic safety assessment (PSA) model developed for the US National Aeronautics and Space Administration (NASA), as part of the lunar program of the Agency. The model uses a phased-based event tree and fault tree logic structure to model a lunar mission, including multiple phases (from launch to return to the Earth surface) and multiple critical systems [for an introduction to the methodology of space PSA, we refer to Stamatelatos et al (2002); see also NASA (2005).] Numerical results show that knowledge of the interaction contributions becomes essential in the identification of the elements on which to focus managerial attention to insure that target system performance and safety are achieved.

The remainder of the paper is organized as follows. Section 2 presents results for the interaction properties of multilinear functions both at the individual level. Section 3 extends the sensitivity measures to factor groups. Section 4 casts the findings in the context of PSA and reliability models. Section 5 illustrates the graphical representation of results and the derivation of managerial insights in the context of a belief-network model. Section 6 describes the full-fledged PSA model utilized in the numerical experiments. Sections 6.1 and 6.2 report the case-study results at the basic event and system levels respectively. Section 6.3 discusses the determination of the interaction threshold. Section 7 offers conclusions.

2 Interactions in Multilinear Functions: Finite and Infinitesimal Changes

This section presents an analysis of the interaction properties of multilinear functions.

Multilinearity plays a fundamental role in many OR problems. In reliability, the expression that links the reliability function of any coherent and non-coherent system to the component failure probabilities is multilinear [Borgonovo (2010b)]. In multiattribute utility theory, a key-role is played by multilinear utility functions [Keeney and Raiffa (1993)]. As Bordley and Kirkwood (2004) state "*the target-oriented preference conditions are analogous to reliability theory conditions for series or parallel failure modes in a system* [Bordley and Kirkwood (2004)]." In Bayesian networks, the mathematical relations that binds the model output to the network parameters is multilinear [Park and Darwiche (2004)]. In response surface problems, multilinear metamodels are often selected for reproducing the input-output mapping [Myers and Montgomery (1995)]. One cannot omit the role of multilinear functions in optimization [Crama (1993), Rikun (1997), Sherali and Driscoll (2002), Lambert et al (2005), Floudas and Gounaris (2009)], set function theory [Hammer and Rudeanu (1968)], game theory and economics [Grabisch et al (2000), Grabisch et al (2003), Lambert et al (2005), Alonso-Meijide et al (2008)].

We write a multilinear function $f : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$y = f(\mathbf{x}) = \sum_{k=1}^{n} \sum_{i_1<i_2<...<i_k} \alpha_{i_1,i_2,...,i_k} \cdot x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_k} \tag{1}$$

with $\alpha_{i_1,i_2,...,i_k} \in \mathbb{R}, \quad k = 0, 1, ..., n$. If all $\alpha_{i_1,i_2,...,i_k}$ are non null, f is called a combinatorial multilinear function [Crama (1993), Rikun (1997)] and contains $2^n - 1$ terms. Eq. (1) states that a multilinear function is separately affine in each variable, which is an alternative definition of multilinearity [Marinacci and Montrucchio (2005)]. Furthermore, f is a homogeneous function satisfying Euler's equation of order 1. It is also a Bernstein polynomial of order 1 [Marinacci and Montrucchio (2005)] and coincides with its Maclaurin polynomial [Borgonovo (2010b)]. In particular, the following holds.

Lemma 1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multilinear function, and let $\mathbf{x}^0$ and $\mathbf{x}^1$ any two points in $\mathbb{R}^n$. Then,*

$$\Delta f = f(\mathbf{x}^1) - f(\mathbf{x}^0) = \sum_{i=1}^{n} f_i' \cdot \Delta x_i + \sum_{i_1<i_2} f_{i_1,i_2}'' \cdot \Delta x_{i_1} \cdot \Delta x_{i_2} + ... + \sum_{i_1<i_2,..<i_{k\max}} f_{i_1,i_2,...,i_{k_{\max}}}^{k_{\max}} \cdot \Delta x_{i_1} \cdot \Delta x_{i_2} ... \cdot \Delta x_{i_{k\max}} \tag{2}$$

where k_{max} is the size of the largest product term in f.

Lemma 1 states that the Taylor (Maclaurin) expansion of any multilinear function is exact, with the maximum order determined by k_{max}, the highest number of factors in the product terms of f [for proofs, see Grabisch et al (2000), or Borgonovo (2010b)]. Eq. (2) provides the differential decomposition of a finite change in f. By eq. (2), let us define the following sensitivity indices.

Definition 1

$$\beta^k_{i_1,i_2,...,i_k} := f^k_{i_1,i_2,...,i_k} \cdot \Delta x_{i_1} \cdot \Delta x_{i_2} ... \cdot \Delta x_{i_k} \tag{3}$$

$\beta^k_{i_1,i_2,...,i_k}$ represents the portion of the differential decomposition of Δf associated with the simultaneous change in $x_{i_1}, x_{i_2}, ..., x_{i_k}$. By eq. (3), eq. (2) can be rewritten as follows

$$\Delta f = \sum_{s=1}^{n} \sum_{i_1<i_2<...<i_s} \beta^s_{i_1,i_2,...,i_s} \tag{4}$$

We observe that the number of terms in eq. (4) is $2^n - 1$. This figure coincides with the number of terms in which a generic function can be decomposed according to an integral (instead of differential) decomposition. In fact, given a product measure μ and a $\mu-$measurable function $g(x) : X \subseteq \mathbb{R}^n \to \mathbb{R}$, it holds [Efron and Stein (1981), Rabitz and Alis (1999), Wang (2006)]:

$$g(\mathbf{x}) = g_0 + \sum_{i=1}^{n} g_i(x_i) + \sum_{i<j} g_{i,j}(x_i, x_j) + ... + g_{1,2,...n}(x_1, x_2, ..., x_n) \tag{5}$$

where

$$\begin{cases} g_0 = \mathbb{E}_\mu[g] = \int \cdots \int g(x) d\mu \\ g_i(x_i) = \mathbb{E}_\mu[g|x_i] - g_0 = \int \cdots \int g(x) \prod_{k \neq i} d\mu_k - g_0 \\ g_{i,j}(x_i,x_j) = \mathbb{E}_\mu[g|x_i, x_j] - g_i(x_i) - g_j(x_j) - g_0 \\ ... \end{cases} \tag{6}$$

Eq. (5) contains $2^n - 1$ orthogonal terms. Given any two points x^0,$x^1 \in X$, any change $\Delta g = g(x^1) - g(x^0)$ can then be decomposed in $2^n - 1$ terms by subtracting term by term the expansions of g at $\mathbf{x}^1$ and at $\mathbf{x}^0$:

$$\Delta g = g(x^1) - g(x^0) = \sum_{i=1}^{n} \Delta g_i + \sum_{i<j} \Delta g_{i,j} + ... + \Delta g_{1,2,...n} \tag{7}$$

In particular, when μ is the Dirac-δ measure, one obtains [Borgonovo (2010a); Borgonovo (2010c)]

$$\Delta g = \sum_{s=1}^{n} \sum_{i_1<i_2<...<i_s} \xi^s_{i_1,i_2,...,i_s} \tag{8}$$

where

$$\begin{cases} \xi_i = g(x^1_i, \mathbf{x}^0_{(-i)}) - g(\mathbf{x}^0) \\ \xi_{i,j} = g(x^1_i, x^1_j, \mathbf{x}^0_{(-i,j)}) - \Delta_i g - \Delta_j g - g(\mathbf{x}^0) \\ ... \end{cases} \tag{9}$$

Eqs. (8) and (9) states that Δg for a generic (not necessarily multilinear) function is decomposed in $2^n - 1$ terms. The first order terms [ξ_i in eq. (9)] deliver the individual factor contributions, the second order terms [$\xi_{i,j}$ in eq. (9)] the contribution of the interactions of all factor pairs, etc.. One calls a generic $\xi_{i_1,i_2,...,i_s}$ an s-order finite change sensitivity index [Borgonovo (2010a).] $\xi_{i_1,i_2,...,i_s}$ expresses the contribution to Δg of the residual interaction among the group of factors $x_{i_1}, x_{i_2}, ..., x_{i_s}$. The total effect of factor x_i (ξ_i^T) is obtained by summing all the terms in Δg associated with a change in x_i [Sobol' (1993)]. One writes

$$\xi_i^T := \sum_{s=1}^{n} \sum_{\substack{i_1<i_2...<i_s \\ i\in i_1<i_2...<i_s}} \xi^s_{i_1 i_2 ... i_s} = \xi_i^1 + \sum_{s=2}^{n} \sum_{\substack{i_1<i_2...<i_s \\ i\in i_1<i_2...<i_s}} \xi_{i_1 i_2 ... i_s} \tag{10}$$

Let us now compare eqs. (8) and (4) further. As said, they contain the same number of terms — this coincidence is due to the multilinearity of f; in fact, the number of terms in a Taylor expansion is, in principle, infinite. — Not only, but, the following holds (for the proof, see Appendix A).

Proposition 1 *Let $g = f$, with f a multilinear function. Then, for all $i_1, i_2, ..., i_s$*

$$\xi^s_{i_1,i_2,...,i_s} = f^s_{i_1,i_2,...,i_s} \Delta x_{i_1} \Delta x_{i_2} ... \Delta x_{i_s} = \beta^s_{i_1,i_2,...,i_s}, \qquad s = 1, 2, ..., n \tag{11}$$

Proposition 1 states that, when g is multilinear, each $\xi^s_{i_1,i_2,...,i_s}$ in eq. (9) coincides with the corresponding $\beta^s_{i_1,i_2,...,i_s}$ in eq. (3). Proposition 1 then implies that, when f is multilinear, eqs. (4) and (8) are term by term identical. Proposition 1 together with eqs. (4) and (7) suggests that any finite change in a multilinear function can be equivalently decomposed through a Taylor expansion or through an integral expansion with the Dirac-δ measure.

By Proposition 1 one obtains the interpretation of the sensitivity measures in eq. (3). $\beta^s_{i_1,i_2,...,i_s}$ represents the residual interaction effect among the groups of factors $x_{i_1}, x_{i_2}, ..., x_{i_s}$. One notes that, because f is infinitely many times differentiable, $f^s_{i_1,i_2,...,i_s}$ is invariant for index permutation. Hence, once a group of factors is identified, the interaction effect associated with that group is unique. As far as the total effect of a factor in a multilinear model is concerned, one can define it as the sum of all $\beta^s_{i_1,i_2,...,i_s}$ in eq. (4) containing x_i. Formally:

$$\beta_i^T := \sum_{s=1}^{n} \sum_{\substack{i_1<i_2...<i_s \\ i\in i_1<i_2...<i_s}} \beta^s_{i_1 i_2 ... i_s} \tag{12}$$

By Proposition 1, it is $\xi_i^T = \beta_i^T$ when $g = f$.

Remark 1 *It is a property of multivariate functions that, if a factor is not allowed to vary, not only its individual contribution to the change is cancelled, but all its contributions associated with interactions with the other factors become zero. Hence, if one is able to prevent (cause) the change in a generic factor of a multilinear model, the portion of Δf which is annulled (added) is β_i^T.*

Because the number of interactions increases exponentially with n, the calculation of β_i^T via eq. (12) becomes unfeasible for models with a large number of factors. However, for any generic function, the total effect ξ_i^T of a factor can be retrieved by the following equation [Borgonovo (2010b)]:

$$\xi_i^T = g(\mathbf{x}^1) - g(x_i^0, \mathbf{x}_{(-i)}^1) \tag{13}$$

where $\mathbf{x}^1$ represents the sensitivity case and $(x_i^0, \mathbf{x}_{(-i)}^1)$ the point obtained with all factors at the sensitivity case, but x_i. Because $\xi_i^T = \beta_i^T$ for multilinear functions, eq. (13) states that

$$\beta_i^T = f(\mathbf{x}^1) - f(x_i^0, \mathbf{x}_{(-i)}^1). \tag{14}$$

Therefore, a cost of $n+1$ model runs is necessary to obtain the β_i^T's of all factors. This result is particularly relevant when n is high, because it allows one to estimate all total contributions at the same cost of OFAT methods.

One can separate a factor's interaction contribution from its total effect, by considering the following sensitivity measures

$$\beta_i^{\mathcal{I}} := \beta_i^T - \beta_i^1 \tag{15}$$

In eq. (15), $\beta_i^{\mathcal{I}}$ represents the contribution of x_i to Δf due to interactions. By eqs. (12) and (15), $\beta_i^{\mathcal{I}} = \sum_{s=2}^{k} \sum_{\substack{i_1 < i_2 \ldots < i_s \\ i \in i_1 < i_2 \ldots < i_s}} \beta_{i_1 i_2 \ldots i_s}$. $\beta_i^{\mathcal{I}}$ is the sum of all terms in Δf involving interactions of x_i with the other factors.

By Proposition 1, $\beta_i^1 = \xi_i^1$. Thus, by eq. (9)

$$\beta_i^1 = f(x^1, \mathbf{x}_{(-i)}^0) - f(\mathbf{x}^0) \tag{16}$$

and all β_i^1 $(i = 1, 2, ..., n)$ can be obtained at a cost of $n+1$ model runs. Therefore, $2n+2$ model runs are necessary to estimate the triplet of sensitivity measures β_i^1, $\beta_i^{\mathcal{I}}$ and β_i^T for all factors.

Let us now investigate the relationship between interactions and the structure of a multilinear function. The next Proposition summarizes results proven in previous literature and relevant to our study [Grabisch et al (2000), Borgonovo (2010b)].

Proposition 2 *If f is multilinear, then:*
1)$f_i^k(\mathbf{x}) = 0 \ \forall \mathbf{x}, \forall i, \forall k \geq 2$
2)$f_{i_1, i_2, \ldots, i_k}^k(\mathbf{x}) = 0 \ \forall \mathbf{x} \Longleftrightarrow$ *there is no term in f containing* $\prod_{s=1}^{k} x_{i_s}$
3)$f_{i_1, i_2, \ldots, i_k}^k(\mathbf{x}) = 0 \ \forall \mathbf{x} \Rightarrow f_{i_1, i_2, \ldots, i_{k+1}}^{k+1}(\mathbf{x}) = 0 \ \forall \mathbf{x}$

Item 1 in Proposition 2 states that, once a multilinear function is differentiated with respect to x_i, all of its higher order derivatives are independent of x_i. Item 2 in Proposition 2 implies that, if $f_{i_1, i_2, \ldots, i_k}^k(\mathbf{x})$ is null at all $\mathbf{x}$, then there is no product term in f containing $x_{i_1}, x_{i_2}, ..., x_{i_k}$. The converse is also true. Item 3 implies that, if $f_{i_1, i_2, \ldots, i_k}^k(\mathbf{x})$ is null, then any higher order partial derivative of f with respect to a group of factors containing $x_{i_1}, x_{i_2}, ..., x_{i_k}$, is null. Proposition 2 has the following implication.

Corollary 1 *$f^k_{i_1,i_2,...,i_k}(\mathbf{x}) = 0\ \forall\mathbf{x}$, implies $\beta^s_{i_1,i_2,...,i_s} = 0\ \forall\mathbf{x}$, $\forall\Delta\mathbf{x}$ and $\forall s \geq k$.*

Corollary 1 states that, if the interaction term $\beta^k_{i_1,i_2,...,i_k}$ is null $\forall\mathbf{x}$ and $\Delta\mathbf{x}$, then there all higher order interaction terms containing $x_{i_1}, x_{i_2}, ..., x_{i_k}$ are null.

In some applications, one is interested in the model response for symmetric factor changes. This might happen, for instance, in using tornado diagrams. If f is multilinear, then the following holds — the proof is in appendix A. —

Proposition 3 *Let $\Delta\mathbf{x}$ and $-\Delta\mathbf{x}$ be two symmetric changes around $\mathbf{x}^0$ and f be a multilinear function. Let $\beta^k_{i_1,i_2,...,i_k}$ and $\overline{\beta}^k_{i_1,i_2,...,i_k}$ denote the sensitivity measures associated with $\Delta\mathbf{x}$ and $-\Delta\mathbf{x}$, respectively. Then,*

$$\overline{\beta}^k_{i_1,i_2,...,i_k} = (-1)^k \beta^k_{i_1,i_2,...,i_k} \qquad \forall i_1, i_2, ..., i_k,\ k = 1, 2, ..., n \tag{17}$$

Proposition 3 implies that, given two sets of symmetric factor changes around $\mathbf{x}^0$, then: *i*) $\left|\overline{\beta}^k_{i_1,i_2,...,i_k}\right| = \left|\beta^k_{i_1,i_2,...,i_k}\right|$, namely, all sensitivity measures retain their magnitude, and *ii*) all odd-order terms reverse their sign, while all even order terms do not. The implication is that the response of a multilinear function to symmetric factor changes is not necessarily symmetric.

Let us now consider the normalized version of the sensitivity measures in eqs. (3) and (15). One writes

$$D^1_i := \frac{\beta^1_i}{\Delta f};\ D^{\mathcal{I}}_i := \frac{\beta^{\mathcal{I}}_i}{\Delta f};\ D^T_i := \frac{\beta^T_i}{\Delta f} \text{ and } D^k_{i_1,i_2,...,i_k} := \frac{\beta^k_{i_1,i_2,...,i_k}}{\Delta f}. \tag{18}$$

The first three sensitivity measures in eq. (18) are the fraction of Δf due to individual (D^1_i), interaction ($D^{\mathcal{I}}_i$) and total order (D^T_i) contributions of parameter x_i, respectively. The last one ($D^k_{i_1,i_2,...,i_k}$) is the fraction of Δf due to the residual interaction among the group of factors $x_{i_1}, x_{i_2}, ..., x_{i_k}$. We observe that

$$D^T_i = \sum_{s=1}^{n} \sum_{\substack{i_1<i_2...<i_s \\ i\in i_1<i_2...<i_s}} D^s_{i_1 i_2 ... i_s} = D^1_i + D^{\mathcal{I}}_i \tag{19}$$

The sensitivity measures in eqs. (18) are useful to address small changes. In fact, the following holds (the proof is in Appendix A).

Proposition 4 *Let f be a multilinear function and let $\Delta\mathbf{x} \to \mathbf{0}$. Then:*

1. *$\Delta f \to df$ and interactions do not contribute to the change in f.*
2. *$\lim_{\Delta\mathbf{x}\to 0} D^T_i = \lim_{\Delta\mathbf{x}\to 0} D^1_i = \dfrac{d_i f}{df} = DIM_i$, where DIM_i is the fraction of the differential of f associated with x_i.*

In item 2 of Proposition 4, DIM_i is called differential importance of x_i [Borgonovo and Apostolakis (2001)]. Proposition 4 implies that, albeit formally present f, interactions might not be numerically relevant in the application at hand, if changes are small. This result then raises the question of identifying the magnitudes of the changes at which interactions start to matter in the analysis. In fact, if interactions are numerically negligible, the decision-making process does not need to be informed about their relevance, and OFAT methods can be used. In Section 4, a numerical procedure for identifying the threshold is proposed.

The next section discusses the extension of the sensitivity measures to groups of factors, so that to provide flexibility in choosing the aggregation level of the analysis.

3 Extension to Factor Groups

In several instances, decision-makers are interested in the importance of groups of factors. As a reference, in PSA and reliability applications, risk managers often need to know the importance of systems and structures [Cheok et al (1998)], which are identified by multiple components or basic events in the model. This creates issues in the definition and computation of importance measures for systems, because "*there is no simple relationship between importance measures evaluated at the single component level and those evaluated at the level of a group of components, and, as a result, some of the commonly used importance measures are not realistic measures of the sensitivity of the overall risk to parameter value changes* [Cheok et al (1998); p. 213]." More generally, when decision-support models contain a large number of factors, analysts might choose to communicate results for categories of factors, rather than individual factors [an example is offered in Borgonovo et al (2010)]. One cannot omit reference to the use of factor grouping in dimensionality reduction and screening problems. (A complete review is outside the scope of this paper; we refer to Bettonvil and Kleijnen (1997), Wan et al (2006).) In the remaining paragraphs, we address the extension of the sensitivity measures of Section 4 to factor groups.

At the basis of the extension is the partition of $\mathbf{x}$ in Q groups:

$$\underbrace{x_1\, x_2\, ..\, x_{s_1}}_{\gamma_1}\ \underbrace{x_{s_1+1}\, x_{s_1+2}\, ..\, x_{s_2}}_{\gamma_2} ... \underbrace{x_{s_{kQ-1}+1}\, x_{s_{Q-1}+2}\, ..\, x_n}_{\gamma_Q} \tag{20}$$

Then, by considering all terms in Δf associated with group γ_i one can define the total sensitivity index of group γ_i as follows.

Definition 2 *Let γ be a generic group of factors and f be a multilinear function. Then,*

$$\beta_\gamma^T := \sum_{l\in\gamma} f_l' \Delta x_l + \sum_{k=2}^{T} \sum_{\substack{i_1<i_2,..<i_k \\ l\in i_1<i_2,..<i_k\ \forall l\in\gamma}} f^k_{i_1,i_2,\dots,i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s} \tag{21}$$

and

$$D^T_{\gamma_i} := \frac{\beta^T_{\gamma_i}}{\Delta f} \tag{22}$$

$D^T_{\gamma_i}$ represents the fraction of Δf associated with the change in group γ.

The next question one faces is whether additional burden in the calculation of group importance measures is generated by this definition. The answer is found in the link between D^T and finite change sensitivity indices [see Borgonovo (2010b) and Appendix A]. In particular, the findings in Borgonovo and Peccati (2009) state that the decomposition of a generic function (g) in respect of groups of factors has the same structure as that of individual parameters, if the parameters are partitioned as in eq. (20). Letting $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, ..., \gamma_Q\}$ denote the vector of groups, then any finite change in g can be decomposed as follows [Borgonovo and Peccati (2009)]

$$\Delta g = g(\boldsymbol{\gamma}^1) - g(\boldsymbol{\gamma}^0) = \sum_{i=1}^{Q} \Delta_{\gamma_i} g + \sum_{i<j}^{Q} \Delta_{\gamma_i,\gamma_j} g + ... + \Delta_{\gamma_{i_1},\gamma_{i_2},...,\gamma_{i_Q}} g \tag{23}$$

where

$$\begin{cases} \Delta_{\gamma_i} g = g(\gamma_i^1; \boldsymbol{\gamma}^0_{(-i)}) - g(\boldsymbol{\gamma}^0) \\ \Delta_{\gamma_i,\gamma_j} g = g(\gamma_i^1, \gamma_j^1; \boldsymbol{\gamma}^0_{-(i,j)}) - \Delta_{\gamma_i} g - \Delta_{\gamma_j} g - g(\boldsymbol{\gamma}^0) \\ ... \end{cases} \tag{24}$$

In eq. (24), $\Delta_{\gamma_i} g$ represents the contribution of group γ_i, and is obtained as difference between a) the value attained by g when only group γ_i is at the sensitivity case and b) $g(\boldsymbol{\gamma}^0)$. $\Delta_{\gamma_i,\gamma_j} g$ represents the residual interaction between groups γ_i, γ_j (etc.). By eq. (23), one obtains normalized sensitivity measures for groups as follows [see also Borgonovo and Peccati (2009)]:

$$\begin{cases} D^1_{\gamma_i} := \dfrac{\Delta_{\gamma_i} g}{\Delta g} = \dfrac{g(\gamma_i^1; \boldsymbol{\gamma}^0_{(-i)}) - g(\boldsymbol{\gamma}^0)}{\Delta g} \\ D^k_{\gamma_1,\gamma_2,...,\gamma_k} := \dfrac{\Delta_{\gamma_1,\gamma_2,...,\gamma_k} g}{\Delta g} \\ D^T_{\gamma_i} := \dfrac{\Delta_{\gamma_i} g + \sum_{i \neq j} \Delta_{\gamma_i,\gamma_j} g + ... + \Delta_{\gamma_1,\gamma_2,...,\gamma_Q} g}{\Delta g} \end{cases} \tag{25}$$

Eqs. (23) and (24) are formally identical to eqs. (8) and (9), if one replaces the individual variable x_i with the group of variables γ_i. Thus, once a partition of the model elements in groups is established, each group can be formally treated as an individual factor. In particular, by the multilinearity of f (Proposition 1) eqs. (22) and (26) coincide, and it still holds

$$D^T_{\gamma_i} = \frac{f(\boldsymbol{\gamma}^1) - f(\gamma_i^0, \boldsymbol{\gamma}^1)}{\Delta f} \tag{26}$$

Hence, the computational advantage is maintained also for factors groups. This result gives analysts (risk managers) full flexibility in selecting the levels of investigation and result communication — e.g., categories vs individual factors, systems or components in a reliability application, etc..—

The analysis is different in the presence of small changes. In fact, the following holds (see Appendix A for the proof).

Proposition 5 *Let f be a multilinear function and $\gamma = (x_{i_1}, x_{i_2}, ..., x_{i_s})$ a group of s factors. Then,* *if* $\Delta \mathbf{x} \to \mathbf{0}$

$$D_{\gamma}^{T} = \sum_{r=1}^{s} D_{i_r}^{T} = \sum_{r=1}^{s} DIM_r \tag{27}$$

Proposition 5 states that, when changes are small, the sensitivity measure of a group is given by the sum of the sensitivity measures of the factors in the group.

The next section applies the previous findings to the determination of interactions in reliability and engineering risk assessment models.

4 Interactions in Reliability and Probabilistic Safety Assessment Models

Probabilistic Safety Assessment (PSA) is "*a comprehensive, structured, and logical analysis method aimed at identifying and assessing risks in complex technological systems for the purpose of cost-effectively improving their safety and performance* [Stamatelatos et al (2002)]." Consolidated by the 1975 Reactor Safety Study of the US Nuclear Regulatory Commission[1] [Rasmussen et al (1975)], PSA is nowadays employed by risk managers in the chemical [Boykin et al (1984)], nuclear [Cheok et al (1998), Smith (1998)], energy and aviation industries. Since 2002, NASA has adopted PSA for its standard risk assessment procedures [Stamatelatos et al (2002)].

PSA models assess the risk associated with an engineering system via a combination of fault and event trees [Apostolakis (1990); Singpurwalla (1988); Smith (1998), Papazoglou (1998); for a recent review on event trees, see Sherali et al (2008); for the foundation of fault trees, see Agrawal and Barlow (1984)]. The output of a PSA model is called a risk metric (R) [Papazoglou (1998); Smith (1998)]. R estimates the frequency of the undesired consequence. For instance, in the nuclear industry, R can be a core damage frequency, when measuring the likelihood of events leading to core damage in one year [typical values are 10^{-5} per year], or a large early release frequency, when measuring the likelihood of events leading to the release of radioactive material to the public in one year [Smith (1998)].

In this context an important information in resource allocation and prioritization of maintenance programs is represented by the knowledge of how critical the performance of a given component is in respect to system performance. This information is gained through importance measures. The first work addressing the importance of a component in contributing to system reliability is Birnbaum (1969). Since then, the problem has been widely addressed in the literature, with the development of several techniques tailored to the different applications at hand. Among the most widely employed are — the following is a non-exhaustive list:— the Fussell-Vesely [Fussell (1975)], Criticality, Risk Achievement Worth, Risk Reduction Worth [Cheok et al (1998), Vasseur and Llory (1999)], Differential [Borgonovo and Apostolakis (2001)], joint [Armstrong (1995)] importance measures. These techniques are extended to composite sensitivity measures for multistate systems in Ramirez-Marquez and Coit (2005), where also the mean-absolute deviation importance measure is

[1] "*The Reactor Safety Study was sponsored by the U. S. Atomic Energy Commission to estimate the public risks that could be involved in potential accidents in commercial nuclear power plants of the type now in use. It was performed under the independent direction of professor Norman C. Rasmussen of the Massachusetts Institute of Technology* [Rasmussen et al (1975); p. 1]."

introduced. Ramirez-Marquez and Coit (2005) offer a thorough review of importance measures and their utilization. Previous works addressing reviews on importance measures are Cheok et al (1998) and Vasseur and Llory (1999). We also recall the reviews of Borgonovo and Apostolakis (2001) and Borgonovo (2007), where it is shown that the multilinarity of the reliability polynomial allows one to obtain analytical relations among the Fussell-Vesely, RAW, Birnbaum, Criticality, Differential and RRW importance measures. In Borgonovo (2010b), one finds a literature review of the class of joint reliability importance measures. This set of methods is of particular interest in this work, as joint importance measures address the problem of interactions. From a historical perspective, research on joint reliability importance measures has evolved towards defining importance measures that account for interactions. The joint importance measure is usually attributed to the extension of the Birnbaum importance to order 2 in Hong and Lie (1993) and Armstrong (1995) for coherent systems, although sensitivity indices for addressing interactions in system aging are introduced in Vesely et al (1990). Joint importance measures are extended to non-coherent systems in Lu and Jiang (2007) [the Birnbaum importance is generalized to non-coherent systems in Andrews and Beeson (2003) and Beeson and Andrews (2003).] Gao et al (2007) extend the Birnbaum importance up to order k. A similar path is registered with the historical development of the differential importance measure (DIM). We recall that DIM subsumes the Birnbaum importance measure as a special case, when component reliability changes are supposed uniform. In Zio and Podofillini (2006), DIM is first extended to order 2. Then, it is extended to order k in Do Van et al (2008) and finally to maximum order in the total order importance measure, D^T [Borgonovo (2010b); see also Do Van et al (2010)].

We now come to connecting the mathematical framework developed in Sections 2 and 3 to the analysis of interactions in PSA models. Let ϕ denote the Boolean variable of the end-event. The end-event is determined by the happening of an initiating event, followed by the failure of the barriers (systems) that should oppose the initiating event. The triplet (initiating event, barriers, end-event) forms an accident sequence. We let n_{IE} denote the number of initiating events. A system (barrier) is composed of N components. System failure is caused by the happening of one out of M minimal cut sets (MCS). MCS k contains m_k basic events. A basic event may be shared among MCS's. We let n_{BE} denote the number of basic events. The PSA risk metric[2] is then written as [Stamatelatos et al (2002), Papazoglou (1998), Sherali et al (2008)]

$$R(\boldsymbol{\upsilon}, \mathbf{p}) = \sum_{j=1}^{n_{IE}} \upsilon_j P(\phi = 1 | IE_j) \tag{28}$$

where $P(\phi = 1|IE_j)$ is the conditional probability of the end-event given that initiating event j (IE_j) has happened, υ_j is the frequency of initiating event j, and $\boldsymbol{\upsilon}$ and $\mathbf{p}$ denote the vectors of initiating event frequencies and conditional basic event probabilities, respectively. A generic p_l ($l = 1, 2, .., n_{BE}$) is conditional on: a) the happening of initiating event j, and b) all the basic events that precede basic event j in the sequence.

[2] In a PSA model, the risk metric of interest is the frequency of the undesired consequence. In the nuclear industry typical risk metrics are the core damage frequency (CDF) or the Large Early Release frequency.

The following holds (see Appendix A for the proof).

Proposition 6 *$R(\boldsymbol{v}, \mathbf{p})$ is a multilinear function of $\boldsymbol{v}$ and $\mathbf{p}$.*

Proposition 6 allows one to apply the findings of Section 2 in the analysis of interactions in PSA and reliability models. In the remainder, we write $\mathbf{x} = (\boldsymbol{v}, \mathbf{p})$, to denote the vector of PSA model factors. $\mathbf{x}$ is a vector of size $n = n_{IE} + n_{BE}$. We let $\Delta\mathbf{x} = (\Delta\boldsymbol{v}, \Delta\mathbf{p})$ denote a change in basic event probabilities or initiating event frequencies and ΔR the corresponding risk metric change. The following property characterizes ΔR [see Appendix A for the proof.]

Corollary 2 *The Taylor expansion of ΔR provoked by $\Delta\mathbf{x}$ is exact and at most of order $T \leq n$, where $n = n_{IE} + n_{BE}$:*

$$\Delta R = R(\mathbf{x}^1) - R(\mathbf{x}^0) = \sum_{l=1}^{n} B_l(\mathbf{x}^0) \cdot \Delta x_l + \sum_{k=2}^{T} \sum_{i_1 < i_2, .. < i_k} J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s} \tag{29}$$

In reliability and PSA, the derivatives in eq. (29) are interpreted as importance measures. $B_l(x^0)$ is the Birnbaum importance of x_l. B_l is originally defined for reliability functions as the probability that basic event l becomes critical to system failure [Birnbaum (1969)]. By a mathematical property of the unreliability functions, Birnbaum (1969) proves that

$$B_l(\mathbf{x}) = \frac{\partial R(\mathbf{x})}{\partial p_l} \tag{30}$$

The symbol J in eq. (29) denotes a joint reliability importance [Hong and Lie (1993), Armstrong (1995), Vesely et al (1990)]. Given a group of k model elements, one writes [Gao et al (2007)]

$$J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x}) = \frac{\partial}{\partial x_{i_k}} (J^{k-1}_{i_1, i_2, \ldots, i_{k-1}}(\mathbf{x})) \tag{31}$$

$J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x})$ has the following interpretation: a positive sign of $J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x})$ implies that $J^{k-1}_{i_1, i_2, \ldots, i_{k-1}}(\mathbf{x})$ increases given that the probability of basic event (or initiating event) i_k increases. Therefore, $J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x})$ provides insights on how PSA model elements *"interact with each other when the probability of one event changes* [Lu and Jiang (2007), p. 435]." Because $R(\mathbf{x})$ is regular, its mixed partial derivatives are continuous at every $\mathbf{x}$. Therefore, $J^k_{i_1, i_2, \ldots, i_k}$ is symmetric for permutation of the indices — it is unique, once group $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ is fixed. —

By eq. (29), it is possible to obtain the following relationships between the importance measures β, B and J (the proof is in Appendix A).

Proposition 7 *Given the risk metric in eq. (28), the following relationships hold:*

$$\begin{aligned}
&1)\ \beta_l^1 = B_l(\mathbf{x})\Delta x_l \\
&2)\ \beta^k_{i_1,i_2,...,i_k} = J^k_{i_1,i_2,...,i_k} \cdot \Delta x_{i_1} \cdot \Delta x_{i_2} ... \cdot \Delta x_{i_k} \\
&3)\ \beta_l^T = B_l(\mathbf{x})\Delta x_l + \sum_{k=2}^{T} \sum_{l<i_2,...,<i_k} J^k_{l,i_2,...,i_k} \cdot \Delta x_{i_1} \cdot \Delta x_{i_2} ... \cdot \Delta x_{i_k}
\end{aligned} \tag{32}$$

The properties of generic multilinear functions discussed in Section 2, lead to the following results for B_l and $J^k_{i_1,i_2,...,i_k}$ in PSA models (see Appendix A for the proof).

Proposition 8 *1)* $B_j = P(\phi = 1|IE_j)$
2) The joint importance of any group of PSA model elements which includes two or more initiating event frequencies is null.
3) Let $i_1, i_2, ..., i_{m_k}$ *be the set of indices of the basic event probabilities in minimal cut set* M_k*. Let* $J_{i_1,i_2,...,i_{m_k}}$ *denote the joint reliability importance of* M_k*. Under the rare event approximation,* $J_{i_1,i_2,...,i_{m_k}}$ *equals the sum of the initiating event frequencies associated with* M_k*. Also, there is no interaction of higher order than the order of the largest minimal cut set plus 1.*

Point 1 in Proposition 8 states that the Birnbaum importance of an initiating event equals the conditional probability of the end-event. Point 2 allows one to exclude the presence of interactions among initiating event frequencies. Initiating event frequencies interact only with the basic event probabilities of the corresponding accident sequence(s). Furthermore, because $J_{IE_s,IE_m,BE_l,...,BE_k} = 0$, there are no interactions associated with groups containing both basic events and initiating events if, in the group, there are two or more initiating events. Point 3 describes the structure of the risk metric when the rare event approximation is used. In this case, there are no interactions of order $k > K_M + 1$, where K_M is the order of the largest MCS. Thus, by item 2 in Proposition 2, the multilinear polynomial of R does not contain product terms of order higher than $K_M + 1$. Note that in reliability functions the largest size is K_M [Borgonovo (2010b)], since no initiating event frequencies are present[3].

The presence of initiating event frequencies ($\boldsymbol{\upsilon}$) in $R(\boldsymbol{\upsilon}, \mathbf{p})$ [eq. (49)] also marks a second departure of PSA models from unreliability functions. Because initiating event frequencies have units $1/time$, while basic events probabilities are pure numbers, the Birnbaum importance of a basic event is not comparable to the Birnbaum importance of an initiating event. The same happens to J — the joint importance of two basic events ($J_{BE_l,...,BE_k}$) has different units from the joint importance of a basic event and an initiating event (J_{IE_s,BE_l}). — Hence, neither B nor J can be utilized to obtain a unique ranking of PSA model elements.

[3] The rare event approximation does not hold in general, and has to be adopted on a case-by-case basis, to avoid introducing distortions in the estimation of R. Indeed, it will not be utilized in the numerical application of this work.

Example 1 *Consider a PSA model composed of one initiating event and one barrier. The corresponding expression of the risk metric is*

$$R(\boldsymbol{\upsilon}, \mathbf{p}) = \nu p \tag{33}$$

where ν is the frequency of the initiating event and p the probability of the barrier failure. Let $\nu = 0.1$ [1/year] and $p = 0.1$. The Birnbaum importance measures are

$$B_\nu = \frac{\partial R}{\partial \nu} = p = 0.1 \text{ and } B_p = \frac{\partial R}{\partial p} = \nu = 0.1 \tag{34}$$

However, by the above result one is not allowed to infer that the basic event and initiating event are equally important. In fact, B_ν and B_p are not comparable, because they have different units.

The problem is not present if one utilizes the sensitivity measures in eq. (18). Proposition 6 and Corollary 2 guarantee the formal extension of the sensitivity measures introduced in Section 2 to PSA models as follows.

- $D_l^T := \dfrac{\Delta_l^T R}{\Delta R}$ $(l = 1, 2, ..., n)$, with $\Delta_l^T R$ being the sum of all terms in the Taylor expansion of ΔR associated with Δx_l (being x_l a basic event probability or an initiating event frequency). By D^T one ranks PSA model elements based on their fractional contribution to the risk change, in consideration of both their individual and interaction effects. We note that this definition extends to PSA models the concept of total order importance introduced in Borgonovo (2010b) for reliability models.
- $D_l^1 := \dfrac{\Delta_l R}{\Delta R}$, where $\Delta_l R = R(x_l^1, \mathbf{x}^0_{(-l)}) - R(\mathbf{x}^0)$. By D_l^1 one obtains the fraction of ΔR associated with the individual variation of x_l.
- $D_l^{\mathcal{I}} := D_l^T - D_l^1$. By $D_l^{\mathcal{I}}$ one is informed about the fraction of ΔR contributed by x_l in its interactions with the other basic or initiating events.

Because $D_{BE_i}^T$ and $D_{IE_j}^T$ have the same units, D^T overcomes the dimensionality issues associated with B and J.

Example 2 (Example 1 continued) *Let $\Delta\mathbf{x} = (\Delta\upsilon, \Delta p)$ denote a change in the factors of Example 1. The corresponding change in risk metric is $\Delta R = \nu\Delta p + p\Delta\nu + \Delta\nu\Delta p$. The importance measures are*

$$D_\upsilon^T = \frac{p\Delta\nu + \Delta\nu\Delta p}{\Delta R} \text{ and } D_p^T = \frac{\nu\Delta p + \Delta\nu\Delta p}{\Delta R} \tag{35}$$

By D_υ^T and D_p^T one can compare the importance of basic and initiating events, because they have the same units. For instance, if $\Delta\upsilon = 0.07$ and $\Delta p = 0.08$, one obtains $D_\upsilon^T = 0.61 < D_p^T = 0.66$.

Similar considerations hold for $D_{BE_i}^1$ and $D_{IE_j}^1$, and $D_{BE_i}^{\mathcal{I}}$ and $D_{IE_j}^{\mathcal{I}}$.

As far as the computational cost of the sensitivity measures is concerned, Proposition 6 has the following implication. By construction, D_l^1 is obtained via $n+1$ model runs. Because R is multilinear, by eq. (13)

$$D_l^T = \frac{R(x_l^1, \mathbf{x}_{(-l)}^0) - R(\mathbf{x}^0)}{\Delta R} \tag{36}$$

and D_l^T can be retrieved by $n+1$ model runs. Hence, the overall cost for estimating D_l^T, D_l^1, and $D_l^{\mathcal{I}}$ is $2n+2$ model runs. This cost coincides with the cost for obtaining importance measures that are commonly estimated by standard PSA software [Smith et al (2008)] as Risk Achievement Worth (RAW) and Risk Reduction Worth (RRW) [Cheok et al (1998)]. As we mentioned, it is also the cost of tornado diagrams. However, these OFAT techniques do not convey information about interactions.

Proposition 8 has the following implications: it allows one to exclude certain types of interactions a-priori, based on the sole structure of the risk metric. However, because R is a multilinear function of $\boldsymbol{\upsilon}$ and $\mathbf{p}$, Proposition 4 applies. Thus, albeit present in the risk metric, interactions might not be numerically relevant in the application at hand, if changes are small. This raises the question of determining the magnitudes of changes $\Delta\mathbf{x} = (\Delta\boldsymbol{\upsilon}, \Delta\mathbf{p})$ that can be considered small. The proof of Proposition 4, suggests the following procedure.

1. Define an increasing sequence $\Delta^k\mathbf{x}$, $k = 1...U_{steps}$ (e.g., $\Delta^k\mathbf{x} < \Delta^{k+1}\mathbf{x}$, $\forall k$).
2. At step k, estimate $D_l^T, D_l^{\mathcal{I}}$ and $D_l^1, l = 1, 2, ..., n$.
3. At step k, compare $D_l^{\mathcal{I}}$ to $D_l^T, \forall l$. If $D_l^{\mathcal{I}} << D_l^1$ $\forall l$, then individual effects prevail. Then, go to step $k+1$ and repeat. The step k at which the condition $D_l^{\mathcal{I}} << D_l^1$ $\forall l$ is violated, determines the size of changes ($\widehat{\Delta\mathbf{x}}$) such that, if $\Delta\mathbf{x} > \widehat{\Delta\mathbf{x}}$, interactions cannot be neglected in the analysis.

The decision on whether the inequality $D_l^{\mathcal{I}} << D_l^1$ is violated can be left to experts' opinion (as we are to see in our example, this information is readily obtained by result inspection), or can be made quantitative and automated. A possible automation procedure is illustrated here. At step k, consider the quantities

$$\varepsilon_l = \frac{\left|D_l^{\mathcal{I}}\right|}{\left|D_l^T\right|}, \qquad l = 1, 2, ..., n \tag{37}$$

ε_l is the ratio of the contribution of x_l to the risk change due to interactions to its overall contribution (or, ε_l is the percentage of the importance of x_l associated with interactions.) Then, set an interaction threshold $\widehat{\varepsilon}$ (in the practice, $\widehat{\varepsilon}$ needs to be tailored to the application at hand and set by analysts and experts' opinion.) If, $\varepsilon_l \geq \widehat{\varepsilon}$ for some l, then $D^{\mathcal{I}} << D^1$ is violated at step k.

In Section 6.3, the procedure is illustrated in a quantitative fashion. In the next section, we discuss the graphical representation of results and the derivation of managerial insights.

5 Managerial Insights and Graphical Representation

In this section, we address general aspects concerning interpretation and communication of results obtained with the sensitivity measures of Section 2.

We base our discussion on the concept of sensitivity analysis setting. Settings have been first introduced in Saltelli and Tarantola (2002). A setting is a way of formulating the sensitivity analysis quest so that the goals of the sensitivity exercise are established beforehand and the insights are consistently inferred [Saltelli and Tarantola (2002)].

Because the example used in this section is a belief-network model, let us start with a brief digression about the use of sensitivity analysis in Bayesian networks. Bayesian networks are "*a compact representation of a probability distribution* [Chan and Darwiche (2002); p. 265]." They map the model inputs [conditional probability tables and instatiators in Darwiche (2003)] into the model output (a probabilistic query) through a multilinear polynomial [Darwiche (2003)]. The works of Blackmond Laskey (1995), Chan and Darwiche (2002), Chan and Darwiche (2004), Darwiche (2003), Park and Darwiche (2004), Chan and Darwiche (2005), Brosnan (2006) show that sensitivity analysis is essential in both the model building and inference-making phases. We refer to Chan and Darwiche (2005) for a thorough review of approaches to the sensitivity analysis of Bayesian networks. Brosnan (2006) comparison these alternative methods in the context of a military application. These works demonstrate the use of alternative methods as responses to several sensitivity questions. A first setting is to determine the "*amount of belief-change that occurs when moving from one distribution to another* [Chan and Darwiche (2005); p. 149]." For answering this setting, Brosnan (2006) applies sensitivity measures based on Shannon entropy and the Kullback-Leibler divergence. In Chan and Darwiche (2005), an alternative distance-based sensitivity measure is presented for overcoming limitations in the Shannon entropy and the Kullback-Leibler divergence. A second goal of the sensitivity is to *"bound the change in a probabilistic query in terms of the corresponding changes in a network parameter* [Chan and Darwiche (2002); p. 268]*."* This setting is addressed in Chan and Darwiche (2002), where the sensitivity method based on the differentiation of the belief-network polynomial. The methodology is extended for addressing changes in multiple parameters in Chan and Darwiche (2004).

Differentiation plays an important role in making inference through Bayesian networks, and thanks to the multilinearity of belief-network polynomials partial derivatives assume relevant semantics in terms of probabilistic queries [see Darwiche (2003) for a thorough overview]. In this respect, we note that, from eq. (3), it is possible to obtain the algebraic relationship between partial derivatives and the finite-change sensitivity measures introduced in this work (see also Proposition 7). However, the sensitivity measures introduced in Section 2 correspond to different sensitivity analysis settings in respect to the sensitivity measures described above. In particular, rephrasing the statement by Chan and Darwiche (2002), the sensitivity measures in eqs. (3) and (12) allow one *to explain the change in the output of a belief-network provoked by changes in the network parameters*. The following three settings can be envisioned to guide the inference of managerial insights through the β sensitivity measures [see also Borgonovo (2010a); Borgonovo (2010c)]:

Setting 1 Model structure: understanding whether model response is governed by interactions or by individual effects;

Setting 2 Direction of change: determining whether the changes in factors (groups of factors) have a positive or negative impact on the decision-support criterion;

Setting 3 Key-drivers: identifying the most relevant factors.

Setting 1 allows one to obtain information on whether the model response is additive or driven by interactions. In the latter case, the decision-making process needs to be informed by the relevance of interactions. Setting 2 is the general comparative statics question of understanding the direction of change in the output following a change in the exogenous variables [Samuelson (1947); p. 20]. Setting 3 corresponds to Eschenbach (1992)'s question of identifying the factors on which to focus managerial attention during implementation. These three settings can be seen as three questions that explode in greater detail the general sensitivity quest of [Chan and Darwiche (2005), p. 158]: *what can we say about the effect of changing some parameter x to a new value x'?*

In the belief-network context, eq. (4) represents the decomposition of the change in f due to a change in some of the network parameters. In general, these can possibly belong to different conditional probability tables and be groups of parameters [Chan and Darwiche (2004)]. In certain instances, groups are represented by entire conditional probability tables [Chan and Darwiche (2004)]. Because the sensitivity measures here introduced maintain the same properties also when extended to factor groups [Section 3], analysts can apply them at the aggregation level deemed appropriate.

Let us now come to results communication. Two graphical representations of sensitivity analysis results are widely employed: Pareto charts and tornado diagrams. Pareto charts are bar charts displaying the sensitivity measures of factors, ordered from the most to the least important [Hart and Hart (1989)]. They enable a direct identification of the factors on which to focus managerial attention. Subroutines for their creation is available in popular software.

Tornado diagrams can be attributed to Howard (1988). A tornado diagram is a graphical representation of a series of OFAT sensitivities. They are built as follows [Eschenbach (1992)]. A variation range identified by a lower and an upper limit is assigned to each factor [Eschenbach (1992)]. This is equivalent to determining three points in the input parameter space, namely the base case, $\mathbf{x}^0$, the lower limit, $\mathbf{x}^-$, and the upper limit, $\mathbf{x}^+$. One then shifts the factors one-at-a-time first from $\mathbf{x}^0$ to $\mathbf{x}^+$, and next from $\mathbf{x}^0$ to $\mathbf{x}^-$. The corresponding model output changes are registered and displayed as horizontal bars ranking from the most to the least influential. It is readily seen that the bars of a tornado diagram coincide with the first order sensitivity measures in Section 2 (ξ_i^1 in general, or β_i^1 for multilinear models.) Thus, a tornado diagram does not contain information about interactions.

Let us now illustrate how the sensitivity measures introduced in Section 2 can be cast in the form of a Pareto Chart and/or of a Tornado Diagram for inferring the sensitivity analysis insights discussed above. To support the illustration, we utilize the sample belief-network of Park and Darwiche (2004) (Figure 1).

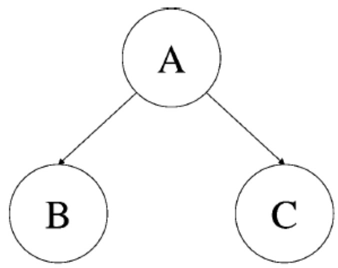

Figure 1: Belief-network from Park and Darwiche (2004), p. 199.

In Park and Darwiche (2004), the probability of getting evidence $B\overline{C}$ in the network of Figure 1 is related to the net parameters by the following multilinear equation:

$$f_{b\bar{c}} = x_1 x_2 x_3 + (1 - x_1)(1 - x_2) x_4 \tag{38}$$

where $x_1 = P(A) = 0.6$, $x_2 = P(B|A) = 0.2$, $x_3 = P(\overline{C}|A) = 0.2$, and $x_4 = P(\overline{C}|\overline{A}) = 0.85$.

Consider now a uniform shift in the probabilities with $\Delta x_i = 0.1$, $i = 1, 2, 3.4$. One has $\Delta f_{b\bar{c}} = -0.0335$. The sensitivity measures (β_i^1,$\beta_i^{\mathcal{I}}$,β_i^T) can be estimated by eqs. (16) and (14), via 10 model runs. We display them in Figure 2.

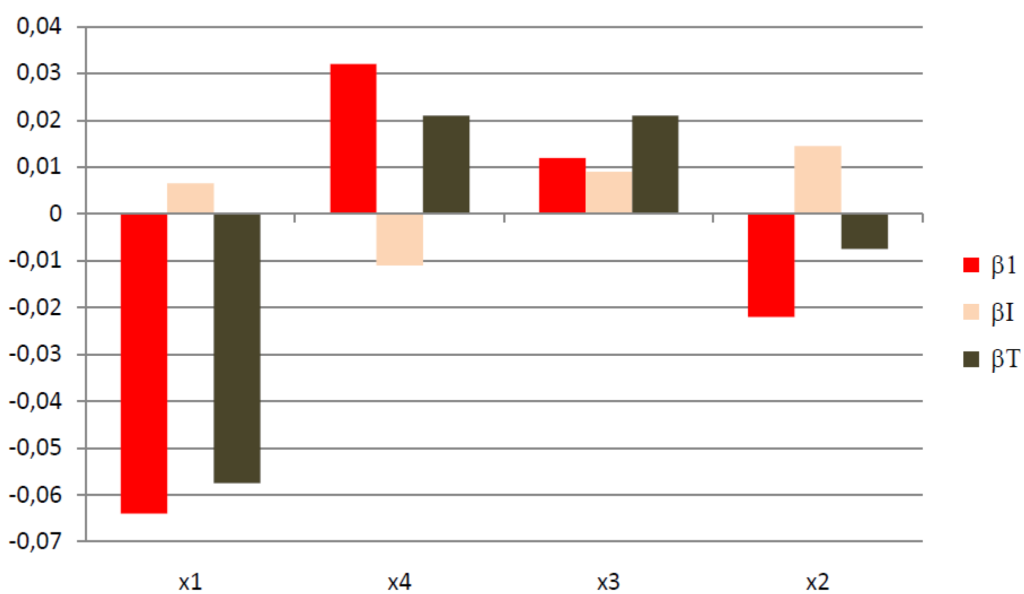

Figure 2: β_i^1, β_i^I and β_i^T arranged in the form of a Pareto Chart for the four factors. The first columns in each triplet represent β_i^1, the second column β_i^I and the third β_i^T.

In Figure 2, factors are ordered from the most to the least relevant, according to $\left|\beta_i^T\right|$. Figure

Table 1: Finite change decomposition results

	β_1^1	β_2^1	β_3^1	β_4^1	$\beta_{3,4}^2$	$\beta_{2,4}^2$	$\beta_{3,2}^2$	$\beta_{4,1}^2$	$\beta_{3,1}^2$	$\beta_{2,1}^2$
$\mathbf{x}^0 \to \mathbf{x}^+$	-0.064	-0.022	0.012	0.032	0	-0.004	0.006	-0.008	0.002	0.0105
	$\beta_{1,2,3}^3$	$\beta_{1,2,4}^3$	$\beta_{1,3,4}^3$	$\beta_{2,3,4}^3$	$\beta_{1,2,3,4}^4$					
$\mathbf{x}^0 \to \mathbf{x}^+$	0.001	0.001	0	0	0					

2 allows us obtain information corresponding to the three sensitivity settings as follows. Setting 1: by comparing the values of β_i^1 and β_i^T, one notes that the model response cannot be attributed solely to individual effects. Setting 2: by referring to the signs of the sensitivity measures, one observes that the variations in x_1 and x_2 impact $f_{b\bar{c}}$ negatively, while the variations in x_1 and x_2 impact $f_{b\bar{c}}$ positively. The model is, therefore, not a monotonic function of the input factors. Also, one notes that interactions have an opposite direction in respect of individual effects for x_1, x_4 and x_2. For these factors, then, $\beta_i^T < \beta_i^1$. Interactions, instead, augment the relevance of factor x_3. Setting 3: the most important factor is x_1, followed by x_4,x_3 and x_2.

The small number of factors allows one to obtain the complete decomposition of $\Delta f_{b\bar{c}}$ in accordance with eq. (8) with 15 model runs ($2^4 - 1$). The sensitivity measures are reported in Table 1.

The sum of the sensitivity measures in Table 1 equals $\Delta f_{b\bar{c}}$. To obtain a tornado diagram, one creates a second set of changes ($\mathbf{x}^0 \to \mathbf{x}^-$) and computes the corresponding sensitivity measures. We choose the symmetric changes ($\Delta x_i = -0.1$, $i = 1, 2, 3, 4$). By this choice, the sensitivity measures are obtained without additional calculations. In fact, Proposition 3 insures that all sensitivity measures maintain their magnitude, with the odd-order ones reversing their signs, and the even-order retaining their signs. The result is the diagram in Figure 3.

The first four bars in Figure 3 form the usual tornado diagram, as they convey the factors individual effects. The remaining eleven bars display interaction effects. In the $\mathbf{x}^0 \to \mathbf{x}^+$ shift (dark bars in Figure 3), the sum of the first order sensitivity measures equals -0.042. Thus, individual contributions account for 125% of the change. This indicates that interactions tend to oppose individual effects, compensating the 25% exceedance. The converse happens in the $\mathbf{x}^0 \to \mathbf{x}^-$ shift, where individual effects account for around 90% of the change and interaction effects contribute the remaining 10%. The most important interaction is the one between x_1 and x_2, with $\beta_{2,1}^2$ one order of magnitude higher than the remaining. Finally, note that because there is no term in eq. (38) containing the product of factors 3 and 4, $\beta_{3,4}^2 = 0$ (see item 2 in Proposition 2) and, in accordance with Corollary 1, all higher order terms containing these two factors are null ($\beta_{1,3,4}^3 = \beta_{2,3,4}^3 = \beta_{1,2,3,4}^4 = 0$.)

The next section presents the application of our findings in an operational case study characterized by a complex numerical code.

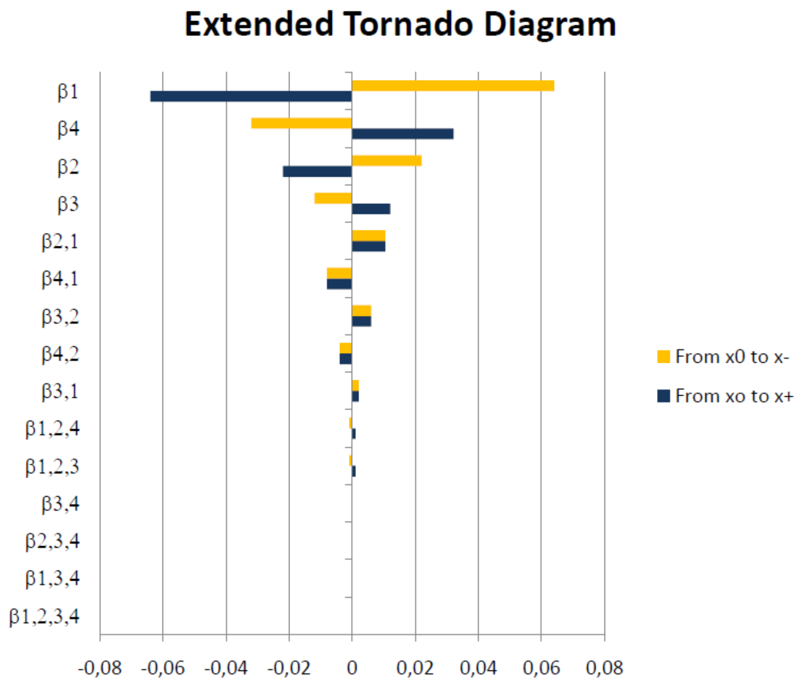

Figure 3: $\beta_{i_1,i_2,..,i_k}$ arranged in an Extended Tornado graph for the decomposition of the finite change $\Delta f_{b\bar{c}}$. Light-coloured bars refer to the shift from $\mathbf{x}^0$ to $\mathbf{x}^-$, dark bars refer to the shift from $\mathbf{x}^0$ to $\mathbf{x}^+$.

6 Case Study: A PSA model for Space Missions

This section demonstrates the use of the previous findings and methodology in the context of a complex operational decision-problem: the design and planning of a lunar space mission.

The National Aeronautics and Space Administration (NASA) has adopted probabilistic safety assessment (PSA) as part of its risk management procedures since 2002. The PSA exercise discussed in this section is part of the comprehensive investigation launched by NASA within its program for the next generation of space missions. Specifically, the model was developed for providing decision-makers with insights to be used for improving risk management and enhancing both mission performance and safety. The PSA model is the result of the work of two teams. The teams worked together to develop consistent modeling and data analysis methods, and to produce an approach to be used as a basis for an Agency-wide PSA for the entire life-cycle of the missions.

The model building process utilizes NASA's Probabilistic Risk Assessment Procedures Guide as the technical basis for the approach [Stamatelatos et al (2002)]. A mission is evaluated according to two aspects: performance and safety. The modeling process starts by representing the launch formulation stage. Following launch, two different end states are considered: loss of crew (LOC), as a safety measure, and loss of mission (LOM), as a performance measure. LOC addresses conducting the mission and returning the crew safely (or not) to earth. LOM addresses performance required to successfully (or not) carry out the lunar activities. Each key part of the mission is decomposed into a fault tree with its top-event representing either LOM or LOC. Note that LOM and LOC are not mutually exclusive. A LOM might result in an attempted return-to-Earth. If the abort or return-to-Earth and landing fail, the LOM leads to an LOC. The details of what has to fail in order to cause either LOM or LOC are addressed in the fault tree for each phase.

Figure 4 shows that the mission is covered by eight phases from launch to earth return.

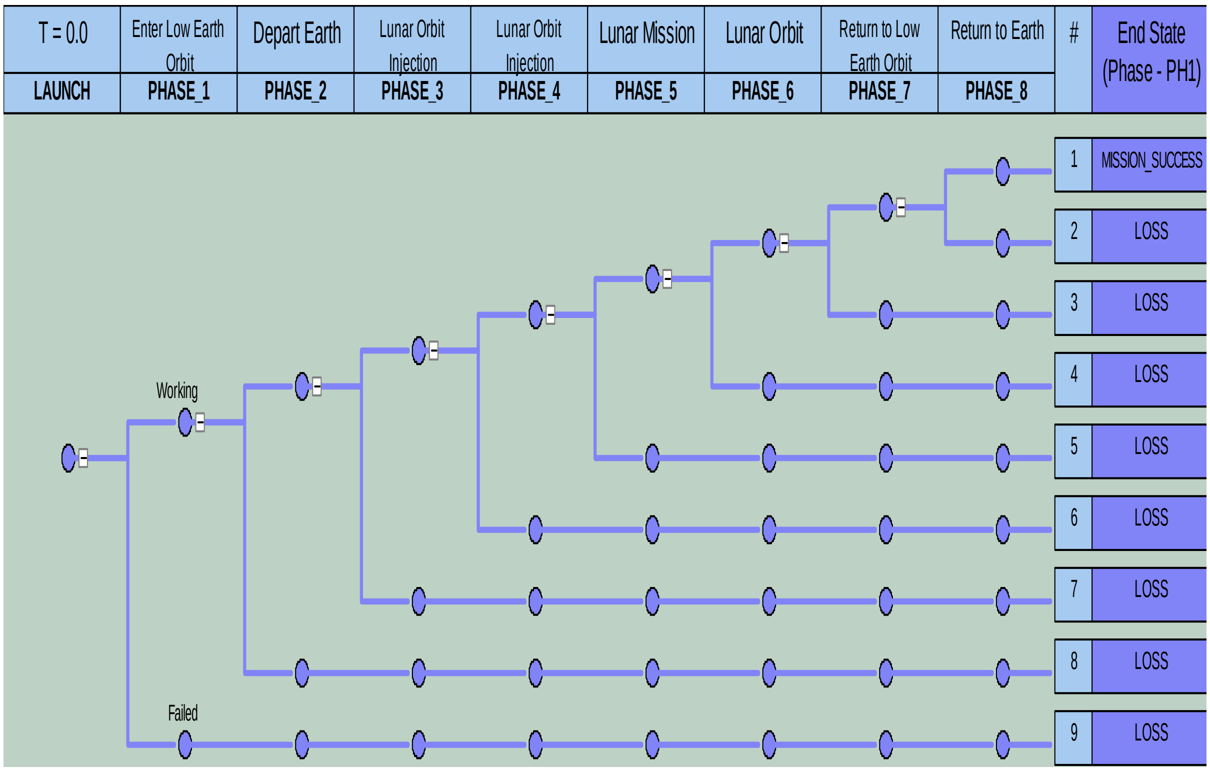

Figure 4: Mission phases for the lunar mission event tree.

The phases are (Figure 4):

Phase 1 – Launches to Low Earth Orbit (LEO). The mission will launch needed materials, vehicles, and the crew to LEO.

Phase 2 – Rendezvous of the vehicles and the crew. The earth departure vehicle is used by the crew to start the journey to the moon.

Phase 3 – Lunar-Orbit Insertion (LOI). The vehicle performs the LOI. All members of the crew transfer to the vehicle going to the lunar surface.

Phase 4 – Vehicle in lunar orbit. The unmanned vehicle remains in a lunar orbit.

Phase 5 – Lunar mission. The crew descends to and from the lunar surface. A typical lunar mission will last up to seven days.

Phase 6 – Re-crew the orbiting vehicle. Following the lunar ascent, the crew docks with the vehicle.

Phase 7 – Return to Earth. The crew returns to earth.

Phase 8 – Earth landing. The crew lands on the Earth.

The model contains seven primary systems that perform the major functions for the mission. These systems are typical of NASA vehicles such as those described by the Exploration Systems Architecture Study NASA (2005) and include:

- Propulsion including the main engine, reaction control system (RCS), mechanical equipment (pumps, valves and controllers), and the propellant/helium tanks.
- Avionics system that receives inputs from the crew, sensors and external communications; perform navigation, guidance, and internal state calculations; and provides control and actuation signals.
- Electric Power System including batteries, solar arrays and electrical distribution and control subsystems.
- Active Thermal Control System including heaters, coolers, condensate controller and mechanical equipment.
- Environmental Control and Life Support System including oxygen tanks, pressure regulators, sensors and mechanical equipment.
- Launch Abort System.
- Pyrotechnic devices that affects component separation.

Since we used a "phased" approach to decomposing portions of the mission, we needed to develop fault trees for each system for each phase. More specifically, under each top event in the event tree, a fault tree is created representing the down branch (i.e., failure) for that specific top event. Once a failure is seen in any one particular phase (say, for example, during Phase 3), that accident sequence goes directly to the LOSS end state (LOM or LOC depending on which one is under scrutiny). The system models in this PSA (i.e., the fault trees) used typical high-level conceptual train design for the respective systems. Included in the system modeling was the interconnected nature of the various systems and subsystems both within a mission phase and across the mission profile. For example, the thermal control system may be dependent on electric power – this dependency is captured in the fault tree logic models. For each system, we assumed a typical operating condition along with the definitions of LOC and LOM in order to create the corresponding failure logic.

In addition to representing assumed system failures via standard fault tree modeling, the model includes dependent failures using common cause failure modeling. For the redundant components in the system fault trees, we model the probability of experiencing a common cause failure using the Multiple Greek Letter method [Mosleh et al (1998)]. The Multiple Greek Letter method is an example of parametric common cause failure modeling that is an extension of the Beta Factor method. This method (as compared to the Beta Factor method) is used to explicitly account for higher order redundancies and to allow for common-cause subgroups.[4]

The resulting model contains 150 fault trees and approximately 800 basic events. The number of MCSs is around 4446 at a truncation of 10^{-15}.

In the next section, we discuss how the importance measures described in Sections 4, and 3 help us to produce insights into what are the important drivers of the overall mission design.

6.1 Interactions at the Basic Event Level

In this section, we present the results of the numerical analysis of interactions carried out at the basic event level. The names of systems and basic events and some details of the analysis results are not provided for in this paper to protect privacy – instead specific basic events are denoted by numbers (e.g., the first event is 1, the second is 2, and so on).

The expression of the risk metric (R) for the PSA model described in Section 6 is obtained in SAPHIRE 7 (Smith et al (2008)) by resolving the MCS's. After truncation at 10^{-15}, 393 basic events survive. The risk metric change of interest (ΔR) is provoked by a finite variation ($\mathbf{\Delta x}$) of the PSA model elements. The variation range is assigned by analysts in such a way to be consistent with the uncertainty in the values of the basic event probabilities and initiating event frequencies. The actual numerical value of ΔR is not reported for privacy reasons.

The sensitivity measures D_l^T for all $\mathbf{p}$'s and $\boldsymbol{\nu}$'s are estimated by utilizing eq. (13) [for the algorithm, see Borgonovo (2010b) and Borgonovo (2010c)]. The algorithm is then augmented so that by additional $n+1$ model runs, one obtains $D^{\mathcal{I}}$ and D^1. Overall, 788 model evaluations are performed to compute D^T, $D^{\mathcal{I}}$ and D^1 for all PSA model elements taking a few seconds on a personal computer. The corresponding Pareto chart is reported in Figure 5. Because of space constraints, only the D_l^T, $D_l^{\mathcal{I}}$ and D_l^1 of the first ten most important basic events are displayed in Figure 5.

Let us now utilize Figure 5 to address the three sensitivity analysis settings of Section 5. Concerning setting 1, model structure, an examination of the interaction contribution of all 393 factors (as said, not displayed due to space constraints), with an interaction threshold $\widehat{\varepsilon} = 10\%$ [eq. (37)], shows that the condition $D_l^{\mathcal{I}} << D_l^1$ is violated in 342 instances. This means that only 51 basic events impact the risk change individually. Therefore, interactions play a crucial role in the risk change. In particular, Figure 5, show that out of the ten most important basic events, seven have their importance driven by interactions ($D_l^{\mathcal{I}} << D_l^1$).

Concerning setting 2, direction of change, by the sign of the importance measures in Figure 5,

[4] As an example, the failure probability of a component in a group of three due to individual and common cause failures is written as [Mosleh et al (1998), p. 263, eq. (8)]

$$x_c = x_1 + 2x_2 + x_3 \tag{39}$$

where x_1, x_2 and x_3 are the probabilities of individual failure, common cause failure with either one of the other two components and failure due to common cause of all the three components, respectively. In turn, x_1, x_2 and x_3 are estimated as $x_1 = (1-\beta)x_c$, $x_2 = (1-\gamma)\beta x_c$, $x_3 = \gamma\beta x_c$, where β and γ represent conditional failure probabilities given that one or two (respectively) of the other components in the group have failed.

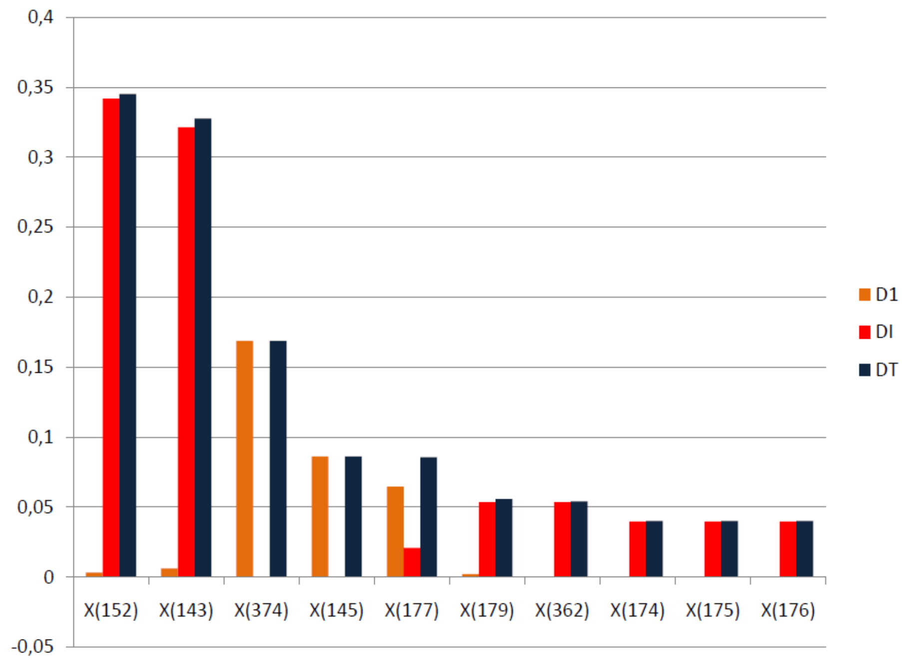

Figure 5: Pareto Chart for the most important PSA model elements. Left bars: D_i^1. Central Bars: D_i^I. Right Bars: D_i^T.

one notes that all individual, interaction and total contributions, are positive. This testifies the monotonicity of R as a function of the basic event probabilities, which stems from the coherency of the system under investigation.

Concerning setting 3, namely, key-driver identification. The proper sensitivity measure for this task is D_l^T. The values of D_l^T of the ten most important basic events are reported in the dark bars at outmost right in each triplet of Figure 5. The most important factor is basic event 152, which contributes to around 35% of the change in risk metric. The contribution is mainly driven by its interactions with the other PSA model elements, as testified by the very low value of its first order sensitivity measure ($D_{152}^1 = 0.0034$). The second most important factor is basic event 143, that accounts for $\sim 33\%$ of the change. Note that its individual contribution is only the 0.6% of the risk change ($D_{143}^T = 0.006$). The third most important basic event, basic event 374, instead, contributes individually to the change in risk, with $D_{374}^T = D_{374}^1 = 0.168$. Let us now come to the risk management implications of these values. By Remark 1, D_l^T is the fraction of the risk change that would be prevented, if basic event probability (or initiating event frequency) l is not allowed to change. Thus, a policy that focusses on basic event 152 has the potential of preventing a risk increase of 34.5%. The second most important basic event on which to focus attention, would be basic event 143, for a potential prevention of 33% of the risk change. Conversely, if one were to select the main risk contributor by the individual sensitivity measures, D_l^1 (as in a tornado diagram), one would indicate basic event 374 as the factor on which to focus managerial attention. However, such a policy would be capable of preventing a risk change of 16% at most.

We note that, by knowledge of the specific nature of the basic events, managers obtain insights on the proper actions to be adopted for insuring that system performance is achieved. As an example, suppose that basic event 152 were related to operator actions (the exact nature of basic event 152 is not revealed for privacy). Specific training would then avoid the increase in p_{152} (or even lead to its reduction), thus eliminating one of the main risk contributors.

In the next section, we presents the results of the analysis at the system level.

6.2 Interactions at the System Level

In this section, we present results at the system level. 15 systems of interest are identified. The 393 basic event probabilities are aggregated in corresponding groups. Groups, therefore, differ in size depending on the number of basic events that describe a given system in the model.

The number of systems ($Q = 15$) makes it is possible to compute all the terms in eq. (25) according to the algorithm proposed by Borgonovo and Peccati (2009). In particular, $2^{15} = 32768$ model evaluation are necessary, taking around one hour on a personal computer. Thus, we have available the $D^T_{\gamma_i}$ importance of all systems, all their individual importance measures ($D^T_{\gamma_i}$), and the interaction effects ($D^k_{i_1,i_2,\ldots,i_k}$) of all orders [eq. (25)].

Figure 6 displays the sensitivity measures of orders 1 and 2. The reason why we limit reporting to these terms is that no significant interactions of order higher than 2 are registered.

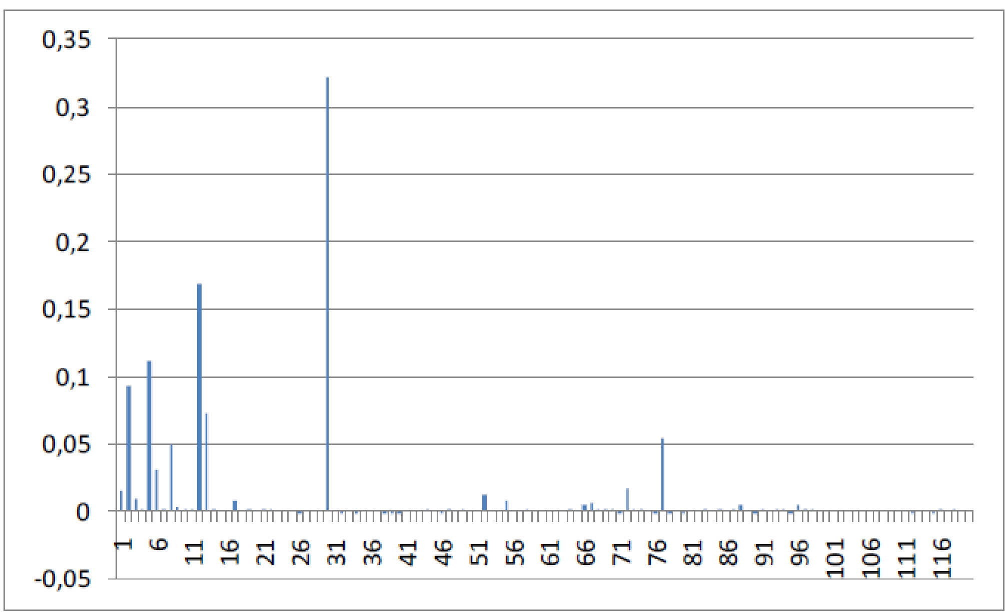

Figure 6: Importance measures up to second order interactions. The first 15 terms concern the importance of each of the 15 systems individually. The terms from 15 to 120 report the importance of the interactions between pairs.

The first 15 terms in Figure 6 portray $D^1_{\gamma_i}$ the individual system contribution to the changes. The terms from 16 to 120 concern the interactions between all possible system pairs [$\binom{15}{2} = 105$].

Figure 6 shows that the most important risk contributor is an interaction of order 2, and in particular the interaction between systems 2 and 3.

The importance measure triplets $D^T_{\gamma_i}, D^{\mathcal{I}}_{\gamma_i}, D^1_{\gamma_i}$ for the 15 systems are reported in Figure 7.

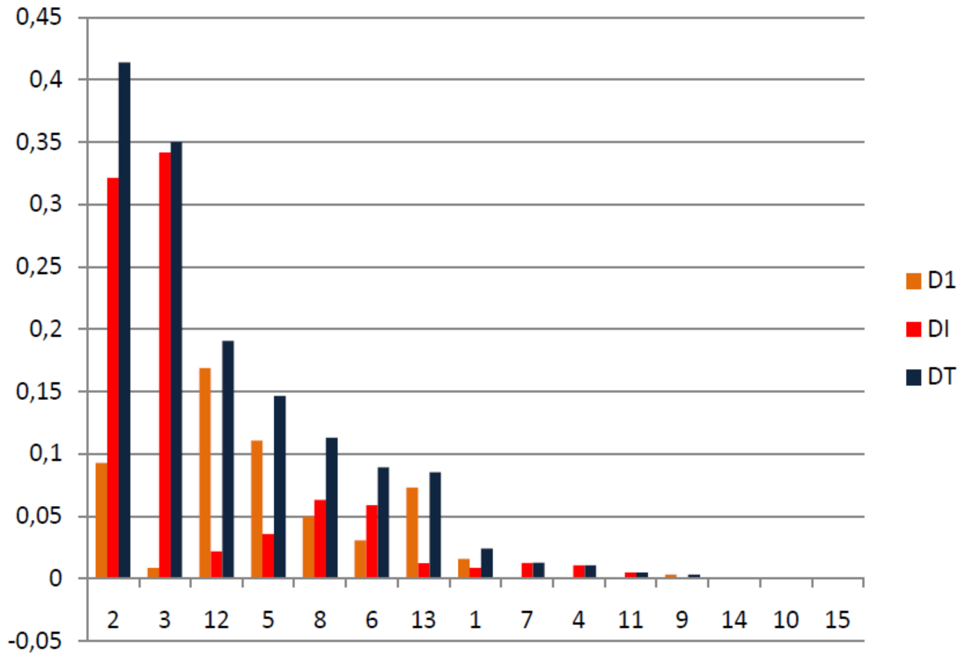

Figure 7: D^1 (left at each column triplet), D^I (middle) and D^T for the 15 systems.

As far as setting 1 is concerned, Figure 7 shows that $D^{\mathcal{I}}_{\gamma_i} > D^1_{\gamma_i}$ for systems $2, 3, 8, 6, 7, 4, 11$, while $D^{\mathcal{I}}_{\gamma_i} < D^1_{\gamma_i}$ for systems $12, 5, 13, 1, 9$. Thus, the contributions to the risk change of systems $2, 3, 8, 6, 7, 4, 11$ are mainly driven by their interactions in the other systems, while the contributions of systems $12, 5, 13, 1, 9$ are, instead, individual. Thus, at the system level, interactions still play an important role. As far as setting 2 is concerned, the signs of the importance measures in Figure 7 denote that all increases in basic event probabilities or initiating event frequencies lead to an increase in the risk metric, consistently with the fact that the system under investigation is coherent. As far as setting 3 is concerned, the magnitudes of $D^T_{\gamma_i}$ in Figure 7 indicate that system 2 is the most important one, followed by systems $3, 12, 5, 8, 6, 13, 1, 7, 4, 11, 9, 10, 14, 15$. In particular, system 2 is associated with around 41% of the risk change, system 3 with around 35%, system 12 with around 17% of ΔR (etc.). By examining the values of D^T and $D^{\mathcal{I}}$ for each system, one notes that the importance of systems 2 and 3 is mainly due to their interaction, with system 3's individual contribution being rather small. This result has the following interpretation: deteriorations that contemporarily affect systems 2 and 3 are amplified by their interactions. Conversely, interventions that are able to fix the reliability of either system 2 or 3 would make their interaction disappear, eliminating the main source of increase in risk.

6.3 Determining the Interaction Threshold

Proposition 4 states that interactions do not necessarily matter in applications in which changes are small, when the model is multilinear. In Section 4, we have set forth a procedure for detecting the interaction threshold in PSA models. In the next paragraphs, we illustrate the numerical results obtained by applying the procedure to the space PSA model under investigation.

Step 1. To obtain the increasing sequence $\Delta^j \mathbf{x}$, we proceed as follows. We note that lognormal distributions are utilized in the PSA to model uncertainty in the parameters (this is a typical PSA choice). We then exploit the link between $\Delta \mathbf{x}$ and the error factor (EF)[5] of a lognormal distribution, namely

$$\Delta \mathbf{x} = \mathbf{x}_{0.95} - \mathbf{x}_{0.05} = 2 \cdot EF \cdot \mathbf{x}_{0.5} \tag{40}$$

A small EF implies a small separation between the 5^{th} and 95^{th} percentiles. Consequently, $\Delta \mathbf{x} \simeq \mathrm{d}\mathbf{x}$, and $\Delta R = R(x_{0.95}) - R(x_{0.05}) \simeq \mathrm{d}R$. As EF increases the separation between the 5^{th} and 95^{th} percentile increases, $\Delta \mathbf{x}$ increases and ΔR becomes finite. By defining an increasing sequence of $EF's$, we obtain a sequence of risk metric changes. For the model at hand, results are reported in Figure 8.

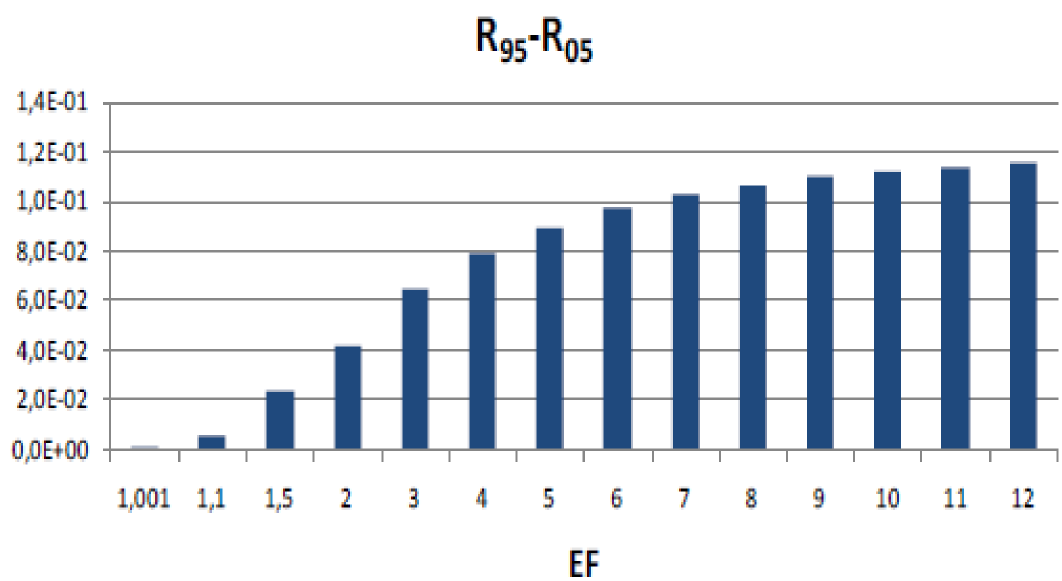

Figure 8: ΔR as the error factor (EF) increases.

Figure 8 displays ΔR as EF is increases from 1.001 to 12 ($U_{steps} = 14$ in this case.) One notes the monotonic increase of ΔR with EF.

Step 2. At each k, we obtain D_l^T, $D_l^{\mathcal{I}}$ and D_l^1 for all 393 PSA model elements. As a reference, Figure 9 reports the values of D_{152}^T, $D_{152}^{\mathcal{I}}$ and D_{152}^1 as EF varies.

[5] The error factor is given by the ratio between the median and the 5^{th} percentile of the distribution, which, in the lognormal case, also equal the ratio of the 95^{th} percentile to the median.

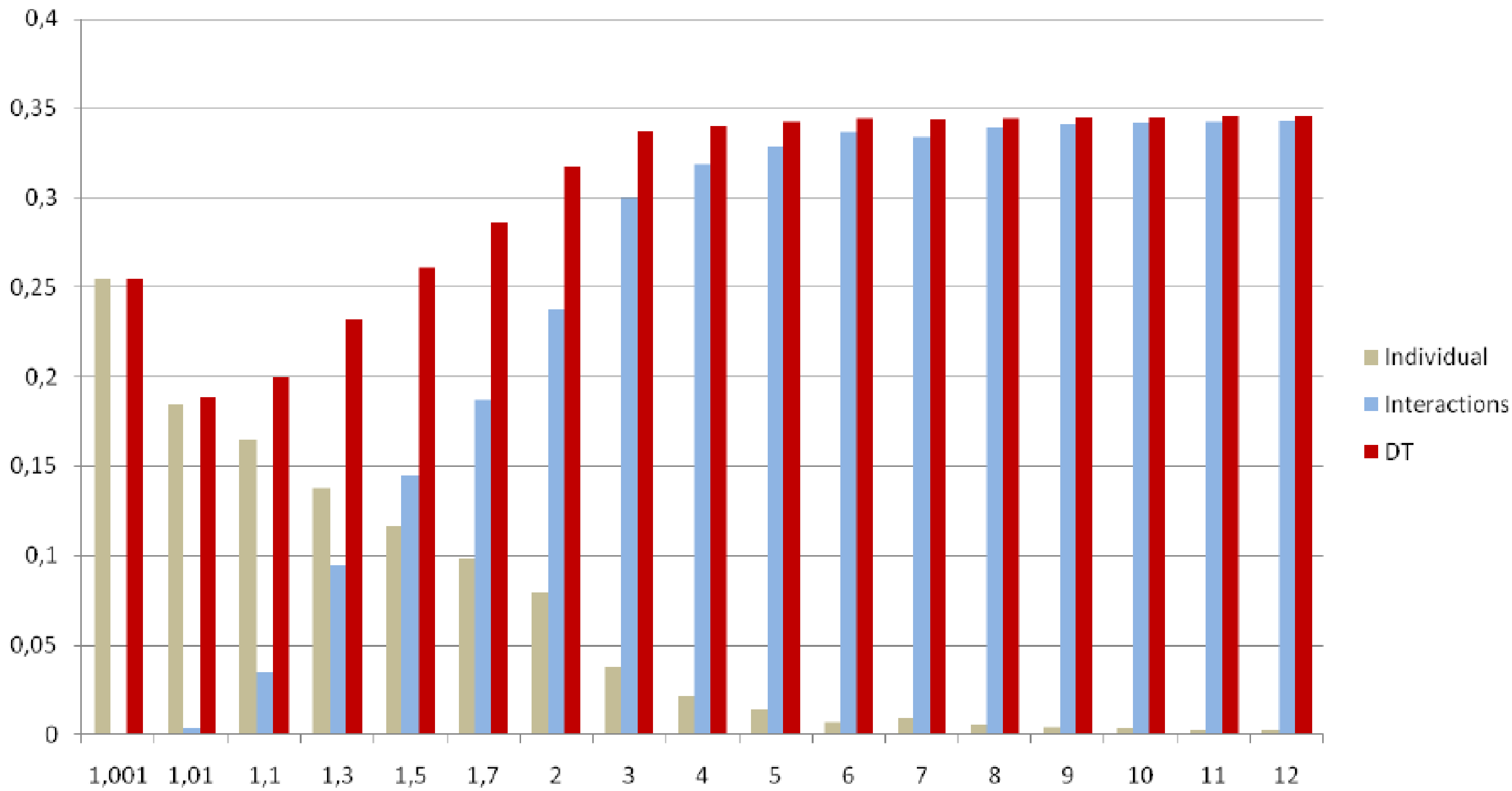

Figure 9: D^1_{152} (left bar in each group of three), D^I_{152} (middel bar) and D^T_{152} (right bar) as d$\mathbf{p}$ and d$\boldsymbol{\nu}$ increase.

By Figure 9, one notes that individual effects prevail over interactions only when EF is smaller than 1.01. For $EF > 1.01$, interaction effects cannot be neglected, and they dominate model behavior as soon as $EF > 1.7$. Conversely, individual effects dominate for $EF < 1.01$. By setting a threshold of $\widehat{\varepsilon} = 10\%$ (see Section 37), for the space mission at hand, we obtain that the condition $D_i^{\mathcal{I}} << D_i^1$ is violated at $k = 3$, corresponding to an $EF = 1.01$. By eq. (40), the corresponding changes $\Delta\mathbf{x}$ are determined. On average, a change in basic event probability of 0.0040 would trigger the presence of interactions. Hence, if the application at hand involves changes in the mission basic event probabilities greater than (indicatively) 0.004, interactions cannot be neglected in judging the effect of the proposed change. Conversely, if the application involves changes smaller than 0.004, it might be treated without giving explicit consideration to interactions, although they are formally present in the risk metric expression.

7 Conclusions

We have proposed an approach for determining the relevance of interactions in operational decision problems.

Our investigation has started with an in-depth analysis of interactions in multilinear models (f). The equivalence of the integral and differential decomposition of a finite change (Δf) has allowed us to introduce sensitivity measures that dissect Δf in the factors' individual and interaction contributions exactly. We have shown that the sensitivity measures can be obtained at the same computational cost of OFAT methods, what makes the approach applicable to full-fledged models.

We have investigated the implications of these findings in reliability and PSA models. We have seen that the sensitivity measures allow one to obtain a unique ranking of PSA model elements (both basic and initiating events), overcoming unit of measure limitations of traditional PSA importance measures. They also allow one to identify the interactions that can be a-priori excluded from the analysis.

Two issues relevant in real-life applications have been addressed next. First, we have proposed a procedure for the identification of the magnitude of changes above which decision-makers need to be informed by the relevance of interactions. Second, we have provided a formal way of extending the sensitivity measures to factor groups. This grants decision-makers with the flexibility of choosing the most suitable aggregation level of analysis.

The inference of managerial insights by the aid of sensitivity analysis settings (model structure, direction of change and key-drivers) has been illustrated next. A belief-network example has been used to demonstrating how these insights can be obtained by casting the sensitivity measures in the form of Pareto charts and extended tornado diagrams.

We have discussed quantitative insights by applying the methodology in the context of a complex operational problem, namely the design phase of space missions. The sensitivity measures have been estimated for a PSA model developed for NASA containing 399 basic events and 4446 MCSs. The analysis has been carried out both at the basic event and at the system levels. Numerical results reveal that the quantification of interactions is crucial in providing guidance to risk managers for correctly identifying the systems and components on which to focus managerial attention towards insuring that target system performance is achieved.

While our application refers to a space PSA, the methodology developed here is applicable to the different engineering sectors where PSA is used (nuclear, chemical, energy and aeronautic industries), and to other O.R. models possessing a multilinear structure.

Acknowledgement 1 *The authors wish to thank the Editor and the Associate Editor for the careful editorial assistance. They also wish to thank the anonymous referees for very perceptive suggestions which have greatly contributed in improving the manuscript. Financial support from the Faculty Staff Exchange program of the Idaho National Laboratory is gratefully acknowledged by the authors. E. Borgonovo also gratefully acknowledges financial support from the ELEUSI Research Center of Bocconi University.*

8 Appendix A: Proofs

Proof of Proposition 1. For notation simplicity, we discuss the equality of $\beta^2_{i,j}$ and $\xi^2_{i,j}$. A similar argument applies to higher order terms. We start with recalling that

$$f''_{i,j} := \lim_{\Delta x_i \Delta x_j \to \mathbf{0}} \frac{g(x_i+\Delta x_i, x_j+\Delta x_j, \mathbf{x}_{(-i,j)}) - g(x_i+\Delta x_i, \mathbf{x}_{(-i)}) - g(x_j+\Delta x_j, \mathbf{x}_{(-j)}) + g(\mathbf{x}_{(-j)})}{\Delta x_i \Delta x_j} \tag{41}$$

Given a multilinear function f, it can be reworked as a function of x_i and x_j as

$$f = a(\mathbf{x}_{-(i,j)})x_i + b(\mathbf{x}_{-(i,j)})x_j + c(\mathbf{x}_{-(i,j)})x_i x_j + d(\mathbf{x}_{-(i,j)}) \tag{42}$$

where $a(\mathbf{x}_{-(i,j)})$, $b(\mathbf{x}_{-(i,j)})$ and $c(\mathbf{x}_{-(i,j)})$ are in general multilinear functions not depending on x_i and x_j. By eqs. (41) and (42)

$$f''_{i,j} = c(\mathbf{x}_{-(i,j)}) \tag{43}$$

Now, we show that for a multilinear function $\xi_{i,j} = c(\mathbf{x}_{-(i,j)})\Delta x_i \Delta x_j$. By eq. (9), we have:

$$\xi_{i,j} = \begin{array}{c} a(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i) + b(\mathbf{x}_{-(i,j)})(x_j+\Delta x_j) + c(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i)(x_j+\Delta x_j) + d(\mathbf{x}_{-(i,j)}) \\ -a(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i) - b(\mathbf{x}_{-(i,j)})x_j - c(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i)x_j - d(\mathbf{x}_{-(i,j)}) \\ -a(\mathbf{x}_{-(i,j)})x_i - b(\mathbf{x}_{-(i,j)})(x_j+\Delta x_j) - c(\mathbf{x}_{-(i,j)})x_i(x_j+\Delta x_j) - d(\mathbf{x}_{-(i,j)}) \\ +a(\mathbf{x}_{-(i,j)})x_i + b(\mathbf{x}_{-(i,j)})x_j + c(\mathbf{x}_{-(i,j)})x_i x_j + d(\mathbf{x}_{-(i,j)}) \end{array} \tag{44}$$

Here one can simplify all the $d(\mathbf{x}_{-(i,j)})$ terms, the terms $a(\mathbf{x}_{-(i,j)})x_i$ and $b(\mathbf{x}_{-(i,j)})x_j$ obtaining

$$\begin{gathered} a(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i)+b(\mathbf{x}_{-(i,j)})(x_j+\Delta x_j)+c(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i)(x_j+\Delta x_j) \\ -a(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i)-c(\mathbf{x}_{-(i,j)})(x_i+\Delta x_i)x_j \\ -b(\mathbf{x}_{-(i,j)})(x_j+\Delta x_j)-c(\mathbf{x}_{-(i,j)})x_i(x_j+\Delta x_j)+c(\mathbf{x}_{-(i,j)})x_ix_j \end{gathered} \tag{45}$$

Then, by expanding and simplifying, one gets

$$\begin{gathered} \xi_{i,j} = \\ a(\mathbf{x}_{-(i,j)})x_i+a(\mathbf{x}_{-(i,j)})\Delta x_i+b(\mathbf{x}_{-(i,j)})x_j+b(\mathbf{x}_{-(i,j)})\Delta x_j+c(\mathbf{x}_{-(i,j)})x_jx_i+c(\mathbf{x}_{-(i,j)})\Delta x_jx_i+c(\mathbf{x}_{-(i,j)})\Delta x_ix_j \\ +c(\mathbf{x}_{-(i,j)})\Delta x_i\Delta x_j-a(\mathbf{x}_{-(i,j)})x_i+a(\mathbf{x}_{-(i,j)}\Delta x_i-c(\mathbf{x}_{-(i,j)})x_ix_j-c(\mathbf{x}_{-(i,j)})\Delta x_ix_j \\ -b(\mathbf{x}_{-(i,j)})x_j-b(\mathbf{x}_{-(i,j)})\Delta x_j-c(\mathbf{x}_{-(i,j)})x_ix_j-c(\mathbf{x}_{-(i,j)})x_i\Delta x_j+c(\mathbf{x}_{-(i,j)})x_ix_j \\ =c(\mathbf{x}_{-(i,j)})\Delta x_i\Delta x_j \end{gathered} \tag{46}$$

By eq. (43), we get $\xi_{i,j} = f''_{i,j}\Delta x_i\Delta x_j = \beta_{i,j}$. ■

Proof of Proposition 3. Follows by eq. (3). ■

Proof of Proposition 4. 1) Because f is differentiable, $\Delta f \to \mathrm{d}f$ as $\Delta\mathbf{x} \to \mathbf{0}$. Because $\mathrm{d}f$ is additive, interactions do not appear in the response of f as $\Delta\mathbf{x} \to \mathbf{0}$.

2) By eq. (2), $\Delta_i^T f \to \Delta_i f \to \mathrm{d}_i f$ and $\Delta f \to \mathrm{d}f$ as $\Delta\mathbf{x} \to \mathbf{0}$. Hence,

$$\lim_{\Delta\mathbf{x}\to 0} D_i^T = \frac{\mathrm{d}_i f}{\mathrm{d}f} = DIM_i \tag{47}$$

$\frac{\mathrm{d}_i f}{\mathrm{d}f}$ is the differential importance of x_i, namely, DIM_i [Borgonovo and Apostolakis (2001)]. ■

Proof of Proposition 5. By item 2 in Proposition 4 $D_i^T = DIM_i$ for small changes. By $DIM_\gamma = \sum_{r=1}^{s} D_{i_r}^T$ [Borgonovo and Apostolakis (2001)], eq. (27) follows. ■

Proof of Proposition 6. By the functional form of eq. (28), $R(\boldsymbol{\upsilon}, \mathbf{p})$ is multilinear if $P(\phi = 1|IE_j)$ is. Letting $\mathbf{M} = (M_1, M_2, ..., M_m)$ denote the collection of minimal cut sets (m denotes the number of MCS's), one writes

$$P(\phi = 1|IE_j) = P(M_1 \cup M_2 \cup ... \cup M_m|IE_j) \tag{48}$$

since the probability of the end-event is the probability of any of the MCS's happening. Let $(BE_{s_1}, BE_{s_2}, ..., BE_{s_{m_s}})$ denote the collection of basic events associated with the s^{th} MCS ($s = 1, 2, ..., m$). By Theorem 1 in Borgonovo (2010b), $P(\phi = 1|IE_j)$ is a multilinear function of the conditional BE probabilities. One therefore writes

$$R(\boldsymbol{\upsilon}, \mathbf{p}) = \sum_{j=1}^{n_{IE}} \upsilon_{IEj}\Big(\sum_{r=1}^{m} \sum_{m_{i_1}<m_{i_2}<...<m_{i_r}} \prod_{s_1=1}^{m_1} p_{i_{s_1}} \prod_{\substack{s_2=1 \\ p_{i_{s_2}}\neq p_{i_{s_1}}}}^{m_2} p_{i_{s_2}} \cdot ... \cdot \prod_{\substack{s_r=1 \\ p_{i_{s_r}}\neq ...\neq p_{i_{s_2}}\neq p_{i_{s_1}}}}^{m_r} p_{i_{s_r}}\Big) \tag{49}$$

By eq. (49), one concludes that the PSA risk-metric is a multilinear function of $\mathbf{p}$ and $\boldsymbol{\upsilon}_{IE}$. ■

Proof of Corollary 2. Follows by the multilinearity of eq. (6). ■

Proof of Proposition 7. Item 1 follows by the first order term of eq. (29) and eq. (3). Item 2 follows by the higher order terms in eqs. (29) and eq. (3). Item 3 follows by items 1,2 and eq. (12). ■

Proof of Proposition 8. 1) Because $B_{IE_j} = \dfrac{\partial R}{\partial \upsilon_j}$, by differentiationg eq. (49), one obtains $B_j = P(\phi = 1|IE_j)$.

2) Let us consider the J of two initiating event frequencies. By definition, it is

$$J_{IE_s,IE_m} := \frac{\partial B_{IE_s}}{\partial \upsilon_s \partial \upsilon_m} \tag{50}$$

Consequently,

$$J_{IE_s,IE_m} = \frac{\partial B_{IE_s}}{\partial \upsilon_m} = \frac{\partial^2 R}{\partial \upsilon_s \partial \upsilon_m} = \frac{\partial}{\partial \upsilon_m} P(\phi = 1|IE_j) = 0 \tag{51}$$

Eq. (51) states that the joint importance of two initiating events is null. By a property of multilinear functions [see, for instance, Section 2 in Borgonovo (2010c)], if a mixed partial derivative with respect to a given group of variables is null, then all partial derivatives of higher order containing the same group of variables are null. Therefore, any joint reliability importance of any group of PSA model elements containing at least two initiating event frequencies is null.

3) Let $i_1, i_2, ..., i_{m_k}$ be the set of indices of the basic event probabilities in minimal cut set M_k. Let S be the set of indices of initiating events that have M_k as minimal cut set in the corresponding sequences. (We allow for the possibility of a MCS to appear in different sequences, as this is the most general case). Then, let $\widehat{R}$ be the risk metric truncated by the rare event approximation and $\widehat{R} \simeq R$. Under the rare event approximation, R is substituted by $\widehat{R}$. Hence, one computes $J_{i_1,i_2,...,i_{m_k}}$ by $J_{i_1,i_2,...,i_{m_k}} = \dfrac{\partial \widehat{R}}{\partial p_{i_1} \partial p_{i_2} ... \partial p_{i_{m_k}}} = \displaystyle\sum_{s \in S} \upsilon_s \dfrac{\partial P(\phi = 1|IE_j)}{\partial p_{i_1} \partial p_{i_2} ... \partial p_{i_{m_k}}}$. By Proposition 5 in Borgonovo (2010b), it is $\dfrac{\partial P(\phi = 1|IE_j)}{\partial p_{i_1} \partial p_{i_2} ... \partial p_{i_{m_k}}} = 1$ and one obtains $J_{i_1,i_2,...,i_{m_k}} = \displaystyle\sum_{s \in S} \upsilon_s$. This proves the first part of Point 3. The second part is proven as follows. Consider $\dfrac{\partial J_{i_1,i_2,...,i_{m_k}}}{\partial x_s}$. If x_s is a basic event probability, then $\dfrac{\partial J_{i_1,i_2,...,i_{m_k}}}{\partial x_s} = 0$, by definition of MCS. If x_s is an initiating frequency, then $\dfrac{\partial J_{i_1,i_2,...,i_{m_k}}}{\partial x_s} = 1$, provided that $s \in S$. Hence, all the partial derivatives from order $M_k + 2$ ($J^{k+2}_{i_1,i_2,...,i_{K_M+2}}$) on are null, where K_M is the order of the largest MCS. ■

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