Functional ANOVA, Ultramodularity and Monotonicity: Applications in Multiattribute Utility Theory

F. Beccacece and E. Borgonovo
Department of Decision Sciences and ELEUSI Research Center
Bocconi University
Via Roentgen 1, 20135, Milano (Italy)

Abstract

Utility function properties as monotonicity and concavity play a fundamental role in reflecting a decision-maker's preference structure. These properties are usually characterized via partial derivatives. However, elicitation methods do not necessarily lead to twice-differentiable utility functions. Furthermore, while in a single-attribute context concavity fully reflects risk aversion, in multiattribute problems such correspondence is not one-to-one. We show that Tsetlin and Winkler's multivariate risk attitudes imply ultramodularity of the utility function. We demonstrate that geometric properties of a multivariate utility function can be successfully studied by utilizing an integral function expansion (functional ANOVA). The necessary and sufficient conditions under which monotonicity and/or ultramodularity of single-attribute functions imply the monotonicity and/or ultramodularity of the corresponding multiattribute function under additive, preferential and mutual utility independence are then established without reliance on the utility function differentiability. We also investigate the relationship between the presence of interactions among the attributes of a multiattribute utility function and the decision-maker's multivariate risk attitudes.

Keywords: Multiattribute Utility Theory; Functional ANOVA; Multi-criteria analysis; Ultramodular Functions.

Acknowledgments

The authors wish to thank the anonymous referees for very perceptive and insightful comments that have greatly helped us in improving the manuscript. We also thank the participants of the XIII International Conference on the Foundations and Applications of Utility, Risk and Decision Theory, IESE, Barcelona, July 2008, and of the AMASES 2007 conference for their insightful comments and observations.

1 Introduction

The solution of several decision-making problems requires the quantitative assessment of multiattribute objective (utility) functions $[u(x)]$. Practical advantages are registered in all those applications in which it is possible to elicit single-attribute functions and next aggregate them. Utility, additive and preferential independence are often assumed in the practice, since they lead to additive and multiplicative forms of the utility functions [Fishburn (1980), Keeney and Raiffa (1993), Keeney (2006), Baucells et al (2008).] The elicitation methods described in Fortemps et al (2008), Greco et al (2008), Figuera et al (2009), Anginella et al (2004), Anginella et al (2010) play a central role for obtaining quantitative results.

Monotonicity and convexity properties on the single-attribute functions are required for the resulting multiattribute utility function to be in one-to-one correspondence with the decision-maker's preference structure and risk aversion attitudes. In the single-attribute case, non-satiation leads to monotonically increasing utility functions; risk aversion combined with non-satiation implies utility function concavity. In multiattribute utility problems, non-satiation still leads to monotonicity, while the notion of risk aversion becomes more general, with several possible extensions. Richard (1975)'s multivariate risk aversion leads to the supermodularity of $u(x)$ [Ortega and Escudero (2010)] while, as we are to see, Tsetlin and Winker (2009) multivariate "preferences for combining good with bad lead to the ultramodularity" of $u(x)$. The question is, then, whether the motoniticity (ultramodularity) of $u(x)$ implies the motoniticity (ultramodularity) of the single-attribute utility functions under the various forms of preference structures. Conversely, one would need to know whether, by eliciting single-attribute utility functions which possess a given property, She is insured that such property is maintained in $u(x)$.

This work introduces an approach for linking analytic and geometric properties of multiattribute objective functions to the properties of their single-attribute constituents, without restrictions on the preference structure.

Ideally, monotonicity, concavity and ultramodularity can all be characterized in terms of partial derivatives (first and second order, respectively). However, methods for multiple criteria ranking [Greco et al (2008), Figuera et al (2009), Jacquet-Lagrèze et al (1987)] build a set of additive value functions compatible with the revealed preference information, which are piecewise defined (mostly piecewise linear). Hence, a mathematical approach based on differentiation does not possess the required generality. We therefore shift the background to the integral expansion of a multivariate function (f) generated by the high dimensional model representation (HDMR) theory (functional ANOVA) [Efron and Stein (1981), Rabitz and Alis (1999), Alis and Rabitz (2001), Sobol' (2001), Sobol' (2003).] For this expansion to hold, in fact, the sole measurability of f is required. Functional ANOVA is a fundamental tool in statistics and global sensitivity analysis [Rabitz and Alis (1999), Wang (2006) Sobol' (2003)]. However, in spite of its widespread utilization, a study of its monotonicity and ultramodularity properties has not been offered yet. We obtain necessary and sufficient conditions that insure that, given the monotonicity and ultramodularity of f, these properties are preserved at the various orders of the expansion. We then specialize these general results to the case of additive and multiplicative functions, since these two functional forms frequently appear in MAUT.

In particular, we obtain the following results: I) Under additive utility independence: *i)* u is monotonic if and only if all the corresponding single-attribute conditional utility functions are;

and *ii)* u is ultramodular if and only if all the conditional utilities are; II) Under mutual utility independence, if and only if u is monotonic and ultramodular, then all single-attribute utility functions are; III) Letting v denote the multiattribute value function for preferences under certainty [on the distinction between u and v, we refer to Keeney and Raiffa (1993)], it is possible to prove that, under preferential independence: *i)* if v is non-decreasing, then u is; *ii)* if v is ultramodular, then u is.

Each of the previous results is also characterized in terms of a decision-maker's multivariate risk attitudes. In particular, it is obtained that u is ultramodular if and only if it reflects the *preferences for combining good with bad* stated in Tsetlin and Winker (2009). The presence of interactions in the functional ANOVA representation of u is also investigated in its connection with multivariate risk attitudes. We show that the absence of interactions in u implies multivariate risk neutrality in the sense of Richard (1975).

The remainder of this work is organized as follows. Section 2 discusses the MAUT implications of ultramodularity. Section 3 provides a review of functional ANOVA and HDMR. Section 4 discusses the monotonicity properties of the functional ANOVA expansion. Section 5 derives results for ultramodularity. Both Sections 4 and 5 apply the findings to separable functions. Section 6 discusses implications of the above findings in MAUT. Conclusions are offered in Section 7.

2 Ultramodularity in MAUT: Multivariate Risk Aversion

The mathematical properties of a multiattribute utility function must be in one-to-one correspondence with the decision-maker's preference structure. For instance, monotonicity reflects the non-satiation property, which states that "*more of an attribute is preferred to less* "[Tsetlin and Winker (2009)*;* p. 1944]*;* see also Ingersoll (1987). Such correspondence is maintained both in single-attribute and in multiattribute problems. In single-attribute problems, given the monotonicity of the utility function, concavity reflects risk aversion; if, in addition, the utility function is regular, risk aversion is univocally characterized by the Arrow-Pratt measure. In multiattribute problems, the one-to-one correspondence between concavity and risk aversion is lost and generalizations of concavity come into play. For instance, multivariate risk aversion as introduced in Richard (1975) implies the submodularity of the utility function [see Ortega and Escudero (2010)]. In this section, we investigate what types of multivariate risk attitudes are linked to ultramodularity.

As discussed in Marinacci and Montrucchio (2005), two alternative definitions of ultramodular functions can be stated, the first one based on the notion of test quadruple, the second one based on increasing differences. We utilize the latter definition, because it is better suited to the purposes of this paper. At the same time, we also introduce the notion of neg-ultramodular functions.

Definition 1 *A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is*
a) ultramodular, if

$$f(x+h) - f(x) \leq f(y+h) - f(y) \tag{1}$$

b) neg-ultramodular, if

$$f(x+h) - f(x) \geq f(y+h) - f(y) \tag{2}$$

for all $x, y \in X$ and $h \geq 0$ with $x \leq y$ and $x + h, y + h \in X$.

By Definition 1, one notes that, if f is ultramodular, then $-f$ is neg-ultramodular. Definition 1 shows that ultramodular (neg-ultramodular) functions possess the property of increasing (decreasing) differences [see Marinacci and Montrucchio (2005) p. 315].

Ultramodular functions are also known as directionally convex functions [Ortega and Escudero (2010)]. Marinacci and Montrucchio (2005) highlight that ultramodularity is a generalization of scalar convexity. In fact, if and only if $f : \mathbb{R} \to \mathbb{R}$ is convex, then it is ultramodular. However, ultramodularity and convexity become distinct notions for multivariate functions [Marinacci and Montrucchio (2005), Marinacci and Montrucchio (2008)].

In Marinacci and Montrucchio (2005), several properties of ultramodular functions are proven. The ones relevant to this work are collected in the following Proposition.

Proposition 1 *Let $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ two ultramodular functions.*

1. *The sum $\alpha f + \beta g$ is ultramodular if α and β are non-negative scalars.*
2. *The sum $f + k$, $k \in \mathbb{R}$ is ultramodular.*
3. *The product $f \cdot g$ is ultramodular if, in addition, f and g are both non-negative and increasing.*
4. *The composition $h \circ f$ is ultramodular, provided f is monotonic and ultramodular, and h is ultramodular and increasing.*

Interpretations of ultramodularity have been offered in Economics and Game Theory. — Marinacci and Montrucchio (2005) illustrate that ultramodularity "*reflects a stronger form of complementary than supermodularity*" [Marinacci and Montrucchio (2005); p. 311]. Indeed, ultramodular functions are also supermodular while the converse is not true [see Topkis (1995), Milgrom and Shannon (1994).] Supermodularity and ultramodularity are also relevant in the theory of pseudo-Boolean functions [see Foldes and Hammer (2005)]. — However, a rigorous investigation of the implications of ultramodularity in MAUT has not been offered yet. The conditions that make ultramodularity appear in multiattribute problems are discussed next.

Ultramodularity is connected with multivariate risk attitudes. Consider a two attribute (x, y) problem for simplicity. The main difference between single-attribute and multiattribute problems is stated in Richard (1975): "*a decision-maker can be risk averse for gambles on x alone or for gambles on y alone, but still be multivariate risk seeking.*" Richard (1975) then introduces the concept of multivariate risk aversion as follows. A decision-maker is multivariate risk averse if She *"prefers getting some of the best and some of the worst to taking a chance on all of the best or all of the worst."* Richard (1975) shows that, provided that $u \in C^2$, a necessary and sufficient condition for multivariate risk aversion is $\partial^2 u / \partial x_i \partial x_j \leq 0$ for $i \neq j$.

Multivariate risk aversion is also characterized by Richard (1975) in terms of the signs of the partial derivatives. The alternating sign condition of Richard (1975) implies the submodularity of the corresponding utility function [see Ortega and Escudero (2010)].

Tsetlin and Winker (2009) show that a decision-maker "*to be consistent with preferring to combine good with bad should have a utility function with alternating signs for successive partial derivatives.*" Theorem 1 in Tsetlin and Winker (2009) [p. 1945] states that a decision-maker prefers to combine good with bad if and only if its utility function belongs to $\mathbb{U}_b^B$. $\mathbb{U}_b^B$ is the class of all "*B-dimensional real-valued functions for which all partial derivatives of a given order up to order b have the same sign, and that sign alternates, being positive for odd orders and negative for even orders.* [Tsetlin and Winker (2009), p. 1945]." $\mathbb{U}_b^B$ plays an important role also in the definition of a new class of utility functions in Prékopa and Mádi-Nagy (2008).

In respect of Richard (1975), Tsetlin and Winker (2009) define the preference for combining good lotteries with bad lotteries as a "*stronger condition, encompassing single-attribute risk aversion and going beyond it*". This condition is expressed by the additional assumption that the decision-maker is single-attribute risk averse, e.g., $\dfrac{-\partial^2 u/\partial x_i^2}{\partial u/\partial x_i} \geq 0$.

Now, we show that the assumptions $u \in \mathbb{U}_b^B$ and $\dfrac{-\partial^2 u/\partial x_i^2}{\partial u/\partial x_i} \geq 0$ stated in Tsetlin and Winker (2009) imply the neg-ultramodularity of u. In Tsetlin and Winker (2009) (but also in Richard (1975)), u is increasing in the attributes, whence $\partial u/\partial x_i \geq 0$, $\forall i$. $u \in \mathbb{U}_b^B$ then implies that the second order partial derivatives are negative, i.e., $\partial^2 u/\partial x_i \partial x_j \leq 0$, $\forall i, j$. Note that i does not need to be different from j due to the single-attribute risk aversion assumption. Now, by Theorem 5.5 in Marinacci and Montrucchio (2005), a function $u \in C^2$ is neg-ultramodular iff $\partial^2 u/\partial x_i \partial x_j \leq 0$, $\forall i, j$. Therefore:

Proposition 2 *Let $u \in C^2(X_1 \times X_2 \times ... \times X_n)$. If and only if the decision-maker is averse to any multivariate risk in the sense of Tsetlin and Winker (2009), then u is neg-ultramodular.*

One notes that, by definition of ultramodularity, the above proposition holds also if u is monotonically decreasing, with neg-ultramodularity replaced by ultramodularity.

As far as elicitation is concerned, we refer to Tsetlin and Winker (2009) [in particular, Section 5, p. 1948], where the assessment of a utility function displaying the above-described multivariate risk attitudes is discussed.

One notes that both our work up to this point, and the preceding literature, has characterized monotonicity and ultramodularity in terms of first and second order derivatives. For this characterization to hold, u must be twice differentiable. Quantitative elicitation methods (see, for instance the GRIP method in Greco et al (2008) and Figuera et al (2009)), however, lead to utility functions that are, generally, non-differentiable. Hence, one needs to obtain a more general characterization of the properties of u, not relying on differentiation. In the next section, the use of functional ANOVA as a way for expanding a multivariate function without resorting to regularity assumptions is discussed.

3 High Dimensional Model Representation Theory and Functional ANOVA

In this section, we describe the foundations of functional ANOVA and offer its interpretation as a tool for expanding multivariate measurable functions.

The origin of functional ANOVA is linked to the problem of decomposing the variance of a square integrable statistics generated by the seminal works of Hoeffding in the 1940's [Hoeffding (1948)]. The "*jackknife*" decomposition is proven in Efron and Stein (1981). The technique utilized by Efron and Stein (1981) consists in the dissection of f via a sequence of nested conditional expectations in accord with a Grahm-Schmidt orthogonalization process. Orthogonality is at the basis of the results by Takemura (1983), where the orthogonal decomposition of any square integrable function is cast in the context of tensor analysis and multilinear algebra. Rabitz and Alis (1999) introduce an alternative generalization of functional ANOVA, called high dimensional model representation (HDMR) theory. Rabitz and Alis (1999) prove the functional ANOVA decomposition through the dissection of the linear space to which the function belong. A fourth and independent way of proving the functional ANOVA expansion is due to Sobol' (1969) and Sobol' (1993). In Sobol' (1969), the decomposition is developed in the context of quadrature methods and called "*the decomposition into summands of different dimensions* [Owen (2003); p. 2]. In Sobol' (1993), the uniqueness of eq. (20) is proven via nested integrations[1].

In this work, we utilize the framework of Rabitz and Alis (1999). Without loss of generality, one refers to a scalar function $f : I^n \subseteq \mathbb{R}^n \to \mathbb{R}$, where I denotes the unit interval $[0, 1]$, and I^n the n-dimensional unitary hypercube. $(I^n, \mathcal{B}(I^n), \mu)$ is a Borel measure space. For the sake of notation simplicity in the remainder, μ denotes the Lebesgue measure [on the standard choices of I^n and the Lebesgue measure, see also Rabitz and Alis (1999), Sobol' (2001), Owen (2003), Sobol' (2003), Wang (2006) and Malliavin (1995), p. 224-227]. f belongs to a linear vector space of functions denoted by F. We require f to be at least measurable [2]. Rabitz and Alis (1999) then utilize the following decomposition of F [a similar approach is also used in Takemura (1983)].

Lemma 1 *[Rabitz and Alis (1999)]*

$$F = F_0 \oplus \sum_{i=1}^{n} F_i \oplus \sum_{i<j} F_{i,j} \oplus ... \oplus F_{123...n} \tag{3}$$

where $\oplus$ is the direct-sum operator and the subspaces are defined as

$$\begin{cases} F_0 \equiv \{f \in F : f = a,\ a \in \mathbb{R}\} \\ F_i \equiv \left\{f \in F : f = f_i(x_i) \text{ with } \int_I f_i(x_i) d\mu_i = 0\right\} \\ F_{i,j} \equiv \left\{f \in F : f = f_{i,j}(x_i, x_j) \text{ with } \int_I f_{i,j}(x_i, x_j)\ d\mu_s = 0,\ s = i, j\right\} \\ ... \end{cases} \tag{4}$$

[1] In the remainder, the terms functional ANOVA, HDMR and integral decomposition shall be regarded as synonyms.

[2] Given the probability space $(\Omega, \mathcal{B}(\Omega), \mu)$, $\mathcal{L}^p(\Omega, \mathcal{B}(\Omega), \mu)$ denotes the set of all $\mu - p-$measurable functions $\psi : \Omega \to \mathbb{R}^n$ such that $\displaystyle\int_\Omega \|\psi(\omega)\|^p\, d\mu(\omega) < \infty$.[Malliavin (1995).]

The following theorem then follows by projection [see Rabitz and Alis (1999)].

Theorem 1 *[Sobol' (1993) and also Rabitz and Alis (1999)]. Let $f : I^n \subseteq \mathbb{R}^n \to \mathbb{R}$, $f \in F$. Under the above assumptions, the following decomposition of f in eq. (5) is unique:*

$$f(x) = f_0 + \sum_{k=1}^{n} \sum_{i_1<i_2<...<i_k} f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \tag{5}$$

with

$$\begin{aligned} &f_0 = \mathbf{M}_0 f = \mathbb{E}_\mu[f(x)] \\ &f_i(x_i) = \mathbf{M}_i f(x) - f_0 \\ &f_{i_1,i_2}(x_{i_1},x_{i_2}) = \mathbf{M}_{i_1,i_2} f(x) - f_{i_1}(x_{i_1}) - f_{i_2}(x_{i_2}) - f_0 \\ &... \end{aligned} \tag{6}$$

and

$$\mathbf{M}_{i_1,i_2,...,i_k}[f] = \mathbb{E}_\mu \left[f | x_{i_1}, x_{i_2}, ..., x_{i_k}\right] \tag{7}$$

In eq. (5), $\mathbf{M}_{i_1,i_2,...,i_k}$ are the conditional expectations of f given $x_{i_1}, x_{i_2}, ..., x_{i_k}$, and are referred to in Rabitz and Alis (1999), Alis and Rabitz (2001), Sobol' (2003) as projection operators. In the remainder, the notation

$$m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) = \mathbb{E}_\mu \left[f | x_{i_1}, x_{i_2}, ..., x_{i_k}\right] \tag{8}$$

shall display the dependence of $\mathbf{M}_{i_1,i_2,...,i_k}[f]$ on $x_{i_1}, x_{i_2}, ..., x_{i_k}$. We call the functions $m_{i_1,i_2,...,i_k}$ in eq. (8) non-orthogonalized terms. The functions f_i are referred to as first order terms [in Rabitz and Alis (1999), Alis and Rabitz (2001), Sobol' (2003)] or main effects [in Huang (1998), Wang (2006) Hooker (2007)], the functions $f_{i_1,i_2,...,i_k}$ as interaction terms or interaction effects.

One needs to recall the crucial role played by the independence assumption in granting uniqueness to the functional ANOVA expansion. In fact, it is proven by Bedford (1998) that, if correlations are present, the decomposition depends on the initial lexicographical ordering of the random variables. The same f then shares multiple representations and eq. (5) looses its uniqueness.

Without claiming exhaustiveness, one can group the numerous applications of functional ANOVA into three interrelated fields: high-dimensional integration, high-dimensional model representation and global sensitivity analysis. In Wang (2006), functional ANOVA is utilized as a dimension-reduction technique for integration problems in financial applications. Functional ANOVA in high-dimensional integration has been studied also in [Sobol' (1969), Sobol' (1993), Sobol' (2001), Sobol' (2003), Owen (2003), Sobol' et al (2007)]. In Rabitz and Alis (1999), Alis and Rabitz (2001) and Li et al (2001), functional ANOVA is employed "*for improving the efficiency of deducing high dimensional input–output system behavior* (Alis and Rabitz (2001); p.1)." In these works, the functional ANOVA expansion is utilized as an interpolation method, in a meta-modelling scheme, and originates the so-called HDMR theory (Rabitz and Alis (1999)). In particular, Rabitz and Alis (1999) and Alis and Rabitz (2001) introduce the cut-HDMR expansion, which provides as one of the most useful tools for complex model output interpolation and dimension reduction. A third and interrelated field where functional ANOVA has been widely applied is global sensitivity analysis of model output (Wagner (1995); Homma and Saltelli (1996); Saltelli et al (2000); Saltelli et al (1999); Saltelli and Tarantola (2002); Borgonovo (2006); Borgonovo (2010)). Indeed, the first application of functional ANOVA in OR is in global sensitivity analysis, by Wagner (1995). Notable efforts have since then been performed towards refining the numerical estimation of the terms in the functional ANOVA expansion, especially in connection with the of Sobol' global Sensitivity indices [see Homma and Saltelli (1996); Saltelli et al (1999) introduce the use of Fourier Amplitude Sensitivity Test to estimation main effects]. A common feature across all the above mentioned work is the assumption of independence among the random variables. Bedford (1998) shows that this assumption is central to obtain the uniqueness of the functional ANOVA representation. The work by Hooker (2007) provides a generalization of functional ANOVA for the diagnostics of high dimensional models in the presence of dependent variables.

In this work, we utilize functional ANOVA as a function expansion method, namely emphasis is posed on eq. (5). In many works related to functional ANOVA, f is required to be square integrable, i.e., $f \in \mathcal{L}^2(\Omega, \mathcal{B}(\Omega), \mu)$, since the variance of f is the goal of the analysis. In our work, because we are concerned with eq. (5), the assumption can be softened to $f \in \mathcal{L}(\Omega, \mathcal{B}(\Omega), \mu)$. As we are to see, this allows us to obtain links between properties of $u(x)$ and those of its conditional utilities, bypassing requirements on the regularity of $u(x)$ implied by a methodology based on first or second order partial derivatives. Such need stems from the fact that, generally, an elicited utility function is not differentiable [see the GRIP method in Greco et al (2008) and Figuera et al (2009).]

However, in spite of the widespread utilization of functional ANOVA, properties as ultramodularity and monotonicity have not been studied via this type of integral expansion yet. It is then the purpose of the next sections to investigate how-whether these properties are maintained in functional ANOVA. We start with monotonicity.

4 Monotonicity Properties in Functional ANOVA

The definition of non-decreasing function of interest for our work is as follows.

Definition 2 *$f : I^n \to \mathbb{R}$ is non-decreasing on I^n, if, $\forall x$ and $\forall y \in I^n$ such that $x \leq y$, then*

$$f(x) \leq f(y) \tag{9}$$

If the strict inequality holds, one says that f is increasing. Similarly, f is non-increasing if $\forall x$ and $\forall y \in I^n$ such that $x \leq y$

$$f(x) \geq f(y) \tag{10}$$

If the strict inequality holds, one says that f is decreasing.

Let us assume that eqs. (5) and (6) hold, e.g., f is at least measurable. The following result holds (see Appendix A for the proof).

Lemma 2 *If f is non-decreasing, then all non-orthogonalized terms $m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k})$ in its ANOVA expansion are.*

By recalling the definition of $m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k})$, Lemma 2 states that conditional expectation preserves the monotonicity of f.

The following result concerns the monotonicity of the main effects of the functional ANOVA expansion (see Appendix A for the proof).

Theorem 2 *If f is non-decreasing, then all first order terms in its ANOVA expansion are non-decreasing.*

Given Definition 2, it is readily seen that results similar to Lemma 2 and Theorem 2 hold in the case of f increasing, non-decreasing and strictly decreasing. Hence, monotonicity is preserved by the main effects (first order terms) of the functional ANOVA expansion. For higher order projections, this property does not hold, in general. The terms $f_{i_1,i_2,...,i_k}$ are the result of an orthogonalization process involving the difference between f and lower order terms. As an example, second order terms are given by:

$$f_{r,l}(x_r, x_l) = m_{r,l}(x_r, x_l) - f_r(x_r) - f_l(x_l) - f_0 \tag{11}$$

In spite of the fact that $m_{r,l}(x_r, x_l), f_r(x_r), f_l(x_l)$ are all non-decreasing individually, — if f is, — the subtraction of $f_r(x_r)$ and $f_l(x_l)$ does not insure that

$$f_{r,l}(x_r, x_l) < f_{r,l}(x_r + h_r, x_l + h_l), \forall h_r, h_l \tag{12}$$

However, the inequality in (12) is satisfied if[3]:

$$\Delta f_r + \Delta f_l \leq \Delta m_{r,l} \tag{15}$$

Ineq. (15) states that $f_{r,l}(x_r, x_l)$ is non-decreasing when the change in f given the simultaneous changes in x_r and x_l averaged over all possible values of the remaining variables ($\Delta m_{r,l}$) is greater than the sum of the average changes due to the variations in x_r and x_l individually ($\Delta f_r + \Delta f_l$). The above result can be extended to higher-order terms as follows (see Appendix A for the proof).

Theorem 3 (Sufficient condition for monotonicity) *If f is non-decreasing, then the generic k^{th} order term $f_{x_{i_1},x_{i_2},...,x_{i_k}}(x_{i_1}, x_{i_2}, ..., x_{i_k})$ $(1 < k \leq n)$ in its ANOVA expansion is non-decreasing*

[3] Starting with the inequality

$$m_{r,l}(x_r, x_l) - f_r(x_r) - f_l(x_l) - f_0 \leq m_{r,l}(x_r + h_r, x_l + h_l) - f_r(x_r + h_r) - f_l(x_l + h_l) - f_0 \tag{13}$$

one obtains:

$$f_r(x_r + h_r) - f_r(x_r) + f_l(x_l + h_l) - f_l(x_l) \leq m_{r,l}(x_r + h_r, x_l + h_l) - m_{r,l}(x_r, x_l) \tag{14}$$

if the following condition holds:

$$\Delta m_{i_1,i_2,\ldots,i_k} \geq \sum_{s=1}^{k-1} \sum_{i_1<i_2\ldots<i_s} \Delta f_{i_1,i_2,\ldots,i_s} \tag{16}$$

where

$$\begin{aligned} \Delta m_{i_1,i_2,\ldots,i_k} &= m_{i_1,i_2,\ldots,i_k}(x_{i_1}+h_{i_1},x_{i_2}+h_{i_2},\ldots,x_{i_k}+h_{i_k}) - m_{i_1,i_2,\ldots,i_k}(x_{i_1},x_{i_2},\ldots,x_{i_k}) \\ \Delta f_{i_1,i_2,\ldots,i_k} &= f_{i_1,i_2,\ldots,i_k}(x_{i_1}+h_{i_1},x_{i_2}+h_{i_2},\ldots,x_{i_k}+h_{i_k}) - f_{i_1,i_2,\ldots,i_k}(x_{i_1},x_{i_2},\ldots,x_{i_k}) \end{aligned} \tag{17}$$

and $h \geq 0$.

Theorem 3 states the conditions on f such that higher order terms in the functional ANOVA expansion retain the monotonicity of f.

In the elicitation of multiattribute utility functions, however, it is often assumed (required) a separable form of the utility function. In particular, additive independence results in an additive form of the utility function, while mutual utility independence results in a multiplicative form [Keeney and Raiffa (1993)]. We recall that separable multiattribute utility functions play also a relevant role in simplifying the algorithms for solution of multiattribute decision-analysis problems represented in the form of influence diagrams, as discussed in Tatman and Shachter (1990).

Let us then investigate monotonicity properties of additive and multiplicative functions via functional ANOVA. We have the following results (the proofs are in Appendix A.)

Corollary 1 *Let $f : I^n \subseteq \mathbb{R}^n \to \mathbb{R}$ be additive [$f(x) = \sum_{i=1}^n z_i(x_i)$.] It is then true that f is non-decreasing if and only if $z_i(x_i)$ are non-decreasing, $\forall i$.*

Corollary 2 *Let $f : I^n \subseteq \mathbb{R}^n \to \mathbb{R}$ be of the form $f(x) = \prod z_i(x_i)$ with $z_i \geq 0$ $\forall i$. Assume that f is continuous on I^n. Then, f is non-decreasing, if and only if $z_i(x_i)$ $\forall i$ are non-decreasing.*

Corollaries 1 and 2 set forth the conditions under which the monotonicity of the univariate functions $z_i(x_i)$ insure the monotonicity of f, when f is additive or multiplicative.

In the next section, we are to address ultramodularity properties.

5 Ultramodularity in Functional ANOVA

This section is devoted to the functional ANOVA expansion of ultramodular functions. In particular, we investigate what are the conditions under which ultramodularity is preserved at the various orders of the expansion.

The next result characterizes first order terms.

Theorem 4 *If f is ultramodular, then the first order terms of eq. (5), f_i, $i = 1,2,,\ldots,n$, are ultramodular.*

Theorem 4 states that the ultramodularity of f is a sufficient condition for the ultramodularity of the first order terms of the integral decomposition. To insure the ultramodularity of higher order terms [e.g., $f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k})$], however, a further assumption on the "strength" of the ultramodularity of f must be added. We start with proving the following result for the non-normalized terms in eqs. (5) and (6).

Lemma 3 *If f is ultramodular, then all non-orthogonalized terms $m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k})$ ($2 \leq k \leq n$) in its ANOVA expansion are ultramodular.*

The sufficient condition under which all the orthogonalized terms in eq. (5), (e.g., the functions $f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}), 2 \leq k \leq n$) are ultramodular is given below.

Theorem 5 (Sufficient condition for ultramodularity of higher order terms) *If f is ultramodular, then the generic k^{th} order term $f_{x_{i_1},x_{i_2},...,x_{i_k}}(x_{i_1}, x_{i_2}, ..., x_{i_k})$ ($1 < k \leq n$) in the decomposition is ultramodular provided that the following condition holds:*

$$\Delta m_{i_1,i_2,...,i_k}(y) - \Delta m_{i_1,i_2,...,i_k}(x) \geq \sum_{s=1}^{k-1} \sum_{i_1<i_2...<i_s} \Delta f_{i_1,i_2,...,i_s}(y) - \sum_{s=1}^{k-1} \sum_{i_1<i_2...<i_s} \Delta f_{i_1,i_2,...,i_s}(x) \quad (18)$$

where

$$\begin{aligned}
\Delta m_{i_1,i_2,...,i_k}(y) &= m_{i_1,i_2,...,i_k}(y_{i_1} + h_{i_1}, y_{i_2} + h_{i_2}, ..., y_{i_k} + h_{i_k}) - m_{i_1,i_2,...,i_k}(y_{i_1}, y_{i_2}, ..., y_{i_k}) \\
\Delta m_{i_1,i_2,...,i_k}(x) &= m_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) - m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \\
\Delta f_{i_1,i_2,...,i_s}(y) &= f_{i_1,i_2,...,i_s}(y_{i_1} + h_{i_1}, y_{i_2} + h_{i_2}, ..., y_{i_s} + h_{i_s}) - f_{i_1,i_2,...,i_k}(y_{i_1}, y_{i_2}, ..., y_{i_s}) \\
\Delta f_{i_1,i_2,...,i_s}(x) &= f_{i_1,i_2,...,i_s}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_s} + h_{i_s}) - f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_s})
\end{aligned} \quad (19)$$

Theorem 5 states the conditions under which all terms in the integral expansion of a ultramodular function are ultramodular. In this respect, it parallels Theorem 3. The conditions implied by Theorem 5 can be softened in the case the utility function is separable.

Concerning the ultramodularity of additive functions, we prove in Appendix A the following results.

Corollary 3 *Let $f : I^n \subseteq \mathbb{R}^n \to \mathbb{R}$ be additive [$f(x) = \sum_{i=1}^{n} z_i(x_i)$.] f is ultramodular if and only if $z_i(x_i)$ are ultramodular $\forall i$.*

Corollary 3 allows us to extend a result in Proposition 4.4. in Marinacci and Montrucchio (2005), where a sufficient condition for an additive function f to be ultramodular and convex is given by the convexity of each z_i $\forall i$.

Corollary 4 *Let $f(x)$ be of the form $\sum_{i=1}^{n} z_i(x_i)$, $f(x)$ is ultramodular if and only if it is convex.*

The next result concerns the ultramodularity of multiplicative functions.

Corollary 5 *If* $f = \prod_{i=1}^{n} z_i(x_i)$ *is ultramodular and* $z_i(x_i) > 0$*, then* $z_i(x_i)$ *is ultramodular* $\forall i = 1, 2, .., n$*. If, in addition,* $z_i(x_i)$ *is increasing* $\forall i = 1, 2, .., n$*, the converse is also true.*

Corollary 5 states that, if $f(x)$ is multiplicative, then its ultramodularity insures the ultramodularity of its univariate functions. However, the ultramodularity of the univariate functions does not insure the ultramodularity of f, unless the univariate functions are also increasing.

The findings of the present section and of Section 4 concern a generic multivariate f. In the next two sections, we discuss the MAUT implications of these findings.

6 Functional ANOVA of Multiattribute Utility Functions

In this section, we discuss the multiattribute utility theory implications of the findings of Sections 3, 4 and 5.

Theorem 1 has the following implication.

Corollary 6 *Any measurable multiattribute utility function* $u(x) : X_1 \times X_2 \times ...X_n \rightarrow \mathbb{R}$*, can be written as:*

$$u(x) = u_0 + \sum_{i=1}^{n} u_i(x_i) + \sum_{i<j} u_{i,j}(x_i, x_j) + ... + u_{1,2,...n}(x_1, x_2, ..., x_n) \tag{20}$$

Corollary 6 [eq. (20)] states that any multiattribute utility function can be expanded in a series of terms in which first order terms concern preferences over individual attributes, the second order terms concern interactions among pairs of attributes, the third order terms interactions among triplets, etc..

We observe that eq. (20) applies without reference to any particular preference structure. Let us investigate this aspect further. We start by an observation of Rabitz and Alis (1999) (p. 198). The observation concerns complex mathematical models and addresses the "*curse of dimensionality*," namely the presence of high-order interactions: "*a dramatic reduction in this scaling* [the number of interactions] *is often expected to arise [...] due to the presence of only low-order correlations amongst the input variables having a significant impact upon the output.*" In applications of functional ANOVA to complex numerical models [Wang (2006), Rabitz and Alis (1999), Li et al (2001)], one can cut eq. (20) at a pre-determined order to reduce model complexity. For instance, in a second order cut, one assumes

$$u(x) \simeq u_0 + \sum_{i=1}^{n} u_i(x_i) + \sum_{i<j} u_{i,j}(x_i, x_j) \tag{21}$$

When financial, physical or chemical systems are concerned, the only way to ascertain the accuracy of the approximation in eq. (21) is by performing numerical experiments. In MAUT, however, one can utilize assumptions on the preference structure of the decision-maker to determine a-priori the order of the expansion. This marks a notable departure between the traditional utilization of functional ANOVA and its utilization in MAUT. Furthermore, as we are to see, the presence or absence of interactions in $u(x)$ is linked to the multivariate risk attitudes of the decision-maker.

In many applications of multiattribute decision-making, it is desirable to quantify u via the elicitation of the conditional utilities in each attribute (Keeney and Raiffa (1993), Keeney (2006), Baucells et al (2008), Greco et al (2008), Figuera et al (2009)). In other words, one assesses $u(x_i|x^0_{(-i)})$ for different levels of x_i, with the remaining attributes $(x^0_{(-i)})$ fixed at a given level $x^0_{(-i)}$. In these applications, a separable form for u is often assumed [Greco et al (2008), Figuera et al (2009), Baucells et al (2008).] In the remainder of the work, we address the questions of whether, given that u is monotonic/ultramodular, all the conditional utilities need to be monotonic/ultramodular and whether the converse statement needs to be true. A positive answer, would insure that, if all the elicited conditional utilities are monotonic/ultramodular, then the decision-maker is characterized by a monotonic/ultramodular utility functions. The converse statement would imply the following. If one assume that a decision-maker is characterized by a monotonic/ultramodular utility function, then none of the elicited conditional utilities can be non-monotonic/non-ultramodular, without contradicting the hypothesis on the decision-maker's preferences.

In the next sections, we show that, using functional ANOVA to decompose u, it is possible to find exact relations between the form of $u(x)$ and the conditional utilities, without relying on the regularity of $u(x)$.

6.1 Additive Independence, Ultramodularity and Monotonicity

Additive independence is invoked by many works in multiattribute decision-making [see Keeney and Raiffa (1993), Zopounidis and Doumpos (2002); Baucells et al (2008).]

The notion of additive independence originates in Fishburn (1965) and refers to the case in which there is no interaction of preference among a set of attributes. For the sake of notation simplicity, let us refer to the two-attribute case. X and Y are additive independent if the comparison of two arbitrary lotteries defined by two joint probability distributions on $X \times Y$ depends only on their marginal probability distributions (Keeney and Raiffa (1993)). As proven in Fishburn (1965), X and Y are additive independent if and only if the utility function $u(x,y)$ is additive, i.e., if and only if

$$u(x,y) = u(x_0,y) + u(x,y_0) \tag{22}$$

where $u(x_0,y)$ and $u(x,y_0)$ are called conditional utility functions. All the utility functions are normalized.

The findings in Sections 4 and 5, in particular Corollaries 1 and 3, allow us to derive the following results concerning the monotonicity and ultramodularity of $u(x,y)$ [the proof is in Appendix A].

Theorem 6 *Consider a utility function under additive independence. Then:*

1. *$u(x,y)$ is non-decreasing if and only if $u(x_0,y)$ and $u(x,y_0)$ are.*
2. *$u(x,y)$ is neg-ultramodular if and only if $u(x_0,y)$ and $u(x,y_0)$ are.*
3. *if $u(x,y)$ is neg-ultramodular, then it is concave.*

Point 1 in Theorem 6 implies that, if in an elicitation problem, additive independence is assumed, then non-satiation in each attribute implies multivariate non-satiation of the decision-maker.

Points 2 implies that a decision-maker who is single-attribute risk averse, is also multivariate risk averse in the sense of Tsetlin and Winker (2009). In fact, if and only if the conditional utility functions are concave (one recalls that for univariate functions neg-ultramodularity and concavity coincide), then the multiattribute utility function is neg-ultramodular.

Point 3 states that if a decision maker is multivariate risk averse, then, under additive independence, She is also globally risk averse (in fact, her multiattribute utility function, when neg-ultramodular is also concave). We recall that, in general, single-attribute risk aversion or multivariate risk-aversion are not sufficient for global risk aversion.

Keeney and Raiffa (1993) [p. 253] suggest additive independence as way for approximating utility functions. This idea can be read through eq. (20) as follows: additive independence is equivalent to a cut at order 1 of the functional ANOVA expansion of a generic u:

$$u(x) \simeq \sum_{i=1}^{n} u_i(x_i) \tag{23}$$

Under this cut, no interactions are present in the utility function. We recall that by Theorem 9 in Richard (1975) [p. 20] the decision-maker is multivariate risk neutral if and only if $u(x) = \sum_{i=1}^{n} u_i(x_i)$. Hence, multivariate risk-neutrality is associated with the absence of interactions among attributes.

In the next section, we explore the case in which interactions among attributes are present, namely the case of utility independence.

6.2 Mutual Utility Independence, Ultramodularity and Multi-Attribute Risk Aversion

In multiattribute theory, the assumption of additive independence may result too restrictive [Hogart and Karelaia (2005)]. A first relaxation of such assumption is represented by utility independence. Utility independence has attracted a widespread attention in literature for several reasons. On the one hand, the utility independence assumptions are appropriate in many realistic problems [Keeney (2006)]. On the other hand, utility independence helps in structuring the problem and in performing sensitivity analysis.

Utility independence allows the multiplicative representation of the utility function $u(x, y)$. If X and Y are mutually utility independent, then $u(x, y)$ can be expressed by the multilinear representation [Fishburn (1980); see also Keeney and Raiffa (1993), Theorem 5.2, p. 234]

$$u(x, y) = u(x, y_0) + u(x_0, y) + ku(x, y_0)u(x_0, y) \tag{24}$$

where u, $u(x, y_0)$, $u(x_0, y)$ are based on the same common origin and consistently scaled by the constant $k \in \mathbb{R}$. By algebraic manipulation, one obtains [Keeney and Raiffa (1993), p. 238, eq.

5.31]

$$ku(x,y)+1=[ku(x,y_0)+1]\,[ku(x_0,y)+1] \tag{25}$$

The above procedure is extended to the $n-$attribute case [Keeney and Raiffa (1993), eq. (6.21), p. 291], and leads to

$$ku(x)+1=\prod_{i=1}^{n}\left[ku(x_i,\hat{x}^0)+1\right] \tag{26}$$

where $(x_i,\hat{x}^0)=(x_1^0,x_2^0,...,x_i,...,x_n^0)$. We also recall the equivalent multiplicative representation

$$u(z)=a+b\prod_{i=1}^{n}u_i(z_i) \tag{27}$$

$(b>0)$ studied in Richard (1975).

In summary, under utility independence the multiattribute utility function is multiplicative. Its monotonicity and ultramodularity properties can, then, be derived by exploiting the results of Sections 4 and 5 on the monotonicity and ultramodularity properties of generic multiplicative functions [Corollaries 2 and 5 are utilized in the Appendices to prove the next results.]

The following result holds.

Theorem 7 *Let u be increasing in the attributes. Then, $u(x)$ is ultramodular (neg-ultramodular) if and only if $u_i(x_i)$ is ultramodular (neg-ultramodular).*

We recall that, by Proposition 2, a decision-maker associated with a multiattribute ultramodular function displays the multivariate risk attitudes of Tsetlin and Winker (2009). By comparing eqs. (22) and (24), one notes that interactions are present in eq. (24). For simplicity, consider a 3 attribute case. In this case, the functional ANOVA expansion of $u(x)$ writes:

$$u=\sum_{i=1}^{3}u_i(x_i)+u_{1,2}(x_1,x_2)+u_{1,3}(x_1,x_3)+u_{2,3}(x_2,x_3)+u_{123}(x_1,x_2,x_3) \tag{28}$$

The terms $u_{1,2}(x_1,x_2)+u_{1,3}(x_1,x_3)+u_{2,3}(x_2,x_3)+u_{123}(x_1,x_2,x_3)$ are null if u is additive, and the decision-maker is multivariate risk-neutral. Conversely, they all appear as soon as interactions among attributes are present. In this latter case, the decision-maker is no more multivariate risk neutral. In fact, given any twice-differentiable function f, then a non-additive form of f implies non-null mixed partial derivatives.

In the next section, we present results for a second relaxation of the additive utility assumption, namely, preferential independence.

6.3 Preferential Independence, Ultramodularity and Monotonicity

A further relaxation of the additive independence assumption is represented by preferential independence. Keeney and Raiffa (1993) provide the procedure for obtain a multiattribute utility function given that an additive value function has been assessed. The corresponding utility function can be of an additive or multiplicative form. The next result sets forth the conditions under which, in preferential independence framework, ultramodularity and monotonicity of the marginal value functions insure ultramodularity and monotonicity of the multiattribute utility function. The proof is in Appendix A.

Theorem 8 *Let $X = X_1 \times X_2 \times ... X_n$, be a set of attributes over which the decision-maker expresses preferential independence. Then, let*

$$v(x) = \sum_{i=1}^{n} v_i(x_i) \tag{29}$$

the value function. (The additive form follows by preferential independence.) Let $u(x)$ the corresponding utility function. Let X_i be utility independent. One has

1. *If $v_i(x_i)$ is non-decreasing $\forall i$, then $u(x)$ is non-decreasing*
2. *If $v_i(x_i)$ is ultramodular $\forall i$, then $u(x)$ is ultramodular.*

We recall that, by theorem 6.11 (page 330 in Keeney and Raiffa (1993)), the utility function corresponding to the preference function in eq. (29) is either of the multiplicative type

$$u(x) \sim e^{cv(x)} = \prod_{i=1}^{n} e^{cv_i(x)} \tag{30}$$

or proportional to $v(x)$, i.e.,

$$u(x) \sim v(x) \tag{31}$$

Theorem 8 then states that monotonicity and ultramodularity of the assessed value function are transferred into the utility function, in both the multiplicative and additive representations. Therefore, the multivariate risk attitudes of the decision-maker are characterized directly by the value function.

7 Conclusions

In this work, we have presented a formal approach for connecting the properties of a multiattribute utility function to those of its conditional utilities, without relying on regularity assumptions.

In the quantitative assessment of multiattribute utility functions, procedures involving elicitation by aggregation of conditional utilities play a central role. The elicited univariate utility functions are not necessarily regular. As a consequence, the characterization of non-satiation (monotonicity) and risk aversion (concavity, ultramodularity) properties of u by first and second order partial derivatives is not possible. We have seen that the high dimensional model representation theory, by requiring the sole measurability of u, provides the required generalization of the mathematical framework. However, in spite of the widespread utilization of functional ANOVA, its monotonicity and ultramodularity properties have not been formalized yet. It has then been necessary to investigate monotonicity and ultramodularity properties for generic multivariate functions

(f) first. As far as non-orthogonalized terms in the integral expansion of f are concerned, findings show that they possess the same monotonicity (ultramodularity) properties of f. As far as orthogonalized terms are concerned, main effects (f_i) always possess the same monotonicity properties of f. However, this is not the case for higher order terms due to the orthogonalization procedure. We have then derived the sufficient conditions on f that insure that all terms in the ANOVA expansion possess the same monotonicity (ultramodularity) properties of f.

The differences between the traditional utilization of functional ANOVA and its utilization in MAUT have been investigated next. In particular, in MAUT, it is possible to determine a-priori the order of the expansion by the assumptions on the decision-maker preferences.

The link between the presence of interactions in the functional ANOVA expansion of u and the multivariate risk attitudes of the decision-maker has also been discussed. We have seen that multivariate risk neutrality is equivalent to a cut of order 1 in the expansion, i.e., it implies the absence of interactions among attributes. When some degree of interactions is allowed, we have shown that multivariate risk attitudes of Tsetlin and Winker (2009) — preferences for combining good with bad — imply the neg-ultramodularity of the utility function.

The findings have then been specialized to the cases of separable (additive and multiplicative) utility functions. We have seen that under utility independence, u is monotonic (ultramodular) if and only if all the corresponding single-attribute conditional utility functions are. Under mutual utility independence, if u is increasing, then it is ultramodular (monotonic) if and only if all the conditional utility functions are. Under preferential independence, letting v denote the multiattribute value function for preferences under certainty, we have seen that if v is non-decreasing, then u is; if v is ultramodular, then u is. The above results hold without restrictions on the regularity of u.

8 Appendix A: Proofs

Proof of Lemma 2. Let $y = x + h$. Then, $\forall h \geq 0$, (i.e., $h_i \geq 0$, $i = 1, 2, ..., n$), by (9), we have:

$$f(x) \leq f(x+h) \tag{32}$$

In particular, when $h = \left[0 \,...\, h_{i_1} \,...\, h_{i_k} \,...\, 0\right]$, with $h_{i_s} \geq 0$, $s = 1, 2, ..., k$ $(k < n)$, eq. (32) holds. Let us then integrate both sides of eq. (32) with respect to all x_i's but $x_{i_1}, x_{i_2}, ..., x_{i_k}$. By the monotonicity property of the integration operation, we get:

$$\int\limits_{I^{n-k}} f(x) \prod_{i \neq i_1, i_2, ..., i_k} dx_i \leq \int\limits_{I^{n-k}} f(x+h) \prod_{i \neq i_1, i_2, ..., i_k} dx_i \tag{33}$$

which implies:

$$m_{i_1, i_2, ..., i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \leq m_{i_1, i_2, ..., i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) \tag{34}$$

■

Proof of Theorem 2. Let $y = x + h$. Then, by (9), we have:

$$f(x) \leq f(x+h) \tag{35}$$

$\forall h \geq 0$, (i.e., $h_i \geq 0$, $i = 1, 2, ..., n$). Eq. (32) holds, in particular, when $h = \left[0 \ldots h_i \ldots 0\right]$, with $h_i \geq 0$. Let us then integrate both sides of eq. (32) with respect to all x_k's but x_i. We get:

$$\int\limits_{I^{n-1}} f(x) \prod_{k \neq i} dx_k \leq \int\limits_{I^{n-1}} f(x+h) \prod_{k \neq i} dx_k \tag{36}$$

which implies:

$$m_i(x_i) - f_0 \leq m_i(x_i + h) - f_0 \tag{37}$$

which proves the assertion. ∎

Proof of Theorem 3. By eq. (6), we have:

$$f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) = m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) - \sum_{s=1}^{k-1} \sum_{i_1 < i_2 ... < i_s} f_{i_1,i_2,...,i_s}(x_{i_1}, x_{i_2}, ..., x_{i_s}) - f_0 \tag{38}$$

$$\begin{aligned} & f_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) = \\ & = m_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) - \sum_{s=1}^{k-1} \sum_{i_1 < i_2 ... < i_s} f_{i_1,i_2,...,i_s}(x_{i_1} + h_{i_1}, ..., x_{i_s} + h_{i_s}) - f_0 \end{aligned} \tag{39}$$

Thus:

$$f_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) \geq f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \tag{40}$$

is equivalent to

$$\begin{aligned} & m_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) - \sum_{s=1}^{k-1} \sum_{i_1 < i_2 ... < i_s} f_{i_1,i_2,...,i_s}(x_{i_1} + h_{i_1}, ..., x_{i_s} + h_{i_s}) - f_0 \geq \\ & m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) - \sum_{s=1}^{k-1} \sum_{i_1 < i_2 ... < i_s} f_{i_1,i_2,...,i_s}(x_{i_1}, x_{i_2}, ..., x_{i_s}) - f_0 \end{aligned} \tag{41}$$

which, rearranged leads to:

$$\begin{aligned} & m_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) - m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \geq \\ & \sum_{s=1}^{k-1} \sum_{i_1 < i_2 ... < i_s} [f_{i_1,i_2,...,i_s}(x_{i_1} + h_{i_1}, ..., x_{i_s} + h_{i_s}) - f_{i_1,i_2,...,i_s}(x_{i_1}, x_{i_2}, ..., x_{i_s})] \end{aligned} \tag{42}$$

∎

Proof of Theorem 4. Let $y = x + h$. Then, eq. (1) holds for any h, and, therefore, in particular when $h = \left[0 \ldots h_i \ldots 0\right]$, with $h_i \geq 0$. Thus, it holds that

$$f(x_i + h_i, \widehat{x}) - f(x_i, \widehat{x}) \leq f(y_i + h_i, \widehat{x}) - f(y_i, \widehat{x}) \tag{43}$$

where $\widehat{x} = x/x_i = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $h_i > 0$, $x_i \leq y_i$. Let us then integrate both sides of (43) on all x_k, but $x_k = x_i$. Since the component of the Sobol' decomposition are given by $m_i(x_i) = f_0 + f_i(x_i) = \int\limits_{I^{n-1}} f(x) \prod_{k \neq i} dx_k$, it holds

$$m_i(x_i + h_i) - m_i(x_i) \leq m_i(y_i + h_i) - m_i(y_i)$$

Since $f_i(x_i) = m_i(x_i) - f_0$, by point (b) of Proposition 1, we get

$$f_i(x_i + h_i) - f_i(x_i) \leq f_i(y_i + h_i) - f_i(y_i) \tag{44}$$

■

Proof of Lemma 3. Let $y = x + h$, with $h = \left[0 \ldots h_{i_1} \ldots h_{i_2} \ldots h_{i_k} \ldots\right]$, where $h_{i_s} > 0$, $s = 1, ..., k$, and $2 < k < n$. Then, by eq. (1), we have:

$$f(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}, \widehat{x}) - f(x) \leq f(y_{i_1} + h_{i_1}, y_{i_2} + h_{i_2}, ..., y_{i_k} + h_{i_k}, \widehat{y}) - f(y) \tag{45}$$

where $\widehat{x} = x/x_{i_1}, x_{i_2}, ..., x_{i_k}$ in this case. Integrating both sides of eq. (45) with respect to all x_i's but x_{i_1},x_{i_2},...,x_{i_k}, thanks to the monotonicity property of the integration operation, we get:

$$\int\limits_{I^{n-k}} [f(x+h) - f(x)] \prod_{i \neq i_1, i_2, ..., i_k} dx_i \leq \int\limits_{I^{n-k}} [f(y+h) - f(y)] \prod_{i \neq i_1, i_2, ..., i_k} dy_i \tag{46}$$

which implies:

$$\begin{aligned} &m_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) - m_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \leq \\ &\leq m_{i_1,i_2,...,i_k}(y_{i_1} + h_{i_1}, y_{i_2} + h_{i_2}, ..., y_{i_k} + h_{i_k}) - m_{i_1,i_2,...,i_k}(y_{i_1}, y_{i_2}, ..., y_{i_k}) \end{aligned} \tag{47}$$

■

Proof of Theorem 5. Let us start with eq. (47). In order for $f_{i_1,i_2,...,i_k}$ to be ultramodular, it must hold that:

$$\begin{aligned} &f_{i_1,i_2,...,i_k}(x_{i_1} + h_{i_1}, x_{i_2} + h_{i_2}, ..., x_{i_k} + h_{i_k}) - f_{i_1,i_2,...,i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) \leq \\ &\leq f_{i_1,i_2,...,i_k}(y_{i_1} + h_{i_1}, y_{i_2} + h_{i_2}, ..., y_{i_k} + h_{i_k}) - f_{i_1,i_2,...,i_k}(y_{i_1}, y_{i_2}, ..., y_{i_k}) \end{aligned} \tag{48}$$

For the sake of notation simplicity, let us rewrite this inequality as:

$$\Delta f_{i_1,i_2,...,i_k}(x) \leq \Delta f_{i_1,i_2,...,i_k}(y) \tag{49}$$

where x and y are here intended as a more synthetic way to represent $x_{i_1}, x_{i_2}, ..., x_{i_k}$ and $y_{i_1}, y_{i_2}, ..., y_{i_k}$ respectively. Then, recalling the definition of $f_{i_1,i_2,...,i_k}$ in eq. (6), i.e.,

$$f_{i_1,i_2,...,i_k}(x) = m_{i_1,i_2,...,i_k}(x) - \sum_{s=1}^{k-1} \sum_{i_1<i_2...<i_s} f_{i_1,i_2,...,i_s}(x) - f_0 \tag{50}$$

one gets:

$$\Delta m_{i_1,i_2,...,i_k}(y) - \Delta m_{i_1,i_2,...,i_k}(x) \geq \sum_{s=1}^{k-1} \sum_{i_1<i_2...<i_s} \Delta f_{i_1,i_2,...,i_s}(y) - \sum_{s=1}^{k-1} \sum_{i_1<i_2...<i_s} \Delta f_{i_1,i_2,...,i_s}(x) \tag{51}$$

■

Proof of Corollary 1. If all $z_i(x_i)$ are non-decreasing, then f is non-decreasing as it is the sum of non-decreasing functions. Conversely, suppose that f is non-decreasing. Thanks to eq. (6), all $z_i(x_i) = f_i + f_0$, where f_0 is a constant. By Theorem 2 if f is non-decreasing, f_i is non-decreasing, and all z_i are non-decreasing as they differ from the corresponding f_i by a constant. ■

Proof of Corollary 3. Since f is additive, then $z_i = f_i + f_0$, as in the previous proof. The necessary condition is assured by Theorem 4. The sufficient condition follows from Proposition 1, points (a) and (b). ■

Proof of Corollary 4. Since f is additive, if it is ultramodular (convex) $z_i(x_i)$ $\forall i$ are ultramodular (convex) as well. As shown by **?** and confirmed in Marinacci and Montrucchio (2005), in the scalar case ultramodularity is equivalent to convexity, provided z_i is continuous. It follows that $f(x)$ is convex (ultramodular), since it is the sum of the convex (ultramodular) functions $z_i(x_i)$. ■

Proof of Corollary 2. If all the z_i are increasing and positive, it is immediate to see that f is increasing. Conversely, by and Lemma 2 noting that

$$\frac{m_i(x_i)}{\prod_{j \neq i} t_j} = z_i(x_i) \tag{52}$$

where $t_j = \int_I z_j(x_j)dx_j$ $\forall j$ and $t_j \geq 0$ by assumption, the thesis follows. ■

Proof of Corollary 5. If f is ultramodular, by Lemma 3 the components of the integral decomposition $m_i(x_i) = \int\limits_{I^{n-1}} f(x) \prod_{k \neq i} dx_k$ are ultramodular. If, in addition, f is of the form $\prod z_i(x_i)$, then one can write

$$m_i(x_i) = z_i(x_i) \cdot \prod_{j \neq i} t_j \tag{53}$$

where $t_j = \int_I z_j(x_j)dx_j$ $\forall j$ and $t_j \geq 0$ by assumption. Ultramodularity of $z_i(x_i)$ $\forall i$ follows. The converse statement follows from Proposition 4.4 in Marinacci and Montrucchio (2005) [see Proposition 1, property (c)]. ■

Proof of Theorem 6. Point 1. Corollary 1 implies that if $f = \sum z_i(x_i)$, it is non-decreasing if and only if all the z_i are non-decreasing. Now, in the case of additive independence, $u(x,y) = \sum k_i u_i(t_i)$, where $k_i > 0$.

Point 2. Corollary 3, assures that if f is of the form $\sum z_i(x_i)$, then f is neg-ultramodular if and only if all z_i are neg-ultramodular. (Recall that f is neg-ultramodular if $-f$ is ultramodular.) Now, in the case of additive independence, $u(x,y) = \sum k_i u_i(t_i)$, where $k_i > 0$.

Point 3. Follows from Corollary 4. ■

Proof of Theorem 7. Under mutual utility independence, $ku(x) + 1 = [\prod_{i=1}^{n} \left[ku(x_i, \hat{x}^0) + 1\right]$.
If $k > 0$, and u is ultramodular, then $ku(x)+1$ is ultramodular. Hence, by Corollary 5, each of the functions $\left[ku(x_i, \hat{x}^0) + 1\right]$ is ultramodular. By Proposition 1, since k is positive, each $u(x_i, \hat{x}^0)$ is ultramodular. The converse statement is proven as follows. If each $u(x_i, \hat{x}^0)$ is ultramodular and k is positive, $ku(x_i, \hat{x}^0) + 1$ is ultramodular for all i. Then, by the second part of Corollary 5, the thesis follows.
If $k < 0$, and u is ultramodular, then $ku(x) + 1$ is neg-ultramodular. Hence, by Corollary 5, each of the functions $\left[ku(x_i, \hat{x}^0) + 1\right]$ is neg-ultramodular. By Proposition 1, since k is negative, each $u(x_i, \hat{x}^0)$ is ultramodular. The converse statement is proven as follows. If each $u(x_i, \hat{x}^0)$ is ultramodular and k is negative, $ku(x_i, \hat{x}^0) + 1$ is neg-ultramodular for all i. Then, by the second part of Corollary 5, the thesis follows.

■

Proof of Theorem 8. Point 1. If $v_i(x_i)$ are increasing $\forall i$, then $v(x)$ is increasing. By theorem 6.11 (page 330 in Keeney and Raiffa (1993)), the utility function corresponding to the preference function in eq. (29) is either of the multiplicative type

$$u(x) \sim e^{cv(x)} = \prod_{i=1}^{n} e^{cv_i(x)} \tag{54}$$

or proportional to $v(x)$, i.e.,

$$u(x) \sim v(x) \tag{55}$$

In the case of eq. (55), the thesis is, then, immediate. In the case of eq. (54), the thesis follows from Corollary 1.

Point 2. From eq. (54), $u(\cdot)$ is an increasing and convex transformation of $v(\cdot)$. If $v_i(x_i)$ is ultramodular $\forall i$ then $v(x)$ is ultramodular by Corollary 3 and the thesis follows by (d) of Proposition 1. ■

References

[Alis and Rabitz (2001)] Alis O. F. and Rabitz H., 2001: "Efficient Computation of High Dimensional Model Representations," *Journal of Mathematical Chemistry*, 29 (2), pp. 127-142.

[Anginella et al (2004)] Anginella S., Greco S., Lamantia F. and Matarazzo B, 2004: "Assessing non-additive utility for multicriteria decision aid," European Journal of Operational

Research, 158, pp. 734–744.

[Anginella et al (2010)] Angilella S., Greco S. and Matarazzo B., 2010: "Non-additive robust ordinal regression: A multiple criteria decision model based on the Choquet integral," *European Journal of Operational Research*, 201 (1), pp. 277-288.

[Baucells et al (2008)] Baucells M., Carrasco J. and Hogarth R.M., 2008: "Cumulative Dominance and Heuristic Performance in Binary Multiattribute Choice", *Operations Research*, 56 (5), pp. 1289-1304.

[Bedford (1998)] Bedford T., 1998: "Sensitivity Indices for (Tree)- Dependent Variables," *Proceedings of the Second International Symposium on Sensitivity Analysis of Model Output*, Venice (Italy), 1998, pp.17-20.

[Borgonovo (2006)] Borgonovo E., 2006: "Measuring Uncertainty Importance: Investigation and Comparison of Alternative Approaches," *Risk Analysis*, 26 (5), pp. 1349-1362.

[Borgonovo (2010)] E. Borgonovo, 2010: "Sensitivity Analysis with Finite Changes: An Application to Modified EOQ models," *European Journal of Operational Research*, 200, pp. 127–138

[Efron and Stein (1981)] Efron B. and Stein C., 1981: "The Jackknife Estimate of Variance," *The Annals of Statistics*, 9 (3), pp.586-596.

[Figuera et al (2009)] Figueira J., Greco S. and Slowinski R., 2009: "Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method," *European Journal of Operational Research*, 195, pp. 460-486.

[Fishburn (1965)] Fishburn P.C., 1965: "Independence in utility theory with whole product sets", *Operations Research*, 13, pp. 28-45.

[Fishburn (1980)] Fishburn P.C., 1980: "Multilinear Expected Uility", *Mathematics of Operations Research*, 5 (4), pp. 502-509.

[Foldes and Hammer (2005)] Foldes S. and Hammer P. L., 2005: "Submodularity, Supermodularity, and Higher-Order Monotonicities of Pseudo-Boolean Functions," *Mathematics of Operations Research*, 30 (2), pp. 453-461.

[Fortemps et al (2008)] Fortemps P., Greco S. and Slowinski R., 2008: "Multicriteria decision support using rules that represent rough-graded preference relations", *European Journal of Operational Research*, 188, pp. 206–223.

[Greco et al (2008)] Greco S., Mousseau V. and Slowinski R., 2008: "Ordinal regression revisited: multiple criteria ranking with a set of additive value functions" *European Journal Operational Research*, 191, pp. 415-435.

[Hoeffding (1948)] Hoeffding W., 1948: "A Class of Statistics With Asymptotically Normal Distributions," *Annals of Mathematical Statistics*, 19, 293-325.

[Hogart and Karelaia (2005)] Hogart R.M. and Karelaia N., 2005: "Simple Models for Multiattribute Choice with Many Alternatives: When It Does and Does Not Pay to Face Trade-offs", *Management Science*, 51 (12), pp. 1860–1872.

[Homma and Saltelli (1996)] Homma T. and Saltelli A., 1996: "Importance Measures in Global Sensitivity Analysis of Nonlinear Models," *Reliability Engineering and System Safety*, 52, 1-17.

[Hooker (2007)] Hooker G., 2007: "Generalized Functional ANOVA Diagnostics for High Dimensional Functions of Dependent Variables", *Journal of Computational and Graphical Statistics*, 16 (3), pp. 709-732.

[Huang (1998)] Huang J.Z., 1998: "Projection Estimation in Multiple Regression with Application to Functional Anova Models", *The Annals of Statistics*, 26 (1), pp. 242-272.

[Ingersoll (1987)] Ingersoll, Jr., J.E., 1987: "Theory of Financial Decision Making" *Cambridge University Press*, Cambridge, U.K., ISBN: 0-847-67359-6.

[Jacquet-Lagrèze et al (1987)] Jacquet-Lagreze E., Meziani R. and Roman Slowinski R., 1987: "MOLP with an interactive assessment of a piecewise linear utility function", *European Journal of Operational Research*, 31 (3), pp. 350-357

[Keeney and Raiffa (1993)] Keeney R. and Raiffa H., 1993: "Decisions with Multiple Objectives" *Rowman & Littlefield Publishers, Inc.*, Totowa, U.S.A., ISBN 0-521-43883-7.

[Keeney (2006)] Keeney R., 2006: "Using preferences for Multi-Attributed Alternatives", *Journal of Multi-criteria Decision Analysis* , 14, pp. 169-174.

[Li et al (2001)] Li G., Wang S.-W., Rosenthal C. and Rabitz H., 2001: "High dimensional model representations generated from low dimensional data samples. I. mp-Cut-HDMR", *Journal of Mathematical Chemistry*, 30 (1).

[Malliavin (1995)] Malliavin P., 1995,: "Integration and Probability," *Springer Verlag Editions,* New York, NY, USA, 322 pages, ISBN 0-387-94409-5.

[Marinacci and Montrucchio (2005)] Marinacci M. Montrucchio L., 2005: "Ultramodular Functions", *Mathematics of Operations Research*, 30(2), May 2005, pp 311-332.

[Marinacci and Montrucchio (2008)] Marinacci M. Montrucchio L., 2005: "On Convexity and Ultramodularity", *Journal of Mathematical Analysis and Applications*, 344, pp. 642–654.

[Meyer and Richard (1972)] Meyer, Richard F., 1972: "Some notes on Discrete Multivariate Utility", *Harvard Business School, Mimeographed Manuscript.*

[Milgrom and Shannon (1994)] MILGROM, P, SHANNON, C., 1994: "Monotone Comparative Statics", *Econometrica,* 62, (1) p. 157-180.

[Owen (2003)] OWEN, A. B. 2003.: "The dimension distribution, and quadrature test functions", *Statistica Sinica*, (13), pp. 1–17.

[Ortega and Escudero (2010)] ORTEGA E., ESCUDERO L.F., 2010: "On expected utility for financial insurance portfolios with stochastic dependencies", *European Journal of Operational Research*, 200, pp.181–186.

[Prékopa and Mádi-Nagy (2008)] PREKOPA A., MADI-NAGY G., 2008: "A class of Multiattribute Utility Functions", *Economic Theory*, 34, 12, pp. 591-602.

[Rabitz and Alis (1999)] RABITZ H. AND ALIS F., 1999: "General foundations of high-dimensional model representations," *Journal of Mathematical Chemistry*, 25, no. 2-3, pp. 197–233.

[Richard (1975)] RICHARD S.F. 1975: "Multivariate risk aversion, utility independence and separable utility functions", *Management Science*, 22, 1, pp. 12-21.

[Saltelli et al (1999)] SALTELLI A., TARANTOLA S. AND CHAN K. P.-S., 1999: 'A Quantitative Model-Independent Method for Global Sensitivity Analysis of Model Output', *Technometrics*, 41, 1, 39-56.

[Saltelli et al (2000)] SALTELLI A., TARANTOLA S. AND CAMPOLONGO F., 2000: "Sensitivity Analysis as an Ingredient of Modelling", *Statistical Science*, 19 (4), pp. 377-395.

[Saltelli and Tarantola (2002)] SALTELLI A. AND TARANTOLA S., 2002: "On the Relative Importance of Input Factors in Mathematical Models: Safety Assessment for Nuclear Waste Disposal," *Journal of the American Statistical Association*, 97 (459), p. 702-709.

[Sobol' (1969)] SOBOL, I. M. 1969: "Multidimensional Quadrature Formulas and Haar Functions", Nauka, Moscow, Russia (in Russian).

[Sobol' (1993)] SOBOL I.M., 1993: "Sensitivity estimates for nonlinear mathematical models," Matem. Modelirovanie, 2(1) (1990) 112-118 (in Russian). English Transl.: MMCE, 1(4) (1993) 407-414.

[Sobol' (2001)] SOBOL I.M., 2001: "Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates," *Mathematics and Computers in Simulation*, 55(1), pp. 271-280.

[Sobol' (2003)] SOBOL I.M., 2003: "Theorem and Examples on High-Dimensional Model Representation," *Reliability Engineering and System Safety*, 79, 187-193.

[Sobol' et al (2007)] SOBOL I.M., TARANTOLA S., GATELLI D., KUCHERENKO S.S. AND MAUNTZ W., 2007: "Estimating the approximation error when fixing unessential factors in global sensitivity analysis", *Reliability Engineering and System Safety*, 92, pp. 957 – 960.

[Takemura (1983)] TAKEMURA A., 1983: "Tensor Analysis of ANOVA Decomposition", *Journal of the American Statistical Association*, 78 (384), pp. 894-900.

[Tatman and Shachter (1990)] TATMAN J. A. AND SHACHTER R.D., 1990: "Dynamic programming and influence diagrams," *IEEE Transactions on Systems Man and Cybernetics*, 20 (2), pp. 365–379.

[Topkis (1995)] TOPKIS D.M., 1995: "Comparative Statics of the Firm", *Journal of Economic Theory*, 67, pp.340-401.

[Tsetlin and Winker (2009)] TSELIN, I., WINKLER R.L., 2009: "Multiattribute Utility Satisfying a Preference for Combining Good with Bad", *Management Science*, 55, **12**, pp 1942-1952.

[Wagner (1995)] WAGNER H.M., 1995: "Global Sensitivity Analysis," *Operations Research*, 43 (6), pp. 948-969.

[Wang (2006)] WANG X., 2006: "On the Effects of Dimension Reduction Techniques on Some High-Dimensional Problems in Finance", *Operations Research*, 54 (6), pp. 1063–1078.

[Zopounidis and Doumpos (2002)] ZOPOUNIDIS C. AND DOUMPOS M., 2002: "Multicriteria classification and sorting methods: A literature review", *European Journal of Operational Research*, 138, pp. 229–246.