Sensitivity Analysis with Finite Changes: an Application to Modified EOQ Models

by

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Abstract

In this work, we introduce a new method for the sensitivity analysis of model output in the presence of finite changes in one or more of the exogenous variables. We define sensitivity measures that do not rest on differentiability. We relate the sensitivity measures to classical differential and comparative statics indicators. We prove a result that allows us to obtain the sensitivity measures at the same cost of one-variable-at-a-time methods, thus making their estimation feasible also for computationally intensive models. We discuss in detail the derivation of managerial insights formulating a procedure based on the concept of "Settings". The method is applied to the sensitivity analysis of a discrete change in optimal order quantity following a jump in the exogenous variables of a nonlinear programming inventory model.

Acknowledgments *The author wishes to thank the anonymous referees for the very insightful and perceptive comments, that have resulted in improvement of the previous versions of the present manuscript. Financial support from the ELEUSI research center is gratefully acknowledged.*

1 Introduction

This work deals with the sensitivity analysis (SA) of model output in the presence of finite changes in the parameters.

Key Words and Phrases: *Inventory; Sensitivity Analysis; Modified Economic Order Quantity Models; Finite Change Sensitivity Indices; Comparative Statics.*

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SA is essential in deriving insights from decision-support models in a wide range of applications[1]. In budgeting, Eschenbach (1992) proposes the use of Tornado Diagrams to convey managerial information and to identify the key-drivers of the decision-making problem. In multi-criteria decision-making, Butler et al (1997) describe the use of simulation techniques for the SA of the decision-support models. In linear programming, SA is tantamount in aiding managers with the interpretation of model results. — We refer to the works of Flavell and Salkin (1975), Jansen et al (1997), and Koltai and Terlaky (2000) and to the references therein contained, since a complete review cannot be contained within the scope of this manuscript. —

In inventory management, Dobson (1988) discusses the sensitivity of the classical economic order quantity (EOQ) model (Harris, 1913 in Erlenkotter (1990)) in the presence of imprecision in the parameter estimates. The starting point of Dobson's analysis is the observation that the total cost per unit in the classical model is "*fairly insensitive to the choice of Q relative to the optimal quantity Q^**" (Dobson (1988)) — see Table 1 for notation. Dobson (1988) quantifies such insensitivity proving that the "*sensitivity of the cost to parameters of the model grows as the fourth root of the uncertainty in the values of the parameters* (Dobson (1988))". Several extensions of the classical EOQ model have been proposed to allow the inclusion of inventory problem facets not comprised in the original model. Such extensions are referred to as modified EOQ models. The works of Berman and Parry (2006), Huang (2007), Goyal et al, 2007, Ben-Daya and Noman (2008), Sana and Chaudhuri (2008) and Soni and Shah (2008) represent some of the most recent generalizations. — These works, in their turn, generalize modified EOQ models introduced in earlier literature, for which we refer to the references therein contained. — Common feature across these works is the presence of an SA exercise that accompanies the numerical determination of Q^*. It is recognized that the value of Q^* is a function of the values assumed by the exogenous variables (denoted by x in this work; see Table 1 for notation). Attention is focused on a subset of the exogenous variables, which are deemed relevant for the analysis. The dependence of Q^* on this subset of parameters is then obtained by plotting or tabulating the values of Q^* with the parameters varying in predetermined ranges. These type of methods are referred to as one-at-a-time methods in the SA literature (Saltelli et al (2004)). As discussed in Saltelli et al (2004), the advantages of these methods are the simplicity of utilization and the low computational cost. However, from one-at-a-time methods it is not possible to derive information on the factors on which "*to focus managerial attention during implementation* (Eschenbach (1992), p. 40-41)". In addition, one-at-a-time methods do not reveal interaction effects. From a more general standpoint, the pre-selection of the parameters on which to perform

[1] Oggier et al (2005) referring to the introduction and use of operations research methods in Nestlè corporation, state: "... *We developed four OR modules: sensitivity analysis, forecasting, simulation, and optimization* ...(Oggier et al, 2005, page 271)".

the SA is questionable. In fact, a group of exogenous variables is deemed important before performing the SA, thus before knowing the model response. This exposes the analyst to the risk of a-priori excluding relevant factors.

In this respect, a methodology for the identification of the key-drivers of optimal inventory policies given small variations in the exogenous variables is carried out in Borgonovo (2008). The approach of Borgonovo (2008) is based on the joint utilization of the classical technique of comparative statics (CS) Samuelson (1947) and of the differential importance measure Borgonovo and Apostolakis (2001). These techniques belong to the family of perturbation approaches. Perturbation analysis has been employed in the works of Glasserman and Tayur (1995), and Bogataj and Bogataj (2004) to analyse the stability of inventory systems. By construction, perturbation techniques entail the requirements of smoothness (differentiability) in the model output and of small changes in the independent variables. In a variety of managerial applications, however, parameters undergo finite variations, inducing discrete/non-smooth changes in the decision-support criterion. We illustrate the issue by means of a sample inventory management problem.

Consider a decision-maker who is selecting an inventory policy supported by a quantitative model. The optimal order quantity (Q^*) is the solution of an optimization problem involving the minimization (maximization) of a given loss (utility) function. — We do not require that the analytical expression of the model is known to the decision-maker; the model can be considered a black-box. — The value of Q^* depends upon the numerical values assigned to the parameters of the inventory model at hand (contractual values for fixed and variable costs, demand, macroeconomic factors, etc.). Let Q_0^* be the value of the optimal order quantity given the values of the exogenous variables at time $t = 0$. The next evaluation of the optimal order policy takes place at a different time, $t = 1$. The external conditions are likely to change with a corresponding variation in the values of the exogenous variables. The model is then updated with the new data. The optimal order quantity shifts to Q_1^*. Correspondingly, actions are taken to implement the new order policy Q_1^* instead of Q_0^*. As the exogenous variables are not under the direct control of the decision-maker, Q^* can increase or decrease and the change $\Delta Q^* = Q_1^* - Q_0^*$ is, in general, finite[2]. In this situation, "*... finding out what it was about the inputs that made the outputs come out as they did* (Little (1970); p. B469)" is an essential information for management purposes. This task, however, cannot be accomplished by means of differentiation-based approaches, as they rest on the assumption of small changes. Thus, they would be inconsistent with the managerial problem at hand.

[2]The previous example can be generalized to all those applications in which a decision-support model is utilized to corroborate the decision-making process. The model estimates the decision support criterion, y. Let y^0 be the value of the criterion at a certain stage. The model is kept alive and adjourned for new information. New numerical values are factored into the model as new data become available. Thus, at a subsequent stage the valuation criterion assumes a new value new value, y^1, which differs, in general, from y^0 by a finite jump. It is even not infrequent that the two values provide very different indications on the decisions to be adopted.

Our purpose is to build a methodology that allows to apportion any model output change to the contribution of individual factors and factor groups. Towards this goal, the first issue one faces is the definition of the mathematical background. It is, in fact, shown in several works (among of the first ones is Flavell and Salkin (1975)) that a Taylor expansion necessarily leads to approximate results when describing finite changes. Therefore, we do not resort to a differential expansion, but to the integral function decomposition of the high dimensional model representation (HDMR) theory (Rabitz and Alis (1999)). We show that any change in model output can be decomposed in a finite number of terms without approximations. Consistently with this result, we introduce new sensitivity measures that allow to appreciate individual contributions and the contributions of parameter interactions and groups. We show that the new sensitivity measures allow us to drop the smoothness requirement in the model output and the assumption of infinitesimal parameter changes. We prove that the measures provide the extension of classical differential approaches by showing that they converge to comparative statics indicators when changes become small.

We next turn to the derivation of managerial insights. Several works highlight issues in communicating SA results to decision-makers (Eschenbach (1992); Jansen et al (1997); Koltai and Terlaky (2000)). Saltelli and Tarantola (2002) propose the concept of "*Setting*" as the key for structuring the information contained in the SA exercise (see Saltelli and Tarantola (2002), page 704). We formulate three Settings (without claiming exhaustiveness) for our problem. The first Setting concerns the determination of the effect (direction of change) on the model output of both individual and simultaneous variations in the parameters. The approach allows the analyst to understand whether the interactions of pairs, triplets and any order group of factors amplify or smoothen individual effects. The second Setting leads to the identification of the key-drivers of the problem. Analysts can then single out the parameters that deserve further attention. Decision-makers are provided with information on the factors that on which to focus during implementation. The third Setting is of a more technical nature. It is related to gaining information on model structure, so as to provide the model user with the indication of whether the model response is additive (i.e. governed by individual effects) or non-additive (i.e., governed by interactions).

The methodology is illustrated via a numerical case study. We tackle the explanation of the discrete change in Q^* provoked by new values of the exogenous variables of the Luciano and Peccati (1999)'s modified EOQ model. The finite change sensitivity indices are computed via implementation on a Matlab Subroutine. We discuss the numerical findings in the light of the three Settings for the derivation of managerial insights. We compare the results to those obtained by applying the differential sensitivity approach developed in Borgonovo (2008). As we are going to see, a relevant discrepancy appears in the identification of the key-drivers. The discrepancy is then explained both from the mathematical and informational viewpoints. The findings underline the inconsistency in employing differential approaches in managerial problems involving the finite changes.

The remainder of the paper is organized as follows. Section 2 provides the mathematical foundations of the approach to SA with finite changes. Section 3 introduces new sensitivity measures for SA with finite changes and illustrates their properties. Section 4 deals with the derivation of managerial insights from the approach. Section 3 relates the new sensitivity measures to classical differential indicators. Section 6 describes the application of the new approach and the derivation of managerial insights in the context of an inventory decision-making problem supported by a non-linear programming modified EOQ model. Conclusions are offered in Section 7.

2 The Decomposition of a Finite Change in a Finite Number of Terms

This Section deals with the technical background related to the decomposition of finite changes.

We consider a decision-making problem supported by the creation of a quantitative support model. The model output estimates the decision-support criterion (y). (see Table 1 for notation) The dependence of y on the parameters (x) is expressed by the relationship:

$$y = f(x), \qquad f : X \subseteq \mathbb{R}^n \to \mathbb{R} \tag{1}$$

where X is the parameter space. y^0 denotes the value of y when the parameters are fixed at point $x^0 \in X$. If external conditions change and new information becomes available (or when, in a scenario analysis, the decision-maker tests the model at new values to corroborate model results) the parameters shift from x^0 to $x^1 \in X$ and a new value of y [$y^1 = f(x^1)$] is obtained. We denote the corresponding changes by $h = x^1 - x^0$ and $\Delta y = y^1 - y^0$. In the literature, the derivation of sensitivity measures is usually accomplished as follows (Helton (1993) provides an accurate description). The assumption of small changes is stated ($h \to \mathrm{d}x$); Δy is approximated through a first order Taylor series; partial derivatives are selected as sensitivity measures (see also Samuelson (1947)). However, when changes are not infinitesimal, the information derived from partial derivatives cannot be utilized (for an early discussion of such a problem in linear programming see Flavell and Salkin (1975)) as Taylor series requires an infinite number of terms to guarantee perfect accuracy[3].

[3] The inclusion of higher order terms in the Taylor expansion is a first possible way to improve the approximation of Δy Griewank et al (2000). Requiring that $f \in C^r$ at least, then one can approximate Δy as:

$$\begin{array}{c} \Delta y = y^1 - y^0 = \sum_{i=1}^{n} f_i'(\mathbf{x}^0) h_i + \sum_{j=1}^{n} \sum_{i=1}^{n} f_{i,j}^{''}(\mathbf{x}^0) h_i h_j + ... + \\ + ... + \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \cdots \sum_{s_r=1}^{n} f_{s_1,s_2,...,s_r}^{r}(\mathbf{x}^0) h_{s_1} h_{s_2} ... h_{s_r} + o(\|h\|^r) \end{array} \tag{2}$$

However, the order r at which to stop the expansion to obtain a given numerical accuracy is not known in advance. Thus, to inspect the effect of adding a further term, one is forced to compute it. As Griewank et

An alternative background is offered by the High Dimensional Model Representation (HDMR) theory. HDMR has been introduced in Rabitz and Alis (1999), and next consolidated in the works of Sobol' (2003) and Sobol' et al (2007). HDMR is the generalization of Functional ANOVA. Historically, Functional ANOVA is related to Hoeffding's work in the 1940's (Hoeffding (1948)). The so called jackknife decomposition of variance is proven in the seminal work of Efron and Stein (1981). A generalization of such a decomposition is next found in Takemura (1983). In 1990, Sobol' (in Russian, see Sobol' (1993)) derives the same result via multi-dimensional nested integrations.

We present the theory in the formulation of Rabitz and Alis (1999) (see Table 1 for notation). The parameter space X is complemented by a Borel algebra and a product measure — the probability space is $(X,\mathcal{A},\mu)$. — The function f is an element of a linear space of functions, $\mathcal{F}$. The following result is proven in Takemura (1983) and next in Rabitz and Alis (1999).

Lemma 1

$$\mathcal{F} = \mathcal{F}_0 \oplus \sum_{i=1}^{n} \mathcal{F}_i \oplus \sum_{i<j}^{n} \mathcal{F}_{i,j} \oplus ... \oplus \mathcal{F}_{1,2,...,n} \tag{3}$$

where $\oplus$ is the direct sum operator and the subspaces are defined as:

$$\begin{cases} \mathcal{F}_0 \equiv \{f \in \mathcal{F} : f = a \in \mathbb{R}\} \\ \mathcal{F}_i \equiv \left\{f \in \mathcal{F} : f = f_i(x_i) \text{ with } \int f_i(x_i) d\mu_i = 0\right\} \\ \mathcal{F}_{i,j} \equiv \left\{f \in \mathcal{F} : f = f_{i,j}(x_i,x_j) \text{ with } \int f_{i,j}(x_i,x_j) d\mu_i = 0 \text{ and } \int f_{i,j}(x_i,x_j) d\mu_j = 0\right\} \\ .... \\ \mathcal{F}_{1,2,...,n} = \left\{f \in \mathcal{F} : f = f_{1,2,..,.n}(x_1,x_2,...,x_n) \text{ with } \int f_{1,2,..,.n}(x_1,x_2,...,x_n) d\mu_k = 0,\ k = 1,2,...,n\right\} \end{cases} \tag{4}$$

In eq. (3), $\mathcal{F}_0$ is the set of functions constant over X; $\mathcal{F}_i$ is the set of functions that depend only upon x_i and with null expectation; $\mathcal{F}_{i,j}$ is the set of bivariate functions with null conditional expectations, and so on so forth (see also Rabitz and Alis (1999)). Any function f can then be decomposed as follows (Sobol' (1993), Rabitz and Alis (1999)).

Theorem 1 *Let $f \in \mathcal{L}^1(X)$ and $d\mu = \prod_{i=1}^{n} d\mu_i$. Then, the following decomposition of f*

al (2000) underlines, one cannot compute the r^{th} term of the expansion by itself, but needs to estimate all orders $s < r$. For each added term, the computational cost is not constant but increases exponentially with s Griewank et al (2000). Thus numerical complications may arise, especially in application to large models. In addition, the use of higher order differentials is applicable only when f possesses the required regularity (e.g., $f \in C^r(X)$). Furthermore, deriving sensitivity measures from eq. (2) is not straightforward, as one cannot combine partial derivatives of different orders (see Helton (1993)) — observe that this problem is not there within a first order approximation framework, as only first order partial derivatives are there. —

is unique (Sobol' (1993), Rabitz and Alis (1999))[4]*:*

$$f(x) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{i,j}(x_i, x_j) + ... + f_{1,2,...n}(x_1, x_2, ..., x_n) \tag{5}$$

where

$$\begin{cases} f_0 = \mathbb{E}_\mu[Y] = \int \cdots \int f(x) d\mu \\ f_i(x_i) = \mathbb{E}_\mu[Y|x_i] - f_0 = \int \cdots \int f(x) \prod_{k \neq i} d\mu_k - f_0 \\ f_{i,j}(x_i,x_j) = \mathbb{E}_\mu[Y|x_i, x_j] - f_i(x_i) - f_j(x_j) - f_0 \\ ... \end{cases} \tag{6}$$

As proven in alternative ways in Efron and Stein (1981), Sobol' (1993), Takemura (1983) and Rabitz and Alis (1999), the functions $f_{i_1,i_2...,i_k}(x_{i_1},x_{i_2}, ..., x_{i_k})$ are orthogonal. Orthogonality implies that the first order terms in eq. (5), $f_i(x_i)$, represent the individual effects of the parameters; the second order terms, $f_{i,j}(x_i,x_j)$, represent the residual effect of the interaction between x_i and x_j; similarly, higher order terms in the expansion represent the synergies of the corresponding parameter groups. When f is square integrable (i.e., $f \in \mathcal{L}^2(X)$), the orthogonality of the terms in eq. (5) enables the complete decomposition of the output variance Efron and Stein (1981)[5]:

$$V = \sum_{i=1}^{n} V_i + \sum_{i<j} V_{i,j} + ... + V_{1,2,...n} = \sum_{k=1}^{n} \sum_{i_1<i_2...<i_k} V_{i_1,i_2,...,i_k} \tag{7}$$

with

$$V_{i_1,i_2,...,i_k} = \int \cdots \int [f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k})]^2 \prod_{m=i_1,i_2,...,i_k} d\mu_m \tag{8}$$

The terms $V_{i_1,i_2,...,i_k}$ $(k = 1, 2, ..., n)$ [eq. (8)] are called partial variances. Eqs. (7) - (8) have been extensively studied in global SA (Sobol' (1993), Wagner (1995), Homma and Saltelli (1996)), both from the computational and theoretical viewpoints. In particular, Homma and Saltelli (1996) introduce the global sensitivity indices of order k $(k = 1, 2, ..., n)$:

$$S^k_{i_1,i_2,...,i_k} = \frac{V_{i_1,i_2,...,i_k}}{V} \tag{9}$$

$S^k_{i_1,i_2,...,i_k}$ is the fraction of the variance associated with the interaction of parameters $x_{i_1}, x_{i_2}, ..., x_{i_k}$. An application of the global sensitivity indices to a modified EOQ model can be found in Borgonovo and Peccati (2007).

Theorem 1 allows to achieve the decomposition of a finite change in a finite number of terms. In fact, the following result holds.

[4]For the notation in the last equality of eq. (5), see Sobol' (2003).

[5]For eq. (7) to hold, the assumption $f \in \mathcal{L}^2(\Omega)$ is required (Sobol', 1993; Efron and Stein, 1981). Such an assumption can be relaxed to $f \in \mathcal{L}^1(\Omega)$ if the existence of the sole function decomposition [eq. (5)] is required. Thus, throughout this paper, the only requirement on f is its integrability.

Theorem 2 *Let $(X, \mathcal{A}, \mu)$ a measure space and $f \in \mathcal{L}^1(X)$ a measurable function. Then, for any x^0 and $x^1 = x^0{+}h$ belonging to X and for any measure μ satisfying the assumptions of Theorem 1, the following holds:*

$$\Delta y = f(x^1) - f(x^0) = \sum_{i=1}^{n} \Delta f_i + \sum_{i<j} \Delta f_{i,j} + ... + \Delta f_{1,2,...n} \tag{10}$$

The proof is in Appendix A. Some observations on Theorem 2 follow.

1. In eq. (10), the summands are the changes in the terms of the function decomposition in eq. (5). There are, therefore, $2^n - 1$ terms. Through eq. (10), one overcomes the limitations of Taylor expansion [eq. (2)], which involves an infinite number of terms to describe a finite change.

2. Theorem 2 requires measurability of f ($f \in \mathcal{L}^1(X)$) while for an exact Talyor decomposition one would need $f \in C^\infty(X)$. Thus, Theorem 2 applies to a vaster class of models (even models with discontinuous output) than Taylor expansion.

In the next Section, we describe the derivation of sensitivity measures for finite changes that stem from Theorem 2.

3 Sensitivity Measures for Finite Changes

In this Section, we derive sensitivity measures that descend from eq. (10) and that describe the model sensitivity to finite changes in the parameters.

Theorem 2 holds for any product measure. In this respect, it is a generalization of a result proven in Rabitz and Alis (1999) and next reported in Sobol' (2003) [eq. 7, p. 188]. In those works, the decomposition of a finite change is obtained with reference to the sole Dirac $\delta-$measure. The choice of the Dirac-δ measure has, however, important operational consequences. The following Corollary holds (the proof is in Appendix A).

Corollary 1 *Let x^0 and $x^1 = x^0{+}h$ be any two points in X. Under the assumptions of Theorem 2, consider the Dirac-δ measure $d\mu = \prod_{i=1}^{n} \delta(x_i^1 - x_i^0) dx_i$. Then,*

$$\Delta y = f(x^1) - f(x^0) = \sum_{i=1}^{n} \Delta_i f + \sum_{i<j}^{n} \Delta_{i,j} f + ... + \Delta_{1,2,...,n} f = \sum_{k=1}^{n} \sum_{i_1<i_2...<i_k} \Delta_{i_1,i_2,...,i_k} f \tag{11}$$

where

$$\begin{cases} \Delta_i f = g_i(x_i) - f(x^0) \\ \Delta_{i,j} f = g_{i,j}(x_i, x_j) - \Delta_i f - \Delta_j f - f(x^0) \\ ... \end{cases} \tag{12}$$

and

$$\begin{cases} g_i(x_i) = f(x_1^0, x_2^0, ..., x_{i-1}^0, x_i^1, x_{i+1}^0, ..., x_n^0) \\ g_{i,j}(x_i, x_j) = f(x_1^0, x_2^0, ..., x_{i-1}^0, x_i^1, x_{i+1}^0, ..., x_{j-1}^0, x_j^1, x_{j+1}^0, ..., x_n^0) \\ \dots \end{cases} \tag{13}$$

Eq. (11) states that the change Δy is the sum of terms of increasing dimensionality ($\Delta_{i_1,i_2,...,i_k} f$, $k = 1, 2, ..., n$). The first order terms $\Delta_i f$ [eq.(12)] are individual contributions, with one parameter at a time varying from x_i^0 to $x_i^0 + h_i$. They are the difference between: *a)* $g_i(x_i)$ [eq. (13)], which represents the value of y with all parameters at their base case value but x_i [$g_i(x_i) = f(x_1^0, x_2^0, ..., x_{i-1}^0, x_i^1, x_{i-1}^0, ..., x_n^0)$] and *b)* the base case value of the model, $f(x^0)$. The second order terms $\Delta_{i,j} f$ account for the additional portion of the change caused by the interaction of all parameters pairs. They are computed as the difference between: *a)* $g_{i,j}(x_i, x_j)$ [eq. (13)], which is the value that y attains when x_i and x_j are moved to x_i^1 and x_j^1, while all other parameters are fixed at their base case value and *b)* the changed caused by x_i and x_j individually (respectively the first order terms $\Delta_i f$ and $\Delta_j f$) and *c)* y^0. The third order terms account for the additional contribution coming from the interaction of triplets and are computed in a similar way. Continuing in the decomposition, higher order terms represent the contributions of higher order interactions, till the residual term of order n.

To introduce sensitivity indicators that capture individual and interaction effects, we propose the following definitions.

Definition 1 *We call the quantity*

$$\varphi_{i_1,i_2,...,i_k}^k = \Delta_{i_1,i_2,...,i_k} f \tag{14}$$

finite change sensitivity index of order k.

$\varphi_{i_1,i_2,...,i_k}^k$ is the contribution to the finite change in y of the interaction of parameters $x_{i_1}, x_{i_2}, ..., x_{i_k}$. This follows from the orthogonality of the terms in eqs. (5) and (11). $\varphi_{i_1,i_2,...,i_k}^k$ plays for a finite change the same role as $V_{i_1,i_2,...,i_k}$ [eq.(8)] for variance decomposition.

As we are to learn shortly, it is also informative to consider the normalized version of the indices:

$$\Phi_{i_1,i_2,...,i_k} = \frac{\Delta_{i_1,i_2,...,i_k} f}{\Delta f} \tag{15}$$

$\Phi_{i_1,i_2,...,i_k}$ is the fraction of the finite change associated with the interaction among parameters $x_{i_1}, x_{i_2}, ..., x_{i_k}$. $\Phi_{i_1,i_2,...,i_k}$ compares to $S_{i_1,i_2,...,i_k}^k$, as $\varphi_{i_1,i_2,...,i_k}^k$ compares to $V_{i_1,i_2,...,i_k}$. From Definition 1, one can then introduce the first and total order sensitivity indices:

Definition 2 *The first order finite change sensitivity indices are defined as*

$$\varphi_l^1 = \Delta_l f \tag{16}$$

and, in normalized version:

$$\Phi_l^1 = \frac{\Delta_l f}{\Delta f} \tag{17}$$

φ_l^1 [eq.(16)] is the contribution to Δf of the change in x_l alone. Φ_l^1 [eq.(17)] is the fraction of the change in f due to the variation of x_l alone.

Definition 3 *the total order sensitivity indices are defined as:*

$$\varphi_l^T = \Delta_l f + \sum_{l \neq j} \Delta_{l,j} f + ... + \Delta_{1,2,...n} f = \sum_{k=1}^{n} \sum_{l \in i_1, i_2, ..., i_k} \varphi_{i_1, i_2, ..., i_k}^k \tag{18}$$

and

$$\Phi_l^T = \frac{\Delta_l f + \sum_{l \neq j} \Delta_{l,j} f + ... + \Delta_{1,2,...n} f}{\Delta f} = \frac{\sum_{k=1}^{n} \sum_{l \in i_1, i_2, ..., i_k} \varphi_{i_1, i_2, ..., i_k}^k}{\Delta f} \tag{19}$$

where the sums are extended to all terms in eq. (11) involving x_l.

φ_l^T [eq.(18)] is the contribution to Δf of the change in x_l by itself and together with its interactions with all other parameters and parameter groups. Φ_l^T [eq.(19)] is the corresponding fraction of the change.

The following Proposition helps in clarifying the meaning of the total order sensitivity indices (φ_l^T) and shares a direct numerical implication (the proof is in Appendix A).

Proposition 1 *The total order sensitivity indices are equal to*

$$\varphi_l^T = \Delta y - \Delta y_{(-l)} = f(x^1) - f(x_{(-l)}^1) \tag{20}$$

where $\Delta y_{(-l)}$ is the jump in model output when all parameters vary but x_l, and $x_{(-l)}^1$ is the point obtained by shifting all parameters at the new value, while keeping only x_l at the base case value [$x_{(-l)}^1 = (x_1^1, x_2^1, ..., x_{l-1}^1, x_l^0, x_{l+1}^1, ..., x_n^1)$].

According to eq. (20), φ_l^T is the difference between the change in y realized when all parameters vary and the change in y realized when all parameters vary but x_l. This gap disappears as soon as x_l moves from x_l^0 to x_l^1. As eq. (19) and the Proof of Proposition 1 in Appendix A show, this gap is exactly equal to the sum of all terms in the decomposition of Δy [eq. (11)] containing x_l, i.e., it equals φ_l^T. Thus, φ_l^T is the portion of Δf is generated by the change in x_l. φ_l^T [or its normalized version Φ_l^T] represents the total effect on the decision-support criterion of a change in an exogenous variable.

Proposition 1 has also a significant computational consequence. In fact, it allows to estimate the total order sensitivity indices at a cost of n model runs. The total cost for computing φ_l^T and φ_l^1 is equal to $2n$, which is the same cost of a Tornado Diagram (Eschenbach, 1992). This result enables the estimation of the finite change sensitivity indices also in the presence of computationally intensive models.

From a more general standpoint, the first order indices φ_l^1 are one-parameter-at-a-time sensitivity measures and contain the same information as Tornado Diagrams Eschenbach (1992). Our discussion shows that, when a model is non-additive, one parameter-at-a-time techniques do not capture the *"total impact"* (Eschenbach (1992); p. 45) of the independent variables, but only their individual effects. The total impact is, instead, captured by φ_l^T.

In the next Section, we discuss a method to formalize the derivation of managerial insights from the finite change sensitivity indices. The method is based on Saltelli and Tarantola's "*Settings*" (Saltelli and Tarantola (2002)).

4 Managerial Information and SA Settings

Eschenbach (1992) highlights the need to organize information obtained from any SA exercise in order not to "*overwhelm managers with data* (Eschenbach (1992))". Little (1970) in his early work on the use of models by managers, states that "*The manager should be able to change inputs easily and obtain outputs quickly*" (Little (1970), p. B470). Jansen et al (1997) and Koltai and Terlaky (2000) point out the differences between mathematical and managerial interpretation of SA information in linear programming. Wallace (2000) and Higle and Wallace (2003) warn modelers about the risk of inconsistency between the managerial problem at hand and the employed SA method. Such issues are recognized and addressed in Saltelli and Tarantola (2002) and Saltelli et al (2004). These works introduce the concept of "*Setting*" (Saltelli and Tarantola (2002); p. 704) as a tool to let SA information match the managerial questions. The first step is a clear statement of the decision-maker/analyst's question. This statement is called a Setting. Given a Setting, one identifies the SA method that delivers the proper insights.

Without the claim of being exhaustive, we state three Settings for this work. The first one stems from Samuelson's (1947) statement *"... to derive definite qualitative restrictions upon the response of our system to changes in certain parameters"* (Samuelson (1947) p. 20). Recalling that in Samuelson (1947) changes are individual and small, while in this work changes can be simultaneous and finite, we then introduce the following Setting 1.

Setting 1: What is the direction of change in the model output due to individual or simultaneous changes in the parameters?

Setting 1 is indeed composed of two parts. The first one concerns individual effects, the second one concerns joint effects. Answering the first part, the decision-maker establishes whether the change in x_l (alone) increases or decreases the value of the decision-support criterion. Answering the second part, she/he determines whether the joint change in two (or more) variables leads to an amplification of their individual effects (cooperation), or softens their individual effects (interference). Sensitivity measures for this Setting are the indices φ [eqs. (14), (16) and (18)]. In particular, the sign of φ_l^1, delivers information on the direction of change in y due to the change in x_l alone; the sign of $\varphi_{i_1,i_2,\ldots,i_k}^k$ indicates whether the simultaneous change results in cooperation or interference; the sign of φ_l^T provides information on the direction of change in y due to the total effect of x_l.

From a risk-management viewpoint, Setting 1 enables the decision-maker to know what factors hedge each other's variations.

The second Setting stems from the managerial question of identifying the factors on which to focus during implementation (see Eschenbach, 1992; p. 40-41, directly quoted in Section 1). We therefore introduce the following Setting 2.

Setting 2: What are the key-drivers of the change?

To answer Setting 2, one needs to determine the importance of the exogenous variables and to rank them. The magnitudes of the total order sensitivity indices φ_l^T [or in normalized version Φ_l^T] are the natural importance measures for this Setting. In fact φ_l^T or (Φ_l^T) convey the total effect of parameter x_l, synthesizing its individual contribution and the contribution deriving from all its interactions with the other parameters (see Proposition 1).

We finally introduce a third Setting.

Setting 3: What is the structure of the model response?

In comparison to Settings 1 and 2, Setting 3 is of a more technical nature and aims at providing the modeler with insights on whether the model response is the superimposition of individual effects or of their interactions. In the first case, one says that the model is additive. In the second case, the model is non-additive. Setting 3 is answered by the decomposition of f in eq. (11), i.e., by the knowledge of the first, second, third order (and so on) effects. The sensitivity measures for Setting 3 are, then, the magnitudes of the terms in the decomposition of eq. (11), i.e., $|\varphi_{i_1,i_2,\ldots,i_k}^k|$. Conversely, in Setting 1 the sign of $\varphi_{i_1,i_2,\ldots,i_k}^k$ is of interest. If the complete decomposition is not achievable due to computational complexity, one can take as indicators the differences $\varphi_l^T - \varphi_l^1$. These differences, if null, signal that interaction effects are irrelevant [see Section 3]. On the other hand, $|\varphi_l^T - \varphi_l^1| >> |\varphi_l^1|$ implies that the relevance of a parameter is attributable to its cooperation with the others, rather than to its individual effect.

We are now left with the question of determining the relationship between the finite change sensitivity indices of eqs. (14) - (19) and the classical indicators of differential analysis and comparative statics. This subject is discussed in the next Section.

5 Relationship to Comparative Statics

In Borgonovo (2008) a methodology for identifying the key-drivers of optimal inventory policies in the presence of small changes is proposed. The purpose of this Section is to investigate the relationship between the finite change sensitivity indices introduced in Section 3 [eqs. (14) - (19)] and the differential indicators of Borgonovo (2008).

Comparative statics has been introduced in Samuelson (1947) (for a generalization, see Caputo and Paris (2008)) and offers the most general framework for the differential SA of model output.

Let

$$t(y,x)=0 \quad t:Y\times X\to T\subseteq\mathbb{R}^m \text{ with } Y\subseteq\mathbb{R}^m,\ X\subseteq\mathbb{R}^n \tag{21}$$

be the relationship that links the endogenous variables $y=(y_1,y_2,...,y_m)$ and the exogenous variables x (see Table 1 for notation). If $t\in C^r(Y\times X)$, the set of points x that solves eq. (21) implicitly defines the C^r-functions, $y=f(x)$ [eq. (1)]. The first order sensitivity is governed by the comparative statics equation (Samuelson (1947)):

$$J_y\mathrm{d}y+J_x\mathrm{d}x{=}0 \tag{22}$$

where J denotes the Jacobian of t. The solution to this equation is the matrix CS

$$CS=-J_y^{-1}J_x \tag{23}$$

whose elements are the partial derivatives of y with respect to x, i.e., $CS=[cs_{j,i}:cs_{j,i}=y\prime_{j,i},\ \begin{matrix} j=1,2,..,m \\ i=1,2,...,n\end{matrix}]$.

The sign of $cs_{j,i}$ indicates the directions of change in endogenous variable y_j after a small increases ($\mathrm{d}x_i$) in exogenous variable x_i. Thus, in the case of small parameter changes, the elements of matrix CS are the appropriate sensitivity indicators to answer Setting 1 (see also Borgonovo (2008)). Matrix CS, however, cannot be utilized to identify the key-drivers of the problem. Geometrically, this is connected with directionality issues. Operationally, this is due to the fact that when parameters have different dimensions, the corresponding derivatives are not comparable (see Borgonovo (2008) for a complete discussion). The importance of an exogenous variable is, instead, appreciated via the elements of matrix Γ, defined as Borgonovo (2008):

$$\Gamma=\left[\gamma_{j,i}:\gamma_{j,i}=\frac{\mathrm{d}_i y_j}{\sum\limits_{s=1}^{n}\mathrm{d}_s y_j}=D_{j,i},\quad \begin{matrix} j=1,2,..,m \\ i=1,2,...,n\end{matrix}\right] \tag{24}$$

In eq. (24), $\gamma_{i,j}$ is the fraction of the differential of output y_j associated with the (small) change in x_i (see Borgonovo (2008)). In other words, an elements of Γ is the differential importance (Borgonovo and Apostolakis (2001)) of parameter x_i with respect to output y_j ($\gamma_{j,i}=D_{j,i}$). Borgonovo (2008) shows that Γ generalizes partial derivatives and elasticities. For optimization problems, Borgonovo (2008) proves that Γ can be derived by the knowledge of the Bordered Hessian of the Lagrangian function of the problem. Matrix Γ allows to identify the key-drivers of the changes in exogenous variables due to differential changes in the parameters. The magnitude of the elements of Γ are therefore appropriate sensitivity indicators to answer Setting 2 in the case of small changes (see also Borgonovo (2008)).

To investigate the relationship between the finite change sensitivity indices and the differential sensitivity indicators, we need to study the behavior of φ and Φ as the magnitude of the changes decreases. We also need to restrict attention to those models that satisfy the differentiability assumptions necessary to estimate CS and Γ. The following result holds (the proof is in Appendix A; for notation simplicity, we consider $m = 1$).

Theorem 3 *Let f be differentiable and such that ∇f is not orthogonal to h. Then:*

1.

$$\begin{cases} \lim_{h \mapsto \mathbf{0}} \dfrac{\varphi_l^T}{h_l} = \lim_{h \mapsto \mathbf{0}} \dfrac{\varphi_l^1}{h_l} \\ \lim_{h \mapsto \mathbf{0}} \Phi_l^T = \lim_{h \mapsto \mathbf{0}} \Phi_l^1 \end{cases} \tag{25}$$

2. and:

$$\begin{cases} \lim_{h \mapsto \mathbf{0}} \dfrac{\varphi_l^T}{h_l} = cs_l \\ \lim_{h \mapsto \mathbf{0}} \Phi_l^T = \dfrac{d_l f}{df} = \gamma_l \end{cases} \tag{26}$$

Theorem 3 shares the following interpretation.

Remark 1 *1) The equalities in eq. (25) state that the total order sensitivity indices tend to the same limit as the first order sensitivity indices when changes becomes small. This implies that, for differentiable models, interaction terms ($\Delta_{i_1,i_2,\ldots,i_k} f$) become significant only when changes are not small. Thus, a differentiable model is always additive when changes are infinitesimal.*
2) The equalities in eq. (26) show that the limits are the differential indicators of comparative statics derived from Taylor expansion. This implies that the finite change sensitivity indices extend the sensitivity measures of comparative statics analysis to the case of finite changes.

Theorem 3 has a further implication: neither matrix CS nor matrix Γ can be utilized to answer Setting 3. Theorem 3, in fact, states that any differentiable model responds additively to perturbations in the parameters. Thus, no interactions effects can be detected by matrices CS and/or Γ.

In the next Section, we compare the use of the finite change sensitivity indices and of matrices CS and Γ in the SA of a non-linear programming inventory management problem.

6 Application: a modified EOQ model Sensitivity Analysis

We consider a firm that selects the optimal order quantity supported by the inventory management model proposed in Luciano and Peccati (1999). This modified EOQ model enables the evaluation of Q^* with explicit consideration of the financial aspects of the problem. We refer the reader to Luciano and Peccati (1999) for the complete illustration of the model, and to Borgonovo (2008) for its comparative statics analysis. We propose a concise presentation of the model main aspects.

Key-features of the model are the focus on the cash flows of the inventory system and their aggregation through the Adjusted Present Value (APV) approach. Demand, R, is assumed constant. This leads to the (usual) constant inventory cycle length $T = \frac{Q}{R}$. Holding costs are assumed proportional to Q, through a (a is a unit order cost per unit time and per unit quantity). Denoting the price of an inventoried good by u and the order cost per cycle by β, the cash flow generated by the inventory cycle is equal to $\left(u + \frac{a}{2}\right) Q + \beta$. In the Luciano-Peccati modified EOQ model, holding costs are exclusive of financial charges, which are implicitly accounted for in the discount process. Denoting by ρ the cost of capital and considering an infinite horizon, the inventory system loss function is:

$$\mathcal{L}(Q, \alpha) = \sum_{s=0}^{+\infty} [\left(u + \frac{a}{2}\right) Q + \beta] e^{-s\dfrac{\rho Q}{R}} = \frac{\left(u + \frac{a}{2}\right) Q + \beta}{1 - e^{-\rho Q/R}} \tag{27}$$

Q^* is then found by solving the first order condition:

$$\mathcal{L}'_Q(Q, x) = \left(\frac{1}{2}a + u\right) \left(e^{\frac{Q}{R}\rho} - 1\right) R - \rho \left(\beta + Q \left(\frac{1}{2}a + u\right)\right) = 0 \tag{28}$$

By eq. (28), Q^* depends on the following five parameters:

$$x = \left[\begin{array}{ccccc} x_1 = u & x_2 = a & x_3 = R & x_4 = \beta & x_5 = \rho \end{array} \right]$$

At $t = 0$ information available to the management allows to set the parameters at

$$x^0 = \left[\begin{array}{ccccc} u^0 \text{ [\$ per item]} & a^0 \text{ [\$ per Item]} & R^0 \text{ [Items]} & \beta^0 \text{ [\$]} & \rho^0 \\ 10 & 1 & 8000 & 30 & 0.07 \end{array} \right] \tag{29}$$

Correspondingly the modified EOQ is $Q_0^* \cong 807$. At $t = 1$, changes in external conditions and new information lead to the following values of the exogenous variables:

$$x^1 = \left[\begin{array}{ccccc} u^1 \text{ [\$ per item]} & a^1 \text{ [\$ per Item]} & R^1 \text{ [Items]} & \beta^1 \text{ [\$]} & \rho^1 \\ 13 & 2 & 8538 & 23 & 0.075 \end{array} \right] \tag{30}$$

With these values, the new modified EOQ is $Q_1^* = 611$. Hence, there is a 24% drop in optimal order quantity (the finite change is $\Delta Q^* = -196$).

We now illustrate how the finite change sensitivity indices enable one to explain the fall. The numerical values of the finite change sensitivity indices of all orders are reported in Table 2.

[Insert Table 2 almost here]

Table 2 shows that first and second order sensitivity indices are relevant, while higher order terms are negligible. The change in u alone accounts for a decrease of 95 units in Q^* ($\varphi_u^1 = -95$). Similarly the individual effects of a, β and ρ correspond to decreases of 19, 100 and 27 units in Q^*. Conversely, the change in R provokes an increase in Q^* of 27 units. The sum of the first order indices equals -214. However, the change in Q^* equals -196 units. This means that individual effects are contrasted by higher order terms. Table 2 shows that the most relevant interaction is the one between unit price (u) and order costs, β: $\varphi_{u\beta}$ equals $+12$. The positive sign means that the interaction between u and β smoothens their individual effects, and is associated with a 12 unit increase in optimal order quantity. The second most important interaction is the one between unit price, u, and holding costs, a. It accounts for a change equal to 6. Also in this case, the positive sign indicates that the interaction between a and u softens their individual effects. The sum of the second order indices equals $+19$. Thus, first and second order terms are associated with a change equal to -195. The residual unit is due to the sole non-null third order interaction term, $\varphi_{a\beta\rho}^3$, which accounts for the remaining 1 unit reduction in Q^*.

The overall effect of the parameter changes is captured by the total order indices (φ_l^T), which are displayed in Table 3.

[Insert Table 3 almost here]

In Table 3 the value $\varphi_u^T = -78$ indicates that unit prices (u) are responsible for a decrease in Q^* of 78 units. The magnitude of the corresponding normalize index, $|\Phi_u^T| = 0.4$, denotes the fraction of the change in Q^* associated with u. Similar considerations hold for the remaining indices reported in Table 3.

Let us then interpret the results in the light of the Settings (Section 4).

As far as the direction of change is concerned (Setting 1), the first order indices (Table 2) show that the individual changes in unit price (u), unit holding costs (a), order costs (β) and cost of capital (ρ) have a negative impact on Q^*. Conversely, a change in demand (R) alone increases Q^*. However, the individual effects are partially softened by interactions, which play in the opposite directions. The overall effect of the parameters is then synthesized in the total order sensitivity indices. We that φ_u^T, φ_a^T, φ_β^T and φ_ρ^T are negative, while φ_R^T is positive (Table 3). In terms of Setting 1, this means that the overall effect of the changes in u, a, β and ρ, respectively, is a decrease in Q^*, while the overall effect of the change in R is an increase in Q^*.

As far as the identification of key-drivers is concerned (Setting 2), the (magnitudes of) the total order sensitivity indices ($|\Phi_l^T|$) are the appropriate sensitivity measures (Section 4). The values of $|\Phi_l^T|$ for the present case study are reported in the second row of Table 3. Table 3 shows that the key-drivers of the change are unit order cost (β) and unit price (u), with the other factors playing a minor role.

Note that the reduction in order costs (β changes from 30 to 23) allows a greater order frequency thus implying a lower value of Q^*. In its turn, the increase in unit price of the goods leads to decreasing the quantities that are purchased. Table 3 shows that it indeed the combination of these two effects explains around 80% of the change.

Finally, in terms of Setting 3, one notes that individual effects prevail over interactions effects in the present application.

We compare our findings to the results obtained by applying matrices CS and Γ to the same model. As far as Setting 1 is concerned, an overall agreement in the signs of the elements of matrix CS and the first order indices is registered. Thus, the first order terms preserve the sign of the comparative statics indicators, in this application. Note that one cannot extend the comparison to higher order indices (see Section 5). As far as Setting 2 is concerned, Table 4 compares the ranking obtained with matrix Γ to the ranking obtained with the total order finite change sensitivity indices. Note that β and u — they ranked first and second by the total order sensitivity indices — are ranked only third and fourth by matrix Γ. Conversely, R and ρ are ranked first and second by matrix Γ, while they ranked fourth and third, respectively, with the finite change sensitivity indices. Thus, the key-drivers of the differential change are not the key-drivers of the finite change. Besides the mathematical explanation (finite vs differential changes) there is an informational aspect to be considered. Matrices CS and Γ are defined in such a way to make their use proper in a perspective mode, i.e., when the decision-maker knows the current value of the parameters, x^0, but is not aware of what the realized change in the parameters will be. At $t = 1$, however, the actual change of the parameters becomes available: one then needs to incorporate this information into the analysis. It is, indeed, the fact that the finite change sensitivity indices give full consideration to actual changes, that makes them the proper sensitivity measure for the types of problems addressed in this work.

7 Conclusions

Decision-support models are essential to the solution of managerial problems in several areas. Numerical results are determined by the assumptions concerning the model parameters. When the corresponding assumptions are monitored over time and numerical values are updated, or when the parameters are varied through different scenarios to corroborate the model results, the decision-criterion is subject to changes that are not necessarily infinitesimal. In these situations, the task of explaining model results cannot be accomplished through SA techniques based on differentiation.

We have presented a methodology that allows one to apportion the change in model output to the changes in the exogenous variables. The approach rests on the HDMR theory, which we have used as a mathematical rationale alternative to Taylor expansion. We have first shown that it is possible to decompose any change in model output in a finite number of terms without approximations. We have introduced sensitivity measures ("finite change sensitivity indices") that allow to appreciate the contribution of factors not only individually but also in groups. We have discussed the properties of the new sensitivity measures and seen that they converge to comparative statics and differential sensitivity measures for small changes. As a result, the finite change sensitivity indices represent an extension of the classical comparative statics indicators. Furthermore, as the sole regularity assumption is measurability, the method is applicable to non-smooth models, thus broadening the class of functions that can be approached with differential SA methods.

The method allows to appreciate the effect of interactions. The total order sensitivity indices [φ_l^T; eq. (18) and Table 1] synthesize the effect of changes in the parameters accounting for both their individual and interaction effects. We have proven a result that allows the estimation of the total order indices at the same computational cost of one-parameter-at-a-time methods. This results in a notable reduction in computational cost, which makes the approach applicable also to complex models.

The derivation of managerial insights has been discussed next. As indicated by relevant SA literature in OR, we have formulated the problem in terms of "Settings", so as to achieve consistency between the SA method and the managerial problem at hand. We have introduced three settings that allow the explanation of the change and the identification of the key-drivers of the problem.

We have applied the finite change sensitivity indices to the Luciano and Peccati (1999)'s modified EOQ model. The approach has allowed us to explain the 24% fall in Q^* provoked by a discrete change in the parameters. Order costs and unit price have been identified as key-drivers of the problem. We have also been able to dissect the change and understand the portion due to the individual changes and to the cooperation of the exogenous variables. Results show that interaction effects indeed soften individual effects in our case. We have seen that differentiation-based methods do not lead to the proper key-driver identification when applied to the same problem. This result confirms that the presence of finite changes makes Taylor-based approaches not consistent with the managerial problem at hand.

We conclude with future research perspectives. A first line of research concerns the generalization of the method. Currently the method is local in nature, as it explores the parameter space in two locations. The conjunction with scenario analysis or Monte Carlo simulation can extend the method to obtain a quasi-global technique. A second line of research concerns the application of the method to other inventory management models (Borgonovo and Peccati (2008)) and to the SA of decision-support models in alternative management areas.

8 Appendix A: Proofs

Proof of Theorem 2. Let us start rewriting eq. (5) as:

$$f(x) - f_0 = \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{i,j}(x_i, x_j) + ... + f_{1,2,...n}(x_1, x_2, ..., x_n) \tag{31}$$

At $x+h \in X$ we have:

$$f(x+h) - f_0 = \sum_{i=1}^{n} f_i(x_i+h_i) + \sum_{i<j} f_{i,j}(x_i+h_i, x_j+h_j) + ... + f_{1,2,...n}(x_1+h_1, x_2+h_2, ..., x_n+h_n) \tag{32}$$

Subtracting term by term eq. (31) from eq. (32), we have:

$$\begin{aligned} f(x+h) - f(x) &= \textstyle\sum_{i=1}^{n} f_i(x_i+h_i) - \sum_{i=1}^{n} f_i(x_i) + \\ &+ \textstyle\sum_{i<j} f_{i,j}(x_i+h_i, x_j+h_j) - \sum_{i<j} f_{i,j}(x_i, x_j) + ... + \\ &+ f_{1,2,...n}(x_1+h_1, x_2+h_2, ..., x_n+h_n) - f_{1,2,...n}(x_1, x_2, ..., x_n) \\ &= \textstyle\sum_{i=1}^{n} [f_i(x_i+h_i) - f_i(x_i)] + \sum_{i<j} [f_{i,j}(x_i+h_i, x_j+h_j) - f_{i,j}(x_i, x_j)] + ... + \\ &+ [f_{1,2,...n}(x_1+h_1, x_2+h_2, ..., x_n+h_n) - f_{1,2,...n}(x_1, x_2, ..., x_n)] \end{aligned} \tag{33}$$

from which, letting $\Delta f_i = f_i(x_i+h_i) - f_i(x_i)$, $\Delta f_{i,j} = f_{i,j}(x_i+h_i, x_j+h_j) - f_{i,j}(x_i, x_j)$, eq. (10) follows. ■

Proof of Corollary 1. If $d\mu = \prod_{i=1}^{n} \delta(x_i^1 - x_i^0) dx_i$, then each of the $f_{i_1,i_2,...,i_k}$ in eq. (10) is the value of f obtained with $x_{i_1}, x_{i_2}, ..., x_{i_k}$ at $x_{i_1}^1, x_{i_2}^1, x_{i_k}^1$ and the remaining parameters at their base case value. ■

Proof of Theorem 3. One needs to regard each of the terms in eq. (12) as functions. We shall let $h = x - x^0$ and use the notation dx when $h \to \mathbf{0}$. We start re-writing eqs.

(12) as:

$$\begin{cases} m_i(x_i) = \Delta_i f(x_i) = f(x_1^0, x_2^0, ..., x_i, ..., x_n^0) - f(x^0) \\ m_{i,j}(x_i, x_j) = \Delta_{i,j} f(x_i, x_j) = f(x_1^0, x_2^0, ..., x_i, ..., x_j, ..., x_n^0) - m_i(x_i) - m_j(x_j) - f(x^0) \\ ... \end{cases} \tag{34}$$

Let us now suppose that $h\mapsto \mathbf{0}$ and that f is smooth (i.e., it can be expanded in Taylor series). We first prove that, when changes are small, the following relationships hold:

$$\begin{cases} m_i(x_i) \simeq f_i'(x^0)dx_i \\ m_{i,j}(x_i, x_j) \simeq f_{i,j}''(x^0)dx_i dx_j \\ m_{i,j,k}(x_i, x_j, x_k) \simeq f_{i,j,k}'''(x^0)dx_i dx_j dx_k \\ etc. \end{cases} \tag{35}$$

We firstly note that eq. (34) implies that the functions $m_{ij...k}$ have the same differentiability properties of f. The first order terms can be expanded as $m_i(x_i) = f_i'(x^0)dx_i + o(dx_i)$, from which the first line in eq. (35) follows. For the second order terms, we have

$$\begin{aligned} & m_{i,j}(x_i, x_j) = f(x_1^0, x_2^0, ..., x_i, ..., x_j, ..., x_n^0) - m_i(x_i) - m_j(x_i) - f(x^0) = \\ & = f(x^0) + f_i'(x^0)dx_i + f_j'(x^0)dx_j + \tfrac{1}{2}[f_i''(x^0)\,(dx_i)^2 + 2f_{i,j}''(x^0)dx_i dx_j + f_j''(x^0)\,(dx_j)^2] + \\ & \quad + o(\|\mathrm{d}x\|^2) - m_i(x_i) - m_j(x_i) - f(x^0) \end{aligned} \tag{36}$$

As the changes dx_i and dx_j tend to zero, we can write:

$$\begin{aligned} & f_i'(x^0)dx_i + f_j'(x^0)dx_j + \tfrac{1}{2}[f_i''(x^0)\,(dx_i)^2 + 2f_{i,j}''(x^0)dx_i dx_j + f_j''(x^0)\,(dx_j)^2] + o(\|\mathrm{d}x\|^2) + \\ & -f_i'(x^0)dx_i - \tfrac{1}{2}f_i''(x^0)\,(dx_i)^2 - o(\|dx_i\|^2) - f_j'(x^0)dx_j - \tfrac{1}{2}f_j''(x^0)\,(dx_j)^2 + o(\|\mathrm{d}x\|^2) \\ & = f_{i,j}''(x^0)dx_i dx_j \end{aligned} \tag{37}$$

The same reasoning leads to the third line in eq. (35), and one can proceed in the same way for higher order terms. Combining eqs. (18) and (34), one gets:

$$\varphi_l^T = m_l(x_l) + \sum_{\substack{j=1 \\ j\neq l}}^{n} m_{j,l}(x_l, x_j) + ... + m_{1,2,...,n}(x_1, x_2, ..., x_n) \tag{38}$$

Utilizing eqs. (35) one can write:

$$\varphi_l^T = f_l'(x^0)h_l + \sum_{\substack{j=1 \\ j\neq l}}^{n} f_{j,l}''(x^0)h_j h_l + ... + f_{1,2,..,n}^n(x^0)h_1 h_2 ... h_n + o(\|h\|^n) \tag{39}$$

Then:
$$\frac{\varphi_l^T}{h_l} = f_l'(x^0) + \sum_{\substack{j=1 \\ j\neq l}}^{n} f_{j,l}''(x^0)h_j + ... + f_{1,2,..,n}^n(x^0)h_1h_2...h_n + o(\|h\|^n) \tag{40}$$

Thus,

$$\lim_{h\to\mathbf{0}} \frac{\varphi_l^T}{h_l} = \lim_{h\to\mathbf{0}} \left[f_l'(x^0) + \sum_{\substack{j=1 \\ j\neq l}}^{n} f_{j,l}''(x^0)h_j + ... + f_{1,2,..,n}^n(x^0)h_1h_2...h_n + o(\|h\|^n) \right] = f_l'(x^0) \tag{41}$$

q.e.d..
Similarly for Φ_l^T, we have:

$$\Phi_l^T = \frac{f_l'(x^0)h_l + \sum_{\substack{j=1 \\ j\neq l}}^{n} f_{j,l}''(x^0)h_lh_j + ... + f_{1,2,..,n}^n(x^0)h_1h_2...h_n + o(\|h\|^n)}{\Delta f} \tag{42}$$

As $h \mapsto \mathbf{0}$, $\Delta f \to \mathrm{d}f$ and the numerator tends to $\mathrm{d}_l f$, and, thus,

$$\lim_{h\mapsto\mathbf{0}} \Phi_l^T = \frac{\mathrm{d}_l f'}{\mathrm{d}f} = \Gamma_l \tag{43}$$

The equality of these limits with the first order indices is readily obtained by recalling that the first term ($f_l'(x^0)h_l$) descend from the first line in eq. (35), and that $\varphi_l^1 = m_l(x_l)$.
■

Proof of Proposition 1. From eqs. (34) and eq. (11), we have that:

$$\Delta y = \sum_{i=1}^{n} \Delta_i f + \sum_{i<j}^{n} \Delta_{i,j} f + ... + \Delta_{1,2,...,n} f = \sum_{i=1}^{n} m_i(x_i) + \sum_{i<j}^{n} m_{i,j}(x_i, x_j) + ... + m_{1,2,...,n}(x_1, x_2, ..., x_n) \tag{44}$$

Now, we have to deal with the situation in which x_l stays fixed. To simplify the notation, let us order the parameters so that x_n is fixed. The change $\Delta y_{(-n)}$ can be still decomposed according to eq. (44), which becomes:

$$\Delta y_{(-n)} = \sum_{i=1}^{n-1} \Delta_i f + \sum_{i<j}^{n-1} \Delta_{i,j} f + ... + \Delta_{1,2,...,n-1} f \tag{45}$$

Taking the term by term difference between eqs. (44) and (45), one gets:

$$\Delta y - \Delta y_{(-n)} = \Delta_n f + \sum_{i=1}^{n-1} \Delta_{in} f + ... + \sum_{n\in i_1,i_2,...,i_{n-1}} \Delta_{i_1,i_2,...,i_{n-1}} f + \Delta_{1,2,...,n} f \tag{46}$$

Now, to complete the proof, it suffices to note that the right hand side of eq. (46) is indeed φ_n^T (by Definition 3).

We also propose an alternative proof. One can show that in eqs. (34) all terms that contain x_l become null when x_l is fixed. Thus, the difference between Δy and $\Delta y_{(-l)}$ equals the sum of all terms in eq. (11) that contain x_l, which is namely φ_l^T. Thus, we need to show that all terms containing x_l become null, if x_l does not vary. In fact, evaluating $m_l(x_l)$ at x_l^0 one gets:

$$m_l(x_l^0) = f(x_1^0, x_2^0, ..., x_l^0, ..., x_n^0) - f(x^0) = 0 \tag{47}$$

Consider then the second order terms in eqs. (34) and (11). From eqs. (12), (13) and (47) one has:

$$f(x_1^0, x_2^0, ..., x_l^0, ..., x_j, ..., x_n^0) = m_j(x_j) + f(x^0) \tag{48}$$

Then, substituting in the second row of eq. (34), one has:

$$m_{j,l}(x_l^0, x_j) = m_j(x_j) + f(x^0) - m_j(x_j) - f(x^0) = 0 \tag{49}$$

and similarly for the higher order terms. This completes the alternative proof. ■

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Table 1: Notation and list of symbols for this work

Symbol	Meaning
Q^*	Numerical value of the Economic Order Quantity
y	Decision-support criterion/model output
$x = (x_1, x_2, ..., x_n)$	Exogenous variables (parameters); factors in Eschenbach (1992)
X	Parameter space
x^0, x^1	Any two sets of values the parameters can assume
h	$x^1 - x^0$
CS	Comparative statics matrix
$\mathcal{F}$	Linear space of functions
μ	Measure
Γ	Differential importance matrix
$D_{j,i}$	Differential Importance of x_i with respect to y_j
$f_{i_1,i_2,...,i_k}$	Generic term in the HDMR expansion [eq.(5)]
$V_{i_1,i_2,...,i_k}$	Partial Variance [eq. (8)]
$S_{i_1,i_2,...,i_k}$	Global sensitivity index of order k [eq.(9)]
$\Delta_{i_1,i_2,...,i_k} f$	Orthogonalized change in f due to the changes in $x_{i_1}, x_{i_2}, ..., x_{i_k}$ [eq. (12)]
$g_{i_1,i_2,...,i_k}$	Value of f with parameters $x_{i_1}, x_{i_2}, ..., x_{i_k}$ shifted [eq. (13)]
$\varphi^k_{i_1,i_2,...,i_k}$	Finite change sensitivity index of order k [eq.(14)]
$\Phi_{i_1,i_2,...,i_k}$	Normalized finite change sensitivity index of order k [eq.(15)]
φ^1_l / Φ^1_l	First order finite change sensitivity indices [eq. (16)/ normalized eq.(17)]
φ^T_l / Φ^T_l	Total order finite change sensitivity indices [eq. (18)/ normalized eq.(19)]
$x^1_{(-l)}$	Point of X obtained by shifting all parameters at the new value but x_l [eq.(20)]

Table 2: Finite change sensitivity indices for the case study

φ^1_u	φ^1_a	φ^1_R	φ^1_β	φ^1_ρ	$\varphi^2_{\rho\beta}$	$\varphi^2_{\rho R}$	$\varphi^2_{\beta R}$	$\varphi^2_{\rho u}$	$\varphi^2_{\beta a}$	φ^2_{Ra}	$\varphi^2_{\rho u}$	$\varphi^2_{u\beta}$
-95	-19	27	-100	-27	3	-1	-3	1	2	-1	3	12
φ^2_{Ru}	φ^2_{au}	$\varphi^3_{\rho\beta R}$	$\varphi^3_{\rho\beta a}$	$\varphi^3_{\rho Ra}$	$\varphi^3_{aR\beta}$	$\varphi^3_{\rho\beta u}$	$\varphi^3_{\rho Ru}$	$\varphi^3_{\beta Ru}$	$\varphi^3_{\rho au}$	$\varphi^3_{a\beta\rho}$	$\varphi^3_{R\beta\rho}$	$\varphi^4_{uaR\beta}$
-3	6	0	0	0	0	0	0	0	0	-1	0	0
$\varphi^4_{uaR\rho}$	$\varphi^4_{ua\beta\rho}$	$\varphi^4_{uR\beta\rho}$	$\varphi^4_{aR\beta\rho}$	$\varphi^5_{uaR\beta\rho}$								
0	0	0	0	0								

Table 3: Total order finite change sensitivity indices

	u	a	R	β	ρ
φ^T	-78	-11	20	-87	-21
Φ^T	0.40	0.06	0.10	0.44	0.11

Table 4: Key-drivers comparison

	u	a	R	β	ρ
Φ	2	5	4	1	3
Γ	4	5	1	3	2