The Reliability Importance of Components and Prime Implicants in Coherent and Non-Coherent Systems Including Total-Order Interactions

by

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Abstract

In the management of complex systems, knowledge of how components contribute to system performance is essential to the correct allocation of resources. Recent works have renewed interest in the properties of the joint (J) and differential (D) reliability importance measures. However, a common background for these importance measures has not been developed yet. In this work, we build a unified framework for the utilization of J and D in both coherent and non-coherent systems. We show that the reliability function of any system is multilinear and its Taylor expansion is exact at an order T. We then introduce a total order importance measure (D^T) that coincides with the exact portion of the change in system reliability associated with any (finite or infinitesimal) change in component reliabilities. We show that D^T synthesizes the Birnbaum, joint and differential importance of all orders in one unique indicator. We propose an algorithm that enables the numerical estimation of D^T by varying one probability at a time, making it suitable in the analysis of complex systems. Findings demonstrate that the simultaneous utilization of D^T and J provides reliability analysts with a complete dissection of system performance.

Keywords: Reliability, Safety; Reliability, Coherent Systems; Probability, Applications.

1 Introduction

Reliability engineers face critical operational decisions such as the determination of optimal maintenance, inspection and replacement policies [Ozekici (1988), Dogramaci and Fraiman (2004), Kubzin and Strusevich (2006), Chun (2008), Castro (2009)][1], the assignment of components to graded quality assurance programs, the categorization of system structures and components [Cheok et al (1998), Vesely (1998), Borgonovo (2008)]. These decision problems are

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[1] One of the first works addressing the problem of optimal replacement policies in multicomponent systems is Ozekici (1988). In Dogramaci and Fraiman (2004), a model is developed to understand how "*a manager* (should) *make replacement decisions for a chain of machines over time* (p. 785)." Kubzin and Strusevich (2006) develop a model to "*initiate research on scheduling models with maintenance periods of controllable length*; Kubzin and

most often characterized by trade-offs between safety levels, system performance and economic viability. As Dillon et al (2003) state, managers of complex systems (projects) "*face a challenge when deciding how to allocate scarce resources to minimize the risks of project (*system*) failure. As resource constraints become tighter, balancing these failure risks is more critical, less intuitive, and can benefit from the power of quantitative analysis* [Dillon et al (2003); p. 354]."

Quantitative models play a central role in the decision-making process, as they provide analysts with crucial insights on how to achieve a given system performance. Several studies have demonstrated that components do not contribute to system performance in the same way [Lambert (1975), Butler (1977), El-Neweihi (1980), Vesely et al (1990), Cheok et al (1998), Borgonovo and Apostolakis (2001), Zio and Podofillini (2006), Lu and Jiang (2007), Gao et al (2007)]. Thus, it is essential for analysts "*to identify ... the critical components* [Gao et al (2007); p. 282]." This information is delivered by importance measures. The recent works of Zio and Podofillini (2006), Lu and Jiang (2007), Gao et al (2007) and Do Van et al (2008) have renewed interest towards the study and utilization of importance measures. In particular, they have extended the joint and differential importance measures [Hong and Lie (1993), Borgonovo and Apostolakis (2001)] so as to capture higher-order interactions. An examination of the literature (see Section 2) shows that the research on these importance measures has sofar proceeded on two parallel but almost independent tracks, with works addressing the properties of either one of the importance measures. The lack of a comprehensive framework limits the insights analysts can obtain from the decision-support model. In fact, as we are to show, it is the simultaneous utilization of the two importance measures that allows analysts to exploit the reliability model information at best.

Our purpose is to build a unified framework for the utilization of the joint and differential reliability importance measures. In this respect, we observe that the distinction between coherent or non-coherent system marks a quite net partition in the literature. For instance, the works of Barlow and Proschan (1965), Birnbaum (1969), Lambert (1975), Agrawal and Barlow (1984), El-Neweihi (1980), Ball and Provan (1988), Hong and Lie (1993), Armstrong (1995), Cheok et al (1998), Borgonovo and Apostolakis (2001), Giglio and Wynn (2004), Zio and Podofillini (2006), Borgonovo (2007), Gao et al (2007), Do Van et al (2008) assume coherent systems, while Inagacki and Henley (1980), Andrews and Beeson (2003), Beeson and Andrews (2003), Lu and Jiang (2007) address non-coherent systems. Thus, we set up our analysis so that our findings hold independently of the system type. For each finding, however, we discuss whether/how the finding is affected by the coherency (or non-coherency) of the system. We begin with the properties of multilinear functions. In particular, we address the coincidence of a multilinear function with its Maclaurin and Bernstein polynomials. By showing that the reliability function of any coherent and non-coherent system is multilinear, we prove that the Maclaurin (or Taylor) expansion of any system reliability function is exact and can be arrested at a finite order T, where $T \leq N$ (see Table 1 for notation and symbols used in this work).

We discuss how this finding relates to classical reliability results (in particular Theorem 3.2

Strusevich (2006), p.790". In Chun (2008), a Bayesian approach is utilized to support the determination of optimal sequential inspection policies a complex product (as a software) "*to further improve its quality and reliability* (Chun (2008); p. 235)." Castro (2009) presents a model to support maintenance policy selection when the system is characterized by two dependent failure modes with imperfect repair.

Table 1: Notation and list of symbols for this work

Symbol	**Meaning**
f	generic function
$\mathbf{x}$	(vector of) independent variables
n	number of independent variables
$\mathbf{Z}$	vector of Boolean variables indicating component states
N	number of components
$(1_i, \mathbf{x})$	vector with component x_i set to 1 and the remaining components at x
$(0_i, \mathbf{x})$	vector with component x_i set to 0 and the remaining components at x
$\phi(\mathbf{Z})$	structure function
$\mathbf{O}$	set of prime implicants
m_i	number of components in a prime implicant
M	number of prime implicants
MCS; MPS	minimal cut set; minimal path set
$\mathbf{q}$	(vector of) component unreliabilities
$\mathbf{p}$	(vector of) component reliabilities
h	reliability
h'_i	partial derivative of the reliability function with respect to x_i
Q	unreliability
$G(\mathbf{x})$	reliability/unreliability function in coherent/non-coherent systems
$J^k_{i_1,i_2,\dots,i_k}$	joint reliability importance of $x_{i_1}, x_{i_2}, ..., x_{i_k}$
T	maximum order of the Taylor expansion of $G(\mathbf{x})$
D_l	differential importance of x_l
B_l	Birnbaum importance
D^k_l	differential importance of order k of x_l
D^T_l	total differential importance of x_l

in El-Neweihi (1980)) and observe that it provides an answer to the research question opened by Do Van et al (2008) concerning the determination of "*the bounds of Maclaurin series to find the minimal k for which the differential importance of order k can provide the true importance ranking* [Do Van et al (2008); p. 7]." It is then possible to prove that, if the rare event approximation applies, then for both coherent and non-coherent systems: *a*) the joint reliability importance of components $i_1, i_2, ..., i_k$ is equal to unity, if they are a prime implicant; and *b*) a group of components has null joint reliability importance if and only if it is not contained in any prime implicant. We also offer an interpretation of these results in terms of the probability of group $i_1, i_2, ..., i_k$ being critical to the system.

The fact that the Taylor expansion of any system reliability function can be arrested at an order T has the following relevant consequence: any finite change in reliability that follows a discrete change in component failure probabilities is exactly apportioned at an order of at most T. We then introduce a new importance indicator, the total order importance measure, denoted by D^T. D^T includes the joint differential importance ($J^k_{i_1,i_2,..,i_k}$) of all groups of components ($k = 1, 2, ..., T$). It is shown that D^T is the exact fraction of the change in system reliability caused by generic changes in component reliabilities/unreliabilities, both for coherent and non-coherent systems, and both in the presence/absence of the rare event approximation. We study the limiting properties of D^T, proving that it ends to the differential importance (D), when changes become small. Furthermore, D^T tends to the Birnbaum importance measure (B), if uniform changes are assumed. Results show that, however, D^T differs from lower orders D^k and from D (or B) significantly as interaction effects become relevant — i.e., when changes are finite. —

The remainder of the paper is organized as follows. Section 2 provides a literature review and the definitions of Birnbaum, differential and joint importance measures. Section 3 lays out the mathematical framework of the work. Section 4 presents a general result for reliability functions of coherent and non-coherent systems. Section 5 introduces the total order importance measure. An algorithm for the numerical estimation of D^T is presented in Section 6. Section 7 offers conclusions.

2 Birnbaum, Differential and Joint Reliability Importance Measures

This section investigates the definitions and relationships between the Birnbaum (B), Joint (J) and Differential (D) reliability importance measures.

The concept of reliability importance stems from the seminal work of Birnbaum (1969) (Table 2).

In Birnbaum (1969) a system of N components is considered. The Birnbaum importance of component i (B_i) is defined as the probability of component i being critical to the system. Letting $\mathbf{p}$ be the (vector of) component success probabilities, and $h(\mathbf{p}) : [0,1]^N \to \mathbb{R}$ the reliability function of the system (see Table 1 for notation), Birnbaum (1969) proves that:

$$P(\text{component } i \text{ is critical}) = \frac{\partial}{\partial p_i} h(\mathbf{p}) = B_i(\mathbf{p}) \tag{1}$$

B is also referred to as marginal reliability importance in later works [Hong and Lie (1993),

Table 2: Synthesis of the lietature review on works concerning joint and differential reliability importance

Work	**Importance Measure**	**System Type**	**Interaction Order**
Birnbaum (1969)	B	Coherent	1
Hong and Lie (1993)	J^{II}	Coherent	2
Armstrong (1995)	J^{II}	Coherent	2
Borgonovo and Apostolakis (2001)	D	Coherent	1
Andrews and Beeson (2003)	B	Non-Coherent	1
Zio and Podofillini (2006)	D^{II}	Coherent	2
Lu and Jiang (2007)	J^{II}	Non-Coherent	2
Gao et al (2007)	J^k	Coherent	k
Do Van et al (2008)	D^k	Coherent	k

Armstrong (1995), Lu and Jiang (2007).] In Andrews and Beeson (2003), it is observed that in a non-coherent system a component can be both failure-critical and repair-critical. Repair-critical means that the system is in a state such that the component is repaired, then the system fails. By utilizing the calculation procedure of Inagacki and Henley (1980), Andrews and Beeson (2003) show that P(component i is critical) is the sum of the probabilities that component i is critical while working and while failed. As a consequence [Andrews and Beeson (2003)], B_i in a non-coherent system is obtained by separate differentiations of the system reliability function with respect to component i reliability and unreliability.

Birnbaum's work has been followed by a wide literature. In the late '70s and early '80s, the theory of importance measures has paralleled the development of reliability theory itself. We recall the works of Lambert (1975), Barlow and Proschan (1976), El-Neweihi (1980), Boland and Proschan (1983). In these works, finding are devoted to the importance of individual components, with independent failure/success in coherent systems.

The works of Hong and Lie (1993), and Armstrong (1995) introduce the joint reliability importance of components i and s (J^{II}) as:

$$J_{i,s}^{II}(\mathbf{p}) := \frac{\partial h(\mathbf{p})}{\partial p_i \partial p_s} \qquad i, s = 1, 2, ..., N;\ i \neq s, \tag{2}$$

The motivation is to provide system analysts with "*information about how component reliabilities affect each other* [Armstrong (1995); p. 408]." This is clarified by the following property of $J_{i,s}^{II}$, proven in Armstrong (1995):

$$J_{i,s}^{II}(\mathbf{p}) = h(1_i, 1_s, \mathbf{p}) - h(0_i, 1_s, \mathbf{p}) - h(1_i, 0_s, \mathbf{p}) + h(0_i, 0_s, \mathbf{p}) \tag{3}$$

Eq. (3) shows that in $J_{i,s}^{II}$ first order effects $h(0_i, 1_s, \mathbf{p})$, $h(1_i, 0_s, \mathbf{p})$ are subtracted from $h(1_i, 1_s, \mathbf{p})+$ $h(0_i, 0_s, \mathbf{p})$. $J_{i,s}^{II}$ then measures a residual (interaction) second order effect. As a consequence, by $J_{i,s}^{II}$ one does not measure the overall importance of the group formed by components i and s. In fact, to measure such importance the sensitivity measure should also account for their individual effects [see Zio and Podofillini (2006)].

According to Lu and Jiang (2007) the sign of J^{II} "*carries critical information for both* *coherent, and non-coherent fault trees* [Lu and Jiang (2007); p. 436]." In particular, if $J^{II}_{i,s} > 0$, component i (or s) becomes more important when component s is perfectly reliable [synergy, as defined in Armstrong (1995).] If $J^{II}_{i,s} < 0$, the converse statement applies. If $J^{II}_{i,j} = 0$, the importance of i is independent of the state of component s. These properties are proven in Armstrong (1995) for coherent systems. The extensions of B and J^{II} to non-coherent systems is achieved in the works of Andrews and Beeson (2003), Beeson and Andrews (2003) and Lu and Jiang (2007).

Gao et al (2007) extend J^{II} to groups involving more than two components defining the joint reliability importance of order k as:

$$J^{k}_{i_1,i_2,\dots,i_k}(\mathbf{p}) := \frac{\partial^k h(\mathbf{p})}{\partial p_{i_1}\partial p_{i_2}...\partial p_{i_k}} \tag{4}$$

Gao et al (2007) prove that formal properties of J^{II} hold also for J^k in coherent systems.

The differential importance measure (D) is introduced in Borgonovo and Apostolakis (2001) as follows:

$$D_i(\mathbf{p};\mathrm{d}\mathbf{p}) := \frac{h'_i(\mathbf{p})\mathrm{d}p_i}{\sum_{s=1}^{N} h'_s(\mathbf{p})\mathrm{d}p_s} \tag{5}$$

where $h'_i(\mathbf{p})$ is the partial derivative of the reliability function w.r.t. the i^{th} probability. D generalizes the Birnbaum and the Criticality importance measures by accounting for the relative changes (direction of change) in component reliabilities. In particular, if one supposes uniform changes ($\mathrm{d}p_i$=$\mathrm{d}p_s$ $\forall i, s = 1, 2, ..., N$, i.e., all component reliabilities are varied by the same amount), then, by eqs. (1) and (5), one obtains:

$$D1_i(\mathbf{p}) = \frac{B_i(\mathbf{p})}{\sum_{s=1}^{N} B_s(\mathbf{p})} \quad i = 1, 2, ..., N \tag{6}$$

Eq. (6) implies proportionality between $D1_i(\mathbf{p})$ and $B_i(\mathbf{p})$. Hence, under the assumption of uniform changes, D provides the same ranking as B. Conversely, by ranking components using B, an analyst is implicitly stating the assumption of uniform changes in component reliabilities [Borgonovo and Apostolakis (2001)]. Borgonovo and Apostolakis (2001) show that, if the assumption of proportional changes is stated, ranking component using D is equivalent to rank them using the Criticality and Fussell-Vesely importance measures [see also Borgonovo (2007)].

We recall that the introduction of D is motivated by the need of computing the importance of simultaneous changes in component reliabilities, issue raised by the work of Cheok et al (1998). In applications as the evaluation of changes in maintenance policies and graded quality assurance programs, groups of components are simultaneously affected by the changes. By utilizing D, the importance of a group of components is found as sum of individual importances of the components in the group [Borgonovo and Apostolakis (2001)]:

$$D_{i_1,i_2,\dots i_k}(\mathbf{p};\mathrm{d}\mathbf{p}) = \sum_{s=1}^{k} D_{i_s}(\mathbf{p};\mathrm{d}\mathbf{p}), \;\; \forall i_1, i_2, ...i_k, \quad k = 1, 2, ..., N \tag{7}$$

Eq. (7) is particularly relevant in the presence of large reliability models as it allows to obtain component importance without further model evaluations. — This is often the case in safety applications, Epstein and Rauzy (2005). —

By comparing $D_{i_1,i_2,...i_k}$ to $J^k_{i_1,i_2,...i_k}$, one notes that $J^k_{i_1,i_2,...i_k}$ does not deliver the overall importance of the group formed by components $i_1, i_2, ...i_k$, but measures the effect of their (residual) interaction. Conversely, $D_{i_1,i_2,...i_k}$ measures the overall importance of the group as the sum of their individual effects. Zio and Podofillini (2006) then introduce a second order D, (D^{II}), to determine component importance including second order interactions. The definition is as follows. Given the second order Taylor expansion of a reliability change,

$$\Delta^{II} h = \sum_{i=1}^{N} B_i(\mathbf{p}) dp_i + \sum_{i=1}^{N} \sum_{s>i}^{N} J_{i,s}(\mathbf{p}) \mathrm{d}p_i \mathrm{d}p_s, \tag{8}$$

Zio and Podofillini (2006) consider the fraction of $\Delta^{II} h$ associated with component i:

$$\Delta^{II} h_i = B_i(\mathbf{p}) dp_i + \sum_{s\neq i}^{N} J_{i,s}(\mathbf{p}) \mathrm{d}p_i \mathrm{d}p_s \tag{9}$$

Then, D^{II}_i is defined as:

$$D^{II}_i(\mathbf{p}; \mathrm{d}\mathbf{p}) := \frac{\Delta^{II} h_i}{\Delta^{II} h} = \frac{B_i(\mathbf{p})\mathrm{d}p_i + \sum_{s\neq i}^{N} J_{i,s}(\mathbf{p})\mathrm{d}p_i \mathrm{d}p_s}{\sum_{s=1}^{N} B_s(\mathbf{p})\mathrm{d}p_s + \sum_{t=1}^{N} \sum_{s>t}^{N} J_{t,s}(\mathbf{p})\mathrm{d}p_t \mathrm{d}p_s} \tag{10}$$

By this definition, D^{II} captures second order interaction effects in determining component i importance. However, as Zio and Podofillini (2006) point out, D^{II} looses the additivity property. We observe that eq. (10) can be regarded as offering the relationship between D^{II}, B and J^{II}.

D and D^{II} are extended to order k ($k < N$) by Do Van et al (2008). Letting

$$\Delta^k h = \sum_{s=1}^{k} \sum_{r_1<r_2<...<r_s} h^s_{r_1,r_2,...,r_s}(\mathbf{p}) \mathrm{d}p_{r_1} \mathrm{d}p_{r_2} ... \mathrm{d}p_{r_s} \tag{11}$$

denote the Taylor approximation of order $k < N$ of the change in system reliability and denoting with

$$\Delta^k h_i = \sum_{k=1}^{k} \sum_{i\in r_1,r_2,...,r_k} h^s_{r_1,r_2,...,r_s}(\mathbf{p}) \mathrm{d}p_{r_1} \mathrm{d}p_{r_2} ... \mathrm{d}p_{r_s} \tag{12}$$

the fraction of the right-hand-side in eq. (11) associated with component l, one has [Do Van et al (2008)]:

$$D^k_i(\mathbf{p}; \mathrm{d}\mathbf{p}) = \frac{\Delta^k h_i}{\Delta^k h} = \frac{\sum_{k=1}^{N} \sum_{i\in r_1,r_2,...,r_k} h^s_{r_1,r_2,...,r_s}(\mathbf{p}) \mathrm{d}p_{r_1} \mathrm{d}p_{r_2} ... \mathrm{d}p_{r_s}}{\sum_{s=1}^{k} \sum_{r_1,r_2,...,r_s} h^s_{r_1,r_2,...,r_s}(\mathbf{p}) \mathrm{d}p_{r_1} \mathrm{d}p_{r_2} ... \mathrm{d}p_{r_s}} \tag{13}$$

As in the case of $D^{II}_i(\mathbf{p};\mathrm{d}\mathbf{p})$, also $D^k_i(\mathbf{p};\mathrm{d}\mathbf{p})$ does not share the additivity property. Eq. (13) can also be regarded as defining the relationship between D^k_l, B and $J^s_{r_1,r_2,...,r_s}$, $s = 2, 3, ..., k$.

One of the questions that Zio and Podofillini (2006), Lu and Jiang (2007), Gao et al (2007) and Do Van et al (2008) leave open is the order k at which one needs to stop the Taylor expansion. In the next sections, we derive a result that allows to answer such question for coherent and non-coherent systems.

3 Multilinear Functions and Maclaurin (Taylor) Expansion

Reliability theory originates with the seminal works of Barlow and Proschan (1965), Birnbaum (1969), Barlow and Proschan (1975), Lambert (1975) [see also Block (2001).] Core of the theory is the representation of a system as a set of N components, each of which can be in two possible states [Birnbaum (1969), Ball and Provan (1988), Boros et al (2000), Giglio and Wynn (2004), Khachiyan et al (2007).] In these instances, multilinear functions play a crucial role and many reliability results are a consequence of the properties of Boolean functions [Birnbaum (1969), Agrawal and Barlow (1984), Ball and Provan (1988), Fishman (1989), Boros et al (2000), Khachiyan et al (2007).] In this section, we discuss the properties of multilinear functions that are essential in studying the relationship between J^k and D^k.

Multilinear functions have been extensively studied in set-function theory [Hammer and Rudeanu (1968), Grabisch et al (2000), Foldes and Hammer (2005)], game-theory [Grabisch et al (2003), Lambert III et al (2005), Alonso-Meijide et al (2008)], multiattribute decision-making [Bordley and Kirkwood (2004), Kirkwood and Sarin (1980)] and optimization [Russell-Philbrick and Kitanidis (2001), Sherali and Driscoll (2002), Floudas and Gounaris (2009).] In set-function language, let $E = [1, 2, ..., n]$ and $\mathcal{P}(E)$ the associated finite power set. A map $m : \mathcal{P}(E) \to \mathbb{R}$ is called a set function. Hammer and Rudeanu (1968) (see Theorem 6, p. 21) prove that "*every pseudo-Boolean function is represented by a unique multilinear polynomial* [Foldes and Hammer (2005); p. 453]." The following statement by Bordley and Kirkwood (2004) (p. 823) well defines the interdisciplinary relevance of multilinear functions: "*In some situations, the target-oriented preference conditions are analogous to reliability theory conditions for series or parallel failure modes in a system*". For a thorough discussion of alternative representations of multilinear functions, we refer to Grabisch et al (2000). We highlight that, from a strictly mathematical viewpoint, a function is multilinear, if it is separately affine in each variable [Marinacci and Montrucchio (2005)]. We utilize the following representation:

$$\begin{gathered} f : \mathbb{R}^n \to \mathbb{R} \\ y = f(\mathbf{x}) = \sum\nolimits_{k=1}^{n} \sum\nolimits_{i_1 < i_2 < \ldots < i_k}^{k} \delta_{i_1, i_2, \ldots, i_k} \cdot x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k} \\ \text{with } \delta_{i_1, i_2, \ldots, i_k} \in \mathbb{R}, \quad k = 0, 1, ..., n \end{gathered} \tag{14}$$

One notes that f in eq. (14) is a homogeneous function and satisfies Euler's equation of order 1:

$$\mathbf{x} \cdot \nabla f = f \tag{15}$$

In the form of eq. (14), f also coincides with its Bernstein polynomial of order 1 [Marinacci and Montrucchio (2005)].

The next example illustrates eq. (14).

Example 1 *$f:\mathbb{R}^3 \to \mathbb{R}$, with*

$$f(x_1,x_2,x_3) = 10x_1x_2 - 5x_1x_3 \tag{16}$$

is multilinear. Using the representation of eq. (14), one has:

$$\delta_0 = \delta_1 = \delta_2 = \delta_3 = \delta_{2,3} = \delta_{1,2,3} = 0 \quad , \quad \delta_{1,2} = 10 \quad \textit{and} \quad \delta_{1,3} = -5 \tag{17}$$

We note that eq. (14) can be regarded as a multilinear function (or Bernstein polynomial) centered at the origin. However, one can translate eq. (14) by centering it at $\mathbf{x}^0$, as the next Remark discusses.

Remark 1 *Consider the function*

$$f:\mathbb{R}^n \to \mathbb{R}$$
$$f(\mathbf{x};\mathbf{x}^0) = \sum\nolimits_{k=0}^{n} \sum\nolimits_{i_1<i_2<...<i_k}^{k} \delta_{i_1,i_2,...,i_k} \cdot (x_{i_1} - x_{i_1}^0) \cdot (x_{i_2} - x_{i_2}^0) \cdot ... \cdot (x_{i_k} - x_{i_k}^0) \tag{18}$$
$$\textit{with } \delta_{i_1,i_2,...,i_k} \in \mathbb{R}, \quad k = 0,1,...,n$$

It is not difficult to see that eq. (18) satisfies Euler's equation under the boundary condition

$$f(\mathbf{x}^0) = \sum_{k=1}^{n} \sum_{i_1<i_2<...<i_k}^{k} \delta_{i_1,i_2,...,i_k} \cdot (-1)^k \prod_{s=1}^{k} x_{i_s}^0 \tag{19}$$

When referring to eq. (18), we shall say that f is a multilinear function centered at $\mathbf{x}^0$. In fact, by shifting the origin of the coordinate system from $(\mathbf{x}=\mathbf{0}, f(\mathbf{0}))$ to $(\mathbf{x}^0, f(\mathbf{x}^0))$ with $f(\mathbf{x}^0)$ as per eq. (19), eq. (18) reduces to eq. (14).

In the remainder, for the sake of notation simplicity, we shall refer to eq. (14). However, by Remark 1, all results applicable to eq. (14) are readily extended to eq. (18).

Before coming to the properties of multilinear functions relevant for this work, we introduce some notation. $\mathbf{x} = (x_1, x_2, ..., x_n)$ denotes the n-dimensional vector of independent variables, $\mathbf{x} \in \mathbb{R}^n$. $(1_{i_1}, 1_{i_2}, ..., 1_{i_k}, \mathbf{x})$ $[(0_{i_1}, 0_{i_2}, ..., 0_{i_k}, \mathbf{x})]$ indicates that the groups of variables $x_{i_1}, x_{i_2}, ..., x_{i_k}$ has been set to unity [zero], while the remaining variables are at $\mathbf{x}$. As an example, in eq. (16), $f(1,1,\mathbf{x}) = 10 - 5x_3$.

In the next Proposition, we collect a set of results concerning multilinear function properties relevant in the remainder of our work. They can be found in set-function and Boolean function theory [see Grabisch et al (2000) and Foldes and Hammer (2005)].

Proposition 1 *Let $f:\mathbb{R}^n \to \mathbb{R}$ be a multilinear function. Then:*

1. *The k^{th} mixed partial derivative with respect to $x_{i_1}, x_{i_2}, ..., x_{i_k}$ of f in eq. (14)-(18) is equal to [El-Neweihi (1980); Foldes and Hammer (2005)]:*

$$f^k_{i_1,i_2,...,i_k} = f(1_{i_1}, 1_{i_2}, ..., 1_{i_k}, \mathbf{x}) + \sum_{s=1}^{k} (-1)^s \sum_{i_1 i_2 ... i_s} f(0_{i_1}, ..., 0_{i_s}, 1_{i_{s+1}}, ..., 1_{i_k}, \mathbf{x}) \tag{20}$$

2. *The k^{th}-order partial derivative of f with respect to any variable is null [Foldes and Hammer (2005)].*

We now turn to the Maclaurin expansion of f (Taylor expansion, if eq. (18) is of concern). For this work, it is relevant to demonstrate that the Maclaurin (Taylor) expansions of a multilinear function coincides with the function itself. We offer an autonomous proof of this result. We start with the following observation.

Proposition 2 *Let f be multilinear and $i_1, i_2, ..., i_k$, $k \leq n$, be a given group of indices. Then, $f^k_{i_1,i_2,...,i_k} = 0 \; \forall \mathbf{x}$, if and only if there is no term in eq. (14) containing $\prod_{s=1}^{k} x_{i_s}$.*

Proof. Point 1. The sufficient condition is immediate. For the necessary condition, let S be set of indices of interest, $S = (i_1, i_2, ..., i_k)$. Then, eq. (14) is partitioned as follows:

$$\begin{aligned} f = {} & \sum_{s=0}^{k-1} \sum_{\substack{i_1<i_2<...<i_s \\ i_u \in S \;\; u=1,2,...,s}} \delta_{i_1,i_2,...,i_s} \cdot x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_s} + \delta_{i_1,i_2,...,i_k} \cdot x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_k} + \\ & + \sum_{s=0}^{k} \sum_{\substack{l_1<l_2<...<l_s \\ l_m \notin S \;\; m=1,2,...,s}} \delta_{l_1,l_2,...,l_s} \cdot x_{l_1} \cdot x_{l_2} \cdot ... \cdot x_{l_s} \\ & + \sum_{s=k+1}^{n} \sum_{\substack{i_1<i_2<...<i_s \\ i_u \in S \;\; u=1,2,...,s}} \delta_{i_1,i_2,...,i_s} \cdot x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_s} \end{aligned} \tag{21}$$

When differentiating w.r.t. x_{i_k} the terms $\sum_{s=0}^{k-1} \sum_{\substack{i_1<i_2<...<i_s \\ i_u \in S \;\; u=1,2,...,s}} \delta_{i_1,i_2,...,i_s} \cdot x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_s}$ drop out. Similarly, all terms in eq. (14) not containing $x_{i_1}, x_{i_2}, ..., x_{i_k}$ ($\sum_{s=0}^{k} \sum_{\substack{l_1<l_2<...<l_s \\ l_m \notin S \;\; m=1,2,...,s}} \delta_{l_1,l_2,...,l_s} \cdot x_{l_1} \cdot x_{l_2} \cdot ... \cdot x_{l_s}$) drop out. Thus, one obtains

$$f^k_{i_1,i_2,...,i_k}(\mathbf{x}) = \delta_{i_1,i_2,...,i_k} + \sum_{s=k+1}^{n} \sum_{\substack{i_1<i_2<...<i_s \\ i_u \in S \;\; u=1,2,...,s}} \delta_{i_1,i_2,...,i_s} \cdot x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_s} \tag{22}$$

One then notes that $f^k_{i_1,i_2,...,i_k}(\mathbf{x})$ is the sum of all and only the terms in f that include S. Thus, if $f^k_{i_1,i_2,...,i_k}(\mathbf{x}) = 0 \; \forall \mathbf{x}$, there is no term in f containing S. ■

Note that $f^k_{i_1,i_2,...,i_k} = 0 \; \forall \mathbf{x} \iff f^{k+r}_{i_1,i_2,...,i_k,i_{k+1},...,i_{k+r}} = 0 \; \forall \mathbf{x}$, for any function. Thus, for multilinear functions, $f^k_{i_1,i_2,...,i_k} = 0 \; \forall \mathbf{x}$ implies that there are no terms of order higher than k containing S. As an example, in eq. (16), $f'''_{1,2,3} = 0$ as there is no term containing all three variables.

Proposition 2 leads to the following properties.

Proposition 3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multilinear function. Then:*

1. *The Maclaurin expansion of f is exact and at most of order T.*
2. *T is $\min(n, r_{largest})$ where $r_{largest}$ is the size of the largest product in f.*
3. *f coincides with its Maclaurin polynomial.*

Proof. A multilinear function is $C^{\infty}(\mathbb{R}^n)$. Then, the Taylor/Maclaurin series can be, in principle, extended to infinity. However, let $k = n$ in Proposition 2. Then, no partial derivative of order $s > n$ can be different from zero. There follows that all terms in the Taylor expansion of order higher than n are null. Furthermore, $T = \min(n, r_{l\arg est})$, where $r_{l\arg est}$ is the maximum size of the terms in eq. (14). This proves points 1 and 2. We now prove point 3. By Maclaurin expansion, one has:

$$\Delta f = f(\mathbf{x}) - f(\mathbf{0}) = \sum_{i=1}^{n} f_i'(\mathbf{0}) \cdot x_i + \sum_{i_1 < i_2} f''_{i_1,i_2}(\mathbf{0}) \cdot x_{i_1} \cdot x_{i_2} + ... + \sum_{i_1 < i_2, .. < i_{k\max}} f^k_{i_1,i_2,...,i_{r\max}}(\mathbf{0}) \cdot x_{i_1} \cdot x_{i_2} \cdots x_{i_{k\max}} \tag{23}$$

By observing that $f(\mathbf{0}) = 0$ and that by eq. (22), one obtains the thesis. ■

Before coming to the reliability implications of the above results, we state the following observation.

Remark 2 *By property 3 in Proposition 3, a multilinear function coincides both with its Maclaurin and Bernstein polynomials.*

We also observe that, if the expansion is centered at $\mathbf{x}^0$, then Proposition 3 holds for Taylor expansion rather than Maclaurin expansion. Remark 2 would then be true for a Bernstein polynomial centered at $\mathbf{x}^0$.

In the next section, we investigate the implications of the above findings in the reliability theory of coherent and non-coherent systems.

4 Finite Changes in Reliability Functions of Coherent and Non-Coherent Systems

In this section, we show that the reliability (unreliability) function of any coherent and non-coherent system is multilinear.

Consider a system of N components (notations and symbols are listed in Table 1). Each component can be working or failed. Let $Z_i = \begin{cases} 1 \\ 0 \end{cases}$ denote the state variable of component i, $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$ the state vector, and $\phi(\mathbf{Z})$ the structure function of the system. $\phi(\mathbf{Z})$ is a Boolean function and $\phi(\mathbf{Z}) = 1$ denotes system success/failure. A structure function is coherent if $\phi(\mathbf{Z}^2) > \phi(\mathbf{Z}^1)$ when $\mathbf{Z}^2 \geq \mathbf{Z}^1$, and $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$. Conversely, it is non-coherent. Let $\mathbf{O} = (O_1, O_2, ..., O_M)$ be the collection of prime implicants (M denotes the number of prime implicants). In general, a prime implicant involves the success or failure of m_i components. We utilize the notation $O_i = (Z_{i_1} \wedge Z_{i_2} \wedge ... \wedge Z_{i_{m_i}})$, where Z_{i_s} represents the state variable of the s^{th} component in prime implicant i. In the fault tree terminology, if system failure is of concern, one would say that O_i is a minimal cut set (MCS) and Z_{i_s} is one of the basic events in MCS$_i$; if system success is of concern, then O_i is a minimal path set (MPS) and Z_{i_s} is one of the basic events in the MPS [Meng (2000)]. If the system is non-coherent, a prime implicant may be including some negations, e.g., $O_i = (Z_{i_1} \wedge \overline{Z_{i_2}} \wedge ... \wedge \overline{Z_{i_k}} \wedge ... \wedge Z_{i_{m_i}})$. Thus, a prime implicant is true for some $Z_{i_l} = 1$ and some $Z_{i_s} = 0$ [Inagacki and Henley (1980), Andrews and Beeson (2003), Beeson and Andrews (2003), Lu and Jiang (2007).] In reliability applications, one usually utilizes h to denote system reliability and Q to denote system unreliability, i.e. $(h = P(\phi = 1)_{success}; Q = P(\phi = 1)_{failure})$. $\mathbf{p}$ is used to denote the vector of component reliabilities and $\mathbf{q} = 1 - \mathbf{p}$ their unreliabilities. If a system is coherent, then h and Q are functions only of $\mathbf{p}$ ($h = h(\mathbf{p})$) and $\mathbf{q}$ ($Q = Q(\mathbf{q})$), respectively. When a system is non-coherent, h and Q become functions of both $\mathbf{p}$ and $\mathbf{q}$ (i.e., $h(\mathbf{p}, \mathbf{q})$, $Q(\mathbf{p}, \mathbf{q})$). Since the results we are going to prove hold for both failure and success logics and for coherent and non-coherent systems, we utilize the symbol $\mathbf{x}$ to denote the generic $\mathbf{p}$ or $\mathbf{q}$ and G to denote h or Q. Thus, we write $P(\phi = 1; \mathbf{x}) = G(\mathbf{x})$.

As discussed in Section 2, throughout the theoretical development of importance measures [from the works of Birnbaum (1969), Barlow and Proschan (1976), El-Neweihi (1980), Boland and Proschan (1983), Beeson and Andrews (2003), Borgonovo and Apostolakis (2001), Zio and Podofillini (2006), Lu and Jiang (2007), to Gao et al (2007)], the assumption of independent component failures has been stated. Under this assumption, a unique p_i/q_i describes the reliability/unreliability of component i in all prime implicants. Thus, there is a one-to-one correspondence between B_i, D_i, $J^k_{i_1,i_2,\ldots,i_k}$ and component i; hence, B_i, D_i, $J^k_{i_1,i_2,\ldots,i_k}$ convey the importance of components. In the case dependencies are present, p_i/q_i become conditional probabilities and vary with the prime implicant in which component i is included. Hence, the one-to-one correspondence between B_i, D_i, $J^k_{i_1,i_2,\ldots,i_k}$ and component i is lost. B_i, D_i, $J^k_{i_1,i_2,\ldots,i_k}$ then assume the interpretation of importance of the conditional failure of component(s) i or, equivalently, of the basic event associated with component i in the given prime implicant.

We let $\mathbf{x}$ denote the vector of conditional reliabilities/unreliabilities. Then, the following holds.

Theorem 1 *Consider a coherent or non-coherent N component system, with dependent or independent failures. Let $\mathbf{x}$ denote the vector of all conditional component (success/failure) probabilities. Then, $G(\mathbf{x})$ is a multilinear function of $\mathbf{x}$.*

Proof. By a fundamental result of Boolean logics [Boros et al (2000)]:

$$\phi = \bigvee_{i=1}^{M} O_i \tag{24}$$

Since ϕ is a Bernoulli variable, then

$$\mathbb{E}[\phi] = P(\phi = 1) = \mathbb{E}[O_1 \vee O_2 \vee \ldots \vee O_M] = P(\cup_{i=1}^{M} O_i = 1) \tag{25}$$

There follows that

$$P(\phi = 1) = \sum_{s=0}^{M} P(O_s = 1) - \sum_{i<s} P((O_i = 1) \cap (O_s = 1)) + \ldots + (-1)^M P(\cap_{i=1}^{M} (O_i = 1)) \tag{26}$$

Then, for each prime implicant, we have:

$$\begin{gathered} P(O_i = 1) = P(Z_{i_1} = 1 \cap Z_{i_2} = 0 \cap \ldots \cap Z_{i_{m_i}} = 1) = \\ = P(Z_{i_{m_i}} = 1 | Z_{i_{m_i-1}} = 1 \cap \ldots \cap Z_{i_2} = 0 \cap Z_{i_1} = 1) \cdot P(Z_{i_{m_i-1}} | Z_{i_{m_i-2}} = 1 \cap \ldots \cap Z_{i_1} = 1) \\ \cdot \ldots \cdot P(Z_{i_2} = 0 | Z_{i_1} = 1) \cdot P(Z_{i_1} = 1) \end{gathered} \tag{27}$$

Eq. (27) is, in general, a product of conditional probabilities. Let $P(Z_{i_1} = 1) = x_{i_1}, P(Z_{i_2} =$

$0|Z_{i_1}=1)=x_{i_2}$ ... and $P(Z_{i_{m_i}}=1|Z_{i_{m_i-1}}=1,...,Z_{i_2}=0,Z_{i_1}=1)=x_{i_{m_i}}$:

$$P(O_i)=\prod_{s=1}^{m_i} x_{i_s} \tag{28}$$

This formalism applies to all the terms in eq. (26) (i.e., to higher order intersections as well). By idempotency he same Z_{i_s} cannot appear twice in $O_i \cap O_j \cap ...$). Hence eq. (26) then takes on the form

$$G(\mathbf{x})=P(\phi=1)=\sum_{r=1}^{M}\sum_{m_{i_1}<m_{i_2}<...<m_{i_r}}\prod_{s_1=1}^{m_1} x_{i_{s_1}}\prod_{\substack{s_2=1\\ x_{i_{s_2}}\neq x_{i_{s_1}}}}^{m_2} x_{i_{s_2}}\cdot ... \cdot\prod_{\substack{s_r=1\\ x_{i_{s_r}}\neq ...\neq x_{i_{s_2}}\neq x_{i_{s_1}}}}^{m_r} x_{i_{s_r}} \tag{29}$$

which is multilinear. ■

We illustrate Theorem 1 by an example.

Example 2 *Consider the non-coherent system in Andrews and Beeson (2003). The prime implicants for system failure are* $(Z_1 \wedge Z_2), (Z_1 \wedge Z_3), (Z_2 \wedge \overline{Z}_3)$*. Correspondingly:*

$$Q=P[(Z_2=1\cap Z_1=1)\cup(Z_3=1\cap Z_1=1)\cup(Z_2=1\cap\overline{Z}_3=1)] \tag{30}$$

Noting that $(Z_1 \wedge Z_3) \wedge$ $(Z_2 \wedge \overline{Z}_3)$ *is necessarily false, and assuming independence, one has:*

$$\begin{aligned} Q(\mathbf{p},\mathbf{q}) &= P(Z_2=1\cap Z_1=1)+P(Z_3=1\cap Z_1=1)+P(Z_2=1\cap\overline{Z}_3=1)- \\ &-P[(Z_2=1\cap Z_1=1\cap Z_3=1)]-P[(Z_1=1\cap Z_2=1\cap\overline{Z}_3=1)= \\ &=q_2q_1+q_3q_1+q_2p_3-q_2q_1q_3-q_1q_2p_3 \end{aligned} \tag{31}$$

which is a multilinear function of 4 *variables (three variables, if one considers that* $p_3=1-q_3$*; however, in non-coherent systems it is usually preferable to keep p and q distinct, as discussed in Andrews and Beeson (2003), Beeson and Andrews (2003) and Lu and Jiang (2007).)*

One of the research questions left open by Zio and Podofillini (2006) and Do Van et al (2008), is the order at which to arrest the Maclaurin expansion of a system reliability function. The combination of Theorem 1 with Proposition 3, provides the answer to this question.

Proposition 4

1. *The Taylor expansion of the reliability/unreliability function of any system is exact. Let* T *be the highest order of the expansion. Then,* $T \leq N$*.*
2. *Any finite change in reliability/unreliability associated with any change in component reliability/unreliability* ($\Delta\mathbf{x}=\mathbf{x}-\mathbf{x}^0$) *in a coherent or non-coherent system is given by:*

$$\Delta G=G(\mathbf{x})-G(\mathbf{x}^0)=\sum_{i=1}^{N}B_i(\mathbf{x}^0)\cdot\Delta x_i+\sum_{k=2}^{T}\sum_{i_1<i_2,..<i_k}J^k_{i_1,i_2,...,i_k}(\mathbf{x}^0)\prod_{s=1}^{k}\Delta x_{i_s} \tag{32}$$

Proof. Proof of Point 1. The term of largest size in eq. (26) is the last term $[P(\cap_{i=1}^{M}(O_i = 1))]$. This term contains the intersection of all prime implicants. By definition of prime implicant:

$$P(\cap_{i=1}^{M}(O_i = 1)) = \mathbb{E}[O_1 \wedge O_2 \wedge ... \wedge O_M] \tag{33}$$

We have the following cases. Consider first a coherent system and suppose that all components are contained in the prime implicants. Then, by idempotency,

$$\mathbb{E}[O_1 \wedge O_2 \wedge ... \wedge O_M] = \mathbb{E}[Z_1 \wedge Z_2 \wedge ... \wedge Z_N] \tag{34}$$

There follows that

$$P(\cap_{i=1}^{M}(O_i = 1)) = P(Z_1 = 1 \cap Z_2 = 1 \cap ... \cap Z_N = 1) \tag{35}$$

is the product of N conditional probabilities. Thus, $r_{largest} = N$. Alternatively, if some of the components is not contained in any of the prime implicants, $r_{largest} < N$. Thus, $T \leq N$ for coherent systems. For non coherent systems the following happens. If at least one component appears in a prime implicant in a working state and in some other prime implicant in a failed state, then $O_1 \wedge O_2 \wedge ... \wedge O_M = 0$. This implies $T < N$. However, a component might appear in only one prime implicant only and in a negated state. Then, $T \leq N$.

Proof of point 2. One notes that $G(\mathbf{x})$ is multilinear by Theorem 1. Applying eq. (23) to $G(\mathbf{x})$, and centering at $\mathbf{x}^0$, one obtains the thesis. ∎

In terms of previous results, Proposition 4 extends Theorem 3.2. in El-Neweihi (1980) to the case of non-coherent systems. In addition, eq. (32) extends eq. (1) in Vesely (1998) [see also Vesely et al (1990)], removing the assumptions of coherent systems and rare events. In this respect, in real life applications the size of a system can be considerable, with $N = 10^3$ or more. However, such a high number of terms might not be needed in the practice for two reasons. The first reason is as follows. In classical reliability theory, failures are regarded as arrivals of a stochastic process: "*Barlow and Proschan examined components as they fail sequentially in time* [Lambert (1975); p. 180)]." This assumption is at the basis of the seminal works of Barlow and Proschan (1976) and many others. The failure of two components at the same time is excluded, as two events cannot happen in the same $\mathrm{d}t$. When applied to system failure, such assumption is equivalent to state that two prime implicants cannot occur at the same time. If system failure is observable, then, when $O_i = 1$, the system undergoes repair. Then, the system is failed due to at most one prime implicant at a time. In other words, $P(O_i = 1 \cap O_s = 1) = 0$. Conversely, system failure might not be observable, as for hot standby systems. However, and here is the second reason for simplification, in the practice of most industrial systems the intersection of two prime implicants is considered a rare event [Vesely et al (1990), Cheok et al (1998), Borgonovo and Apostolakis (2001), Zio and Podofillini (2006), Borgonovo (2007).] In other words, it is assumed that the probability of two prime implicants being true at the same time is negligible: $P(O_i = 1 \cap O_s = 1) \simeq 0$ (the rare event approximation). We note that this option is available in reliability software used for the probabilistic risk assessment of complex systems as SAPHIRE [Smith et al (2008).] In both situations (i.e., sequential and observable failures and rare events), only the first order terms in eq. (26) are retained. In this case, it is possible to obtain further results.

Proposition 5 *Suppose,* $P(O_i = 1 \cap O_s = 1) \simeq 0$, $\forall i, s = 1, 2..., M$ $(i \neq s)$. *Then, for both coherent and non-coherent systems:*

1.

$$T = \max_{i=1,2,...,M}(m_i) \tag{36}$$

2.

$$\forall t > T \quad J^t_{i_1,i_2,...,i_t} = 0 \tag{37}$$

$$t \leq T \quad J^t_{i_1,i_2,...,i_t} = 0 \ \forall \mathbf{x} \neq \mathbf{0} \iff i_1, i_2, ..., i_t \notin O_m, \qquad m = 1, 2, ..., M \tag{38}$$

3.

$$\Delta G = \sum_{s=1}^{T} \sum_{i_1,i_2,..,i_s} J^k_{i_1,i_2,...,i_k}(\mathbf{x}^0)\Delta x_{i_1}\Delta x_{i_2}...\Delta x_{i_s} \tag{39}$$

4. Let J_{O_i} *denote the importance of a prime implicant. Then:*

$$J_{O_i} = 1 \tag{40}$$

Proof. 1) If the rare event approximation holds, then eq. (26) becomes:

$$P(\phi = 1) \simeq \sum_{s=0}^{M} P(O_s = 1) \tag{41}$$

As a consequence, eq. (29) contains only the terms related to individual prime implicants and not to their intersections. One obtains:

$$G(\mathbf{x}) = P(\phi = 1) = \sum_{s=0}^{M} P(Z_{i_1} = 1 \cap Z_{i_2} = 0 \cap ... \cap Z_{i_{m_i}} = 1) = \sum_{r=1}^{M} \prod_{s=1}^{m_r} x_{i_s} \tag{42}$$

Each summand in $G(\mathbf{x})$ is the probability of a prime implicant. Let m_{max} be the size of the largest largest prime implicant ($m_{max} = \max_{r=1,2,...,M}(m_r)$). Then, $T = m_{max}$.
2-Eq. (37). Eq. (37) follows by Theorem 1 and Proposition 3.
2-Eq. (38). By the rare event approximation, only the summands corresponding to prime implicants appear in eq. (42). Thus, if the set $i_1, i_2, ..., i_t$ is not contained in a prime implicant, then there is no term with the corresponding variables in $G(\mathbf{x})$. This implies $J^T_{i_1,i_2,...,i_T} = 0$. Conversely, suppose $J^t_{i_1,i_2,...,i_t}(\mathbf{x}^0) = 0$. Then, $G(\mathbf{x})$ does not depend on the $x_{i_1}, x_{i_2}, ..., x_{i_t}$, i.e., it does not depend on the failure/success probabilities of components $i_1, i_2, ..., i_t$. Then, $i_1, i_2, ..., i_t$ are not included in any prime implicant.
3) Eq. (39) follows from point 1 of this Proposition and Point 3 in Proposition 4.
4) Eq. (40) follows by application of eq. (20) to eq. (42). ■

Let us illustrate the meaning of Proposition 5. Given the rare event approximation: 1) Eq. (36) states that, under the rare event approximation, no terms of order higher than the size of the largest prime implicant appear in the expression of the system reliability/unreliability; 2) Eq.

(37) implies that no joint reliability importance of order higher than $\max_{i=1,2,...,M}(m_i)$ is present; this also implies that the highest order of Taylor approximation is $T = \max_{i=1,2,...,M}(m_i)$; 3) Eq. (38) states that, if a group of components is not included in a prime implicant, then the corresponding joint reliability importance is null, and the converse statement is also true; 4) Eq. (40) states that the joint reliability importance of a prime implicant is unity, if the rare event approximation holds. We note that this holds both for coherent and non-coherent systems. In this respect, for coherent systems let us investigate the meaning of eq. (40) in the light of Lemma 3.1 in El-Neweihi (1980). In order to allow this comparison, we need to resume the independence assumption. One has [El-Neweihi (1980)]:

$$J_{O_i} = \mathbb{E}[\Delta_{i_1,i_2,...,i_{m_i}}\phi(\mathbf{Z})] \tag{43}$$

where

$$\Delta_{i_1,i_2,...,i_{m_i}}\phi(\mathbf{Z}) = \phi(1_{i_1}, 1_{i_2}, ..., 1_{i_{m_i}}, \mathbf{Z}) - \sum_{s=1}^{m_i-1}(-1)^s \sum_{i_1<i_2<...<i_s} \phi(0_{i_1}, ..., 0_{i_s}, 1_{i_{s+1}}, ..., 1_{i_k}, \mathbf{Z}) \tag{44}$$

As ϕ is a Bernoulli variable, then

$$\begin{gathered} J_{O_i} = \mathbb{E}[\Delta_{i_1,i_2,...,i_{m_i}}\phi(\mathbf{Z})] = \\ P[\phi(1_{i_1}, 1_{i_2}, ..., 1_{i_{m_i}}, \mathbf{Z}) - \sum_{s=1}^{m_i-1}(-1)^s \sum_{i_1<i_2<...<i_s} \phi(0_{i_1}, ..., 0_{i_s}, 1_{i_{s+1}}, ..., 1_{i_k}, \mathbf{Z}) = 1] \end{gathered} \tag{45}$$

Eq. (45) states that J_{O_i} is the probability that the group of events $Z_{i_1} = 1, Z_{i_2} = 1, ..., Z_{i_{m_i}} = 1$ is critical to the system [El-Neweihi (1980).] Then, $J_{O_i} = 1$ implies that this is indeed the case when $Z_{i_1}, Z_{i_2}, ..., Z_{i_{m_i}}$ is a prime implicant and $P(O_i = 1 \cap O_s = 1) = 0$.

For non coherent systems eq. (40) is obtained as a consequence of the findings in Andrews and Beeson (2003) and Lu and Jiang (2007), i.e., by treating $\mathbf{p}$ and $\mathbf{q}$ independently "*to retain the physical meanings of the respective basic events* [Lu and Jiang (2007); p. 437]." Let us illustrate this concept by an example.

Example 3 *Consider again Example 2. Recall that the prime implicants are* $(Z_1 \wedge Z_2), (Z_1 \wedge Z_3), (Z_2 \wedge \overline{Z}_3)$. *Under the rare event approximation eq. (31) becomes:*

$$Q(\mathbf{p}, \mathbf{q}) = q_2q_1 + q_3q_1 + q_2p_3 \tag{46}$$

Then, joint reliability importance of the prime implicants is $J_{q_2,q_1} = 1$, $J_{q_3,q_1} = 1$ $J_{q_2,p_3} = 1$.

In the next section, we formalize further the relationship between J and D.

5 The Total Order Importance Measure

This Section introduces a new importance measure. The new indicator provides the relationship between joint and differential reliability importance of all orders.

Consider a generic system, coherent or non-coherent. Let $G(\mathbf{x})$ be the system reliability/unreliability function.

Definition 1 *Let T be the highest order of the Taylor expansion in eq. (32). Let*

$$\Delta^T h_l := B_l \Delta x_l + \sum_{k=2}^{T} \sum_{\substack{i_1<i_2,..<i_k \\ l\in i_1<i_2,..<i_k}} J^k_{i_1,i_2,...,i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s} \tag{47}$$

denote the sum of the terms in eq. (32) containing a contribution from x_l. Then, we call

$$D_l^T := \frac{\Delta^T G_l}{\Delta G} = \frac{B_l \Delta x_l + \sum_{k=2}^{T} \sum_{\substack{i_1<i_2,..<i_k \\ l\in i_1<i_2,..<i_k}} J^k_{i_1,i_2,...,i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s}}{\sum_{i=1}^{N} B_i(\mathbf{x}^0)\cdot \Delta x_i + \sum_{k=2}^{T} \sum_{i_1<i_2,..<i_k} J^k_{i_1,i_2,...,i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s}} \tag{48}$$

the total order importance of x_l.

We have immediately the following property of D_l^T.

Remark 3 *D_l^T is the exact fraction of the change in system reliability/unreliability caused by the change in component l reliability/unreliability (by Proposition 4, the decomposition of order T of ΔG is exact.) D_l^T measures the influence of component l as a result of its individual effects and of all its possible interactions with the other components.*

Eq. (48) provides the relationship between D^T, B and J^k of all orders, $(k = 2, 3, ..., T)$. Thus, D_l^T turns out to be a sensitivity measure that synthesizes in one unique indicator the information obtained by the joint reliability importances of any order.

We prove a limiting property of D^T. When changes become small, interaction effects smoothen and D^T tends to the differential importance measure.

Proposition 6 *Let $1 < k < T$. As changes become small,*

$$D_l^T(\mathbf{x}; \Delta\mathbf{x}) \to D_l^k(\mathbf{x}; \Delta\mathbf{x}) \to D_l(\mathbf{x}; \Delta\mathbf{x}) \tag{49}$$

i.e.,

$$\lim_{\Delta\mathbf{x}\to\mathbf{0}} D_l^T = \lim_{\Delta\mathbf{x}\to\mathbf{0}} D_l^k = D_l \tag{50}$$

Proof. Consider eq. (48). Let $k < T$. As $\Delta\mathbf{x} \to \mathbf{0}$, $\Delta^T h_l \to \Delta^k h_i \to B_l \mathrm{d}x_l$ and $\Delta G \to \mathrm{d}G$. Thus, by the properties of the ratio of limits, as $\Delta\mathbf{x} \to \mathbf{0}$, the ratio $\dfrac{\Delta^T h_i}{\Delta G}$ and $\dfrac{\Delta^k h_i}{\Delta G} \to \dfrac{B_l \mathrm{d}x_l}{\mathrm{d}G} = D_l$. ■

Proposition 6 has the following consequence.

Corollary 1 *Let $\Delta x_l = \Delta x_s \quad \forall l, s = 1, 2, ..., N$ then*

$$\lim_{\Delta\mathbf{x}\to\mathbf{0}} D_l^T = D_l \propto B_l \tag{51}$$

Table 3: Importance measures for the example as in Vesely et al (1990), p. A.5.

Importance Measure	**Value**
B_1	0.003
B_2	0.006
B_3	0.0002
$J_{1,2}$	0.3
$J_{1,3}$	0.01
$J_{2,3}$	0.02
$J_{1,2,3}$	1

Table 4: Importance Measures with changes in component unavailabilities of 10-ş

Component	D_l	D_l^{II}	D_l^T
1	0.314762	0.347288	0.347393
2	0.629525	0.663099	0.663204
3	0.020984	0.024132	0.024237

Proof. From Proposition 6, when $\mathbf{\Delta x} \to \mathbf{0}$, then $D_l^T \to D_l$. If $\Delta x_l = \Delta x_s$, then by eq. (6), eq. (51) holds. ■

Corollary 1, states that, under assumptions of uniform and small changes, D^T and B produce the same ranking.

To illustrate Definition 1, Remark 3 and Proposition 6, we make use of the example in Vesely et al (1990). System failure is of concern, the prime implicant is $Z_1 \wedge Z_2 \wedge Z_3$ and the component unreliabilities are $q_1 = 0.02, q_2 = 0.01$ and $q_3 = 0.3$. The joint reliability importance of various orders are calculated in Table 3, following the analytical results in Vesely et al (1990).

To illustrate Proposition 6, let us write the expressions of component 1 first, second and total order differential importance:

$$\begin{cases} D_1 = \dfrac{B_1 \cdot \Delta q}{\Delta Q} \\ D_1^{II} = \dfrac{B_1 \cdot \Delta q + (J_{1,2} + J_{1,3})(\Delta q)^2}{\Delta Q} \\ D_1^T = \dfrac{B_1 \cdot \Delta q + (J_{1,2} + J_{1,3})(\Delta q)^2 + J_{1,2,3} \cdot (\Delta q)^3}{\Delta Q} \end{cases} \tag{52}$$

Figure 1 shows D_1,D_1^{II} and D_1^T as a function of Δq.

Figure 1 confirms that, as Δq decreases, D_1, D_1^{II} and D_1^T tend to coincide [see also Zio and Podofillini (2006).] In particular, for $\Delta q < 10^{-4}$, their values coincide for all practical purposes. We note that, in this case, D_l^T produces the same ranking as B_l, in accordance with Corollary 1. At $\Delta q_l = 10^{-3}$, however, second order effects start to be felt. Table 4 details the D_l, D_l^{II} and D_l^T for the three components when $\Delta q_l = 10^{-3}$ $(l = 1, 2, 3)$.

Table 4 shows that D_l start differing from D_l^{II} and D_l^T at $\Delta q_l = 10^{-3}$. As Δq_l increases, the values assumed by D_l^{II} and D_l^T remain close until $\Delta q_l = \dfrac{1}{2}10^{-1}$. This result indicates that second order effects prevail over first order effects, but third order effects are negligible up to $\Delta q_l \simeq 0.05$. As $\Delta q_l > 0.1$, third order effects become relevant, and D_l^T starts differing from D_l

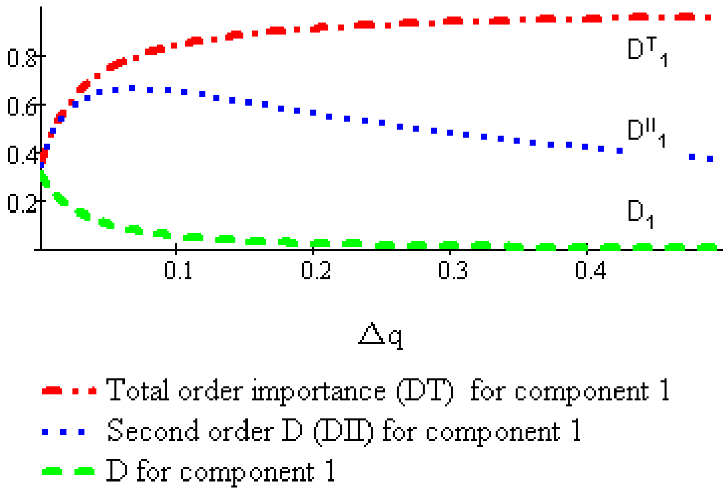

Figure 1: D_1^T (— ·), D_1^2 (· ·) and D_1 (— —) as a function of $\mathbf{\Delta q}$.

and D_l^{II} significantly (Figure 1).

Let us now turn the attention to the information entailed in Table 3. The system is coherent, then $B_l > 0$ for all l. In particular, the component with the highest B_l is 2, followed by component 1, and with component 3 having a very small marginal importance. $J_{i,s}^{II} > 0 \ \forall i, s$ implies that failures reinforce each other worsening system performance. The most relevant interaction is the one between components 1 and 2, followed by 2−3 and 1−3: $J_{1,2}^{II} > J_{2,3}^{II} > J_{1,3}^{II}$. Finally, $J_{1,2,3}^{T} = 1$ in accordance with Proposition 5. Note that Table 3 does not convey the overall importance of a component. Rather, it details the relevance of its individual, second and third order effects. The importance of a component including all its interaction effects is obtained by computing D^T [eq. (48).]. We are then lead to study how D_1^T, D_2^T, D_3^T vary as Δq_l varies. Figure 2 reports the values of D_l^T $(l = 1, 2, 3)$ as Δq varies from 0.0001 to 0.5.

From Figure 2 one notes that D_l^T $(l = 1, 2, 3)$ increase with Δq. Component 2 remains the most important component across the whole variation range. One notes that D_1^T tends to D_2^T, as changes grow. However, D_1^T tends to D_2^T, as changes grow. This is due to the weight of $J_{1,2,3}$, i.e., to the fact that third order interaction effects become dominant when changes become finite. To explain this result, let us write D_l^T $(l = 1, 2, 3)$ explicitly:

$$\begin{cases} D_1^T = \dfrac{B_1 \cdot \Delta q + (J_{1,2} + J_{1,3})(\Delta q)^2 + J_{1,2,3} \cdot (\Delta q)^3}{\Delta Q} \\ D_2^T = \dfrac{B_2 \cdot \Delta q + (J_{1,2} + J_{2,3})(\Delta q)^2 + J_{1,2,3} \cdot (\Delta q)^3}{\Delta Q} \\ D_3^T = \dfrac{B_3 \cdot \Delta q + (J_{1,3} + J_{2,3})(\Delta q)^2 + J_{1,2,3} \cdot (\Delta q)^3}{\Delta Q} \end{cases} \tag{53}$$

Each D_l^T is composed of three terms. The first term concerns the individual effect $(B_l \cdot \Delta q)$, the second term the second order interaction effects $((J_{1,2} + J_{1,3})(\Delta q)^2)$ and the third term the third order effect $(J_{1,2,3} \cdot (\Delta q)^3)$. Let us then observe the values of Table 3. One notes that B_l

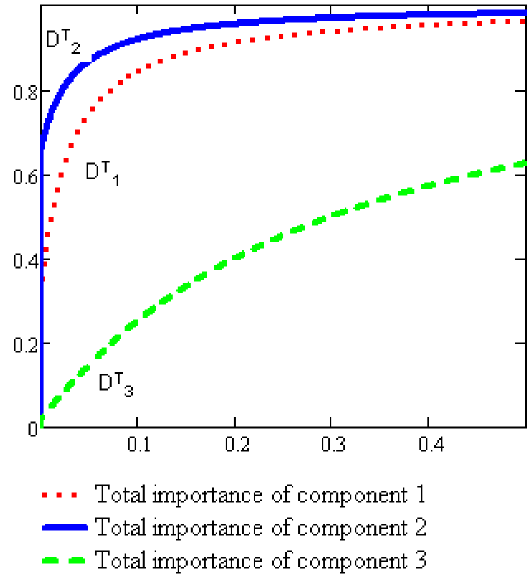

Figure 2: D_l^T, $l = 1, 2, 3$. Component 2 (continuous line) is characterized by the highest importance, followed by component 1, $(\cdot\cdot)$, and component 3, $(--)$.

Table 5: First, second and third order contributions for small (0.001), and finite (0.5) changes in unreliabilities

$\Delta q = 0.001$	$\frac{B_l \Delta q}{\Delta Q}$	$\frac{(J_{s,t} + J_{s,r})(\Delta q)^2}{\Delta Q}$	$\frac{J_{1,2,3}(\Delta q)^3}{\Delta Q}$
Component 1	0.3147	0.0325	$1.0492 \cdot 10^{-4}$
Component 2	0.6295	0.0336	$1.0492 \cdot 10^{-4}$
Component 3	0.0210	$3.1476 \cdot 10^{-3}$	$1.0492 \cdot 10^{-4}$
$\Delta q = 0.5$	$\frac{B_l \Delta q}{\Delta Q}$	$\frac{(J_{s,t} + J_{s,r})(\Delta q)^2}{\Delta Q}$	$\frac{J_{1,2,3}(\Delta q)^3}{\Delta Q}$
Component 1	$7.0721 \cdot 10^{-3}$	0.3654	0.5893
Component 2	0.0141	0.3772	0.5893
Component 3	$4.7146 \cdot 10^{-4}$	0.0354	0.5893

is one or even two order of magnitudes lower than $J_{i,s}^{II}$. Furthermore, $J_{1,2,3}^{T} = 1 > J_{i,s}^{II} >> B_l$. Therefore, when changes are small, it is the small values of $(\Delta q)^2$ or $(\Delta q)^3$, that causes second and third order terms in eq. (53) to be smaller than the individual effects $B_l \cdot \Delta q$. Conversely, when Δq increases, the individual effects become negligible. Given that $J_{1,2,3}^{T} = 1$, for discrete values of Δq, third order effects will become predominant. This expectation is confirmed by the numerical results of Table 5.

The upper portion of Table 5 shows that for small Δq, the individual contributions prevail in determining component importance. The lower portion of Table 5, instead, shows that second and third order effects prevail when Δq is finite – when $\Delta q = 0.5$ individual effects are two orders of magnitude lower than interaction effects.

The above discussion can be summarized as follows. By knowledge of D^T one obtains information on the overall importance of a component including all its individual and interaction

Table 6: Algorithm for the estimation of DT at the cost of N model runs

Step nr.	**Task**
1	Evaluate $G(\mathbf{x}^0)$
2	Evaluate $G(\mathbf{x}^1) = G(\mathbf{x}^0 + \Delta\mathbf{x})$
3	*For* $i = 1, 2, ..., N$ Evaluate $G(x_i, \mathbf{x}^1)$ *end*
4	Substitute in eq. (54)

effects. By knowledge of B and J^k (Table 3) one obtains information on the sign and magnitude of the individual (B_l) and interaction ($J^k_{i_1,i_2,...,i_k}$) effects. Thus, by a simultaneous use of D^T and J^k ($k = 1, 2, ..., T$) analysts obtain a complete dissection of system behavior and determine exactly how each component contributes to system performance.

6 Algorithmic Computation of D^T

In this Section, we address the numerical estimation of D^T. In complex systems the number of components (N) can be of the order of 10^3 or more. A brute-force estimation of D^T, i.e., an estimation procedure based on eq. (48), would entail the computation of all mixed partial derivatives up to T. The cost of such an algorithm grows more than linearly with N. For instance, when $N = 2$, the number of model evaluations necessary to estimate all first and second order partial derivatives is equal to $\frac{N + N^2}{2}$. Hence, for large N, utilization of a brute force approach might impair the estimation of D^T.

However, D^T can be obtained at a cost of N model runs without computation of the partial derivatives, as the next result shows.

Proposition 7 *Consider a system with N components and let $G : \mathbb{R}^N \to \mathbb{R}$ be the corresponding reliability/unreliability function. Let $\mathbf{x}^0$ and $\mathbf{x}^1 = \mathbf{x}^0 + \Delta\mathbf{x}$ denote the values of the reliabilities/unreliabilities before and after the change. Let $(x_i, \mathbf{x}^1)$ the point obtained when all reliabilities/unreliabilities are changed but x_i. Then:*

$$D_i^T = \frac{G(\mathbf{x}^1) - G(x_i, \mathbf{x}^1)}{\Delta G} \tag{54}$$

Proof. By Definition 1 [eq. (48)], D_i^T equals the fraction of the change in G associated with x_i. In Borgonovo (2009), it is proven that the fraction of the change in any function associated with variable x_i is equal to the total order finite change sensitivity index (Φ_i^T) of x_i. Hence, $D_i^T = \Phi_i^T$. By Proposition 1 in Borgonovo (2009) it holds that $\Phi_i^T = \frac{G(\mathbf{x}^1) - G(x_i, \mathbf{x}^1)}{\Delta G}$. Hence, eq. (54) holds. ■

Proposition 7 can be turned into an estimation procedure that allows to estimate D^T via the following algorithm [Table 6.]

To illustrate the procedure numerically, let us consider the example of Section 5, with $\Delta q = 0.2$. $G(\mathbf{x}^0)$ in Table 6 becomes $Q(\mathbf{q})$. Step 1 foresees to estimate the base case value of

Table 7: Illustration of the algorithm of Table 6

i	$Q(q_i, \mathbf{q}^1)$	$Q(\mathbf{q}^1) - Q(q_i, \mathbf{q}^1)$	D_i^T [eq. (54)]
1	0.0021	0.021	0.91146
2	0.0011	0.022	0.95486
3	0.01386	0.00924	0.40104

the system unreliability. One obtains

$$Q(\mathbf{q}^0) = 0.01 \cdot 0.02 \cdot 0.3 = 6 \cdot 10^{-5}$$

The second step is the estimation of the system unreliability at $\mathbf{q}^1 = \mathbf{q}^0 + \Delta\mathbf{q}$:

$$Q(\mathbf{q}^1) = Q(\mathbf{q}^0 + \Delta\mathbf{q}) = \prod_{i=1}^{3}(q_i + \Delta q_i) = 0.0231$$

The third step entails the evaluation of Q at $(q_i,\mathbf{q}^1)$, $i = 1, 2, ..., N$. The points $(q_i,\mathbf{q}^1)$ are obtained by keeping q_i unchanged, while the remaining unreliabilities are fixed at $q_i + \Delta q_i$. Table 7 reports the results for steps 3 and 4 of the algorithm in Table 6. The numerical results in Table 7 coincide with the values of D^T in Figure 2 (one needs to draw a vertical line at $\Delta q = 0.2$.) We recall that the results of Figure 2 were obtained by utilizing Definition 1, i.e., with estimation of all the mixed partial derivatives, while the results in Table 7 make use of Proposition 7, bypassing the estimation of the partial derivatives.

One notes that by Proposition 7, the computational cost of D^T is equal to $N + 2$ model evaluations. This cost is lower than the estimation cost of $J^k_{i_1,i_2,...,i_k}$ for any order k, and equals the cost for estimating the Risk Achievement Worth (RAW) and Risk Reduction Worth (RRW) importance measures, which are two of the most widely utilized importance measures in the safety analysis of complex systems.

7 Conclusions

The recent works of Lu and Jiang (2007) and Gao et al (2007) have renewed interest in reliability importance measures. A literature review, however, reveals that the studies of the properties of the joint and differential importance measures have developed on parallel tracks, an a common background is missing. In this work, we have proposed a unified framework for the utilization of the differential and joint reliability importance measures.

We have first addressed the question of the order T at which to arrest the Taylor expansion of a reliability function. By addressing the fact that a multilinear function coincides with both its Maclaurin and Bernstein polynomials, it has been possible to prove that the Taylor expansion of the reliability function of both coherent and non-coherent systems, with dependent and independent failures, is exact. In addition, any finite change in system reliability is exactly expanded in a Taylor polynomial of order T.

We have then introduced a new importance measure, the total order importance measure (D^T), which is the exact fraction of the change in system reliability related to a change in the reliability properties of component i. We have seen that D^T includes all joint reliability importance measures of orders 1 to T. D^T then extends the definitions of Zio and Podofillini (2006) and Do Van et al (2008)'s of higher order differential importance. D_l^k delivers the importance of a component in consideration of its individual effect and of all its interactions with the other components. We have studied the limiting properties of D^T and shown that it tends to the differential importance when changes become small. If, in addition, uniform changes in reliabilities are considered, then D^T and B produce the same ranking.

We have seen that $J^k_{i_1,i_2,\ldots,i_k}$ and D^T provide a full dissection of the system behavior, with D^T offering the overall importance of a component and $J^k_{i_1,i_2,\ldots,i_k}$ providing detailed information on interactions.

Finally, we have introduced a result that allows the estimation of D^T by varying one-probability-at-a-time. By the corresponding algorithm, one estimates D^T at the same computational cost of reliability importance measures utilized in the analysis of complex systems.

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