# Moment Calculations for Piecewise-Defined Functions: An Application to Stochastic Optimization with Coherent Risk Measures

E. Borgonovo* and L. Peccati

ELEUSI Research Center and Department of Decision Sciences
Bocconi University, Via Roentgen 1, 20136 Milano, Italy

**Abstract**

This work introduces a new analytical approach to the formulation of optimization problems with piecewise-defined (PD) objective functions. First, we introduce a new definition of multivariate PD functions and derive formal results for their continuity and differentiability. Then, we obtain closed-form expressions for the calculation of their moments. We apply these findings to three classes of optimization problems involving coherent risk measures. The method enables one to obtain insights on problem structure and on sensitivity to imprecision at the problem formulation stage, eliminating reliance on ad-hoc post-optimality numerical calculations.

**Acknowledgement 1** *The authors wish to thank the anonymous referees for the very insightful comments, that have greatly contributed in the final realization of the present paper. A special thank to Jacques Carette for the very perceptive and useful suggestions on an earlier version of the present paper. Financial support from the ELEUSI Research Center of Bocconi University is gratefully acknowledged.*

## 1 Introduction

In this work, we present a new methodology for supporting the formulation of stochastic programming problems involving piecewise-defined (PD) functions.

Referring to the solution of managerial problems in the presence of uncertainty, Dantzig (1999) ($p.1$) states: "*By "solve", I mean in the practical sense of determining strategic decisions that are demonstrably superior to those obtained by ways that avoid having to properly take uncertainty fully into account.*" After the seminal works of Dantzig (1955) and Beale (1955), stochastic programming (SP) has been broadly applied to support managerial

*emanuele.borgonovo@unibocconi.it; lorenzo.peccati@unibocconi.it

decisions in several sectors. Quoting directly from Dupačova (2002), p. 288: "*There are many excellent recent papers on successful real-life applications of stochastic programming; it is impossible to list them all...*". SP applications range from project management[1], to revenue management[2], to the chemical[3] and energy sectors[4] (for a recent review, see also Antunes and Dias (2007)).

SP with coherent risk measures is nowadays pervasive in Finance [Artzner et al (1999), Rockafellar and Uryasev (2002), Benati (2004), Siegmann and Lucas (2005)] and has gained recent interest in Inventory Management [Gotoh and Takano (2007); Ahmed et al (2007); Borgonovo and Peccati (2008)]. Convexity properties, duality aspects and risk-coherence of the optimization problems have been thoroughly discussed in the literature. Ruszczynski and Shapiro (2005) offer a comprehensive theoretical approach to optimization with coherent risk measures.

In this work, we focus on problem formulation. The presence of a PD objective function can limit the analytical treatment of SP problems, leaving the derivation of important decision-making insights to post-optimality analysis. Additional sensitivity tests (for example on the selection of different distributions) often rely on ad-hoc numerical experiments and provide only case-by-case answers. We develop a method to exploit the problem formulation stage for the derivation of additional decision-making insights, thus eliminating reliance on numerical tests. The knowledge of the stochastic properties that affect the managerial choice plays a crucial role towards this goal.

Our first step is the setup of the required theoretical framework. Properties as continuity and differentiability naturally emerge in O.R. applications. Nonetheless, for PD functions they have not been discussed comprehensively. The first work that formalizes the notion of a multivariate PD function is Herrera (2007). In Herrera (2007) integrability is of interest and the definition is formulated $\mu.a.e.$. As a consequence, continuity and differentiability cannot be addressed. We extend Herrera's definition and derive results concerning continuity and differentiability of PD functions. We introduce the notion of distinct constituents, to reinforce their piecewise character. For operational reasons, we distinguish two types of PD

[1] Nozick et al (2004) present a model to support decision making in the "*allocation and management of scarce resources across many projects*" while facing "*uncertainty in the duration and outcomes of specific tasks*".

[2] We refer to Bertsimas and Shioda (2003) and Cooper and Gupta (2006) for the use of stochastic optimization to support decision-making in revenue management.

[3] Tomazi (2004), Shah and Madhavan (2004) apply SP in the optimization of batch reactions to address the presence of uncertainty.

[4] Chaton and Doucet (2003) utilize an SP model to support decision making in "*new investments in generation and transmission capacity*"; Schaefer and Schaefer (2004) present a model to minimize the costs of distributed hybrid generation while incorportating "*uncertainty in customer demand, weather, and fuel costs*" in the analysis.

functions, in accordance with their assignment type. This investigation provides us with all the tools necessary to the calculation of PD function moments. We derive closed form expressions for the decomposition of PD function central and non-central moments in terms of the moments of their constituents. The analysis reveals that extended constituents play a central role. In particular, any moment of a PD function is the sum of the moments of its extended constituents; and any central moment (of order $p$) is the linear combination of all the moments up to $p$ of the extended constituents. We apply the findings to PD functions generated by the $max(\cdot)$ and $|\cdot|$ operations, as their appear frequently in O.R. problems.

The above findings allow to unveil the stochastic properties involved in SP problems whose objective functions are PD. We study mean-deviation, mean-upper-semideviation and Conditional Value at Risk (CVAR) SP problems. As far as stochastic properties are concerned, the decomposition of the objective functions shows that mean-deviation and mean-upper-semideviation problems are closely related. The stochastic properties involved in CVAR decisions, instead, differ substantially. As far as the invariance of optimal policies is concerned, the analysis reveals that the set of distributions over which the solution of a mean-deviation problem is insensitive to imprecision is included in the set of distributions that do not alter the solution of mean-upper-semideviation problem of the same order. Furthermore, sensitivity to imprecision increases with the order of the deviation measures.

The remainder of the paper is organized as follows. Section 2 presents a concise literature review, highlighting the role of PD functions in O.R.. Section 3 offers a revised definition of PD functions and deals with their analytic properties. Section 4 presents results for the calculation of the moments of PD functions. Section 5 specializes the results for the cases of two widely applied PD functions: the max$(\cdot)$ and $|\cdot|$ PD functions. Section 6 derives results for SP problems with mean-deviation, mean-upper-semideviation and CVAR. Section 7 offers conclusions.

## 2 Literature Review and problem Statement

Optimization with coherent risk measures has found widespread use in Finance and recent interest in Inventory Management. As Ruszczynski and Shapiro (2005) point out, in spite of the application area, a risk-neutral problem is formulated as:

$$\min_{x \in S} \mathbb{E}[Z(x, \omega)] \tag{1}$$

where $Z(x, \omega)$ is the loss function of the system at hand, $x$ is the vector of choice variables, $S$ the feasible set, $\omega \in \Omega \subseteq \mathbb{R}^n$ the stochastic variable(s) (see Table 1 for notation). As in Ruszczynski and Shapiro (2005), we assume that $Z$ is an element of the linear space of functions $\mathcal{Z} := \mathcal{L}_p(\Omega)$[5] for each $x \in S$.

[Insert Table 1 about here]

Ruszczynski and Shapiro (2005) [p.2] underline that the solution of problem 1 represents "*an optimal decision on average*" and... "*for these reasons, quantitative models of risk and risk aversion are needed.*" One then solves the problem:

$$\min_{x \in S}[\rho\{Z(x,\omega)\}] \tag{2}$$

where $\rho : \mathcal{Z} \to \mathbb{R}$. Ruszczynski and Shapiro (2005) refer to $\rho(\cdot)$ as a risk-function. If $\rho(\cdot)$ satisfies the axioms of Artzner et al (1999), then it is a coherent measure of risk [for a description of the axioms, see Artzner et al (1999), Rockafellar and Uryasev (2002), Benati (2004); for the relationship between risk functions and utility functions, see Section 4 in Ruszczynski and Shapiro (2005)].

Three examples of risk functions follow. The CVAR coherent risk measure has been introduced in Rockafellar and Uryasev (2002).

**Example 1** *Let* $\alpha \in (0,1)$. $CVAR_\alpha$ *is defined as* "the mean of the $\alpha-$tail of the distribution *of* $Z$ *(Rockafellar and Uryasev (2002); p. 1448)". In optimization with* $CVAR_\alpha$, *one solves the problem:*

$$\min_{\mathbf{x},\zeta \in S \times \mathbb{R}} H_\alpha(x,\zeta) \tag{3}$$

*with*

$$H_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha}\mathbb{E}\left\{[Z(x,\omega)-\zeta]_+\right\} \tag{4}$$

*and*

$$CVAR_\alpha = H_\alpha(x^*, \zeta_\alpha(x^*)) \tag{5}$$

$VAR_\alpha(x) = \arg\min_\zeta H_\alpha(x,\zeta)$ *is the Value at Risk (*$VAR$*) of the loss function.*

The presence of the $max(\cdot)$ operation[6] in eq. (4) makes the objective function of $CVAR$ optimization PD.

Besides the CVAR risk-measure, mean-deviation and semi-deviation measures have been utilized in the literature. We report here the analytical expressions following the notation of Ruszczynski and Shapiro (2005), p. 10.

[5] Given the probability space $(\Omega, \mathcal{B}(\Omega), \mu)$, we denote by $\mathcal{L}_p(\Omega)$ the set of all $\mu - p-$measurable functions [Ruszczynski and Shapiro (2005)], i.e., the set of all functions $\psi : \Omega \to \mathbb{R}^n$ such that $\int_\Omega \|\psi(\omega)\|^p \, d\mu(\omega) < \infty$.

[6] $[Z(x,\omega)-\zeta]_+ = \max(0, Z(x,\omega)-\zeta)$

**Example 2** *Mean-deviation of order p:*

$$\rho(Z) := \mathbb{E}[Z] + c(\mathbb{E}[|Z - \mathbb{E}[Z]|^p])^{1/p} \tag{6}$$

**Example 3** *Mean-upper-semideviation of order p:*

$$\rho(Z) := \mathbb{E}[Z] + c(\mathbb{E}[[Z - \mathbb{E}[Z]]_+^p])^{1/p} \tag{7}$$

In eqs. (6) and (7), $c$ is an arbitrary and strictly positive real number. Ruszczynski and Shapiro (2005) provide a rigorous formalization of stochastic optimization with risk-functions and derive the conditions under which they become coherent measures of risk in the sense of Artzner et al (1999). Siegmann and Lucas (2005) discuss portfolio optimization with shortfall measures [eq. (7) with $p = 1$] and quadratic shortfall [eq. (7) with $p = 2$]. These works are generalized by Rockafellar et al (2006) — see also Rockafellar et al (2003), — where a complete theoretical treatment of deviation measures and of their application to portfolio optimization is offered.

The optimization problems connected to Examples 2 and 3 are characterized by PD objective functions. In fact, $|Z - \mathbb{E}[Z]|$ makes the mean-deviation objective function PD [eq. (6)]. Similarly, [eq. (7)], $[Z - \mathbb{E}[Z]]_+$ makes the objective function of a mean-upper-semideviation problem PD. Not only, but in supply chain management, the objective function of Arcelus et al (2006) [eq. (3) p. 51] entails the presence of the $\max(\cdot)$ operation and is, therefore, PD. In revenue management, the objective function of Cooper and Gupta (2006) [eq. (1), p. 222] is PD. In inventory management, the recent works of Ahmed et al (2007), Gotoh and Takano (2007), and Borgonovo and Peccati (2008) formulate the decision-making question as a risk coherent problem. All the objective functions of these works are PD. In Ahmed et al (2007), the loss function is an extension of the newsvendor model. Ahmed et al (2007) derive the convexity properties of the SP problem for mean $p^{th}$ semi-deviation and CVAR coherent risk measures. Sensitivity and monotonicity properties for increasing risk aversion are also obtained. Gotoh and Takano (2007) propose a multi-item version of the inventory system and study its solution when the coherent risk measure is CVAR. Borgonovo and Peccati (2008) study the risk coherent choice problems with alternative risk measures for a multi-item inventory system characterized by a the loss function

$$Z(x, \omega) = \sum_{i=1}^{N} \left( -p_i x_i + a_i + \frac{h_i x_i^2}{2\omega_i} \right) \tag{8}$$

where $N$ is the total number of items in the inventory system, $x_i$ is the order quantity of item $i$, $h_i$ is the holding cost per unit item and unit period, $\omega = \{d_i,\ \ i = 1, 2, ..., N\}$ is item $i$ demand [we refer to Borgonovo and Peccati (2008) for the complete derivation of eq. (8)]. Then the SP problems in Examples 1, 2 and 3 become:

$$\min_{x,\zeta \in S \times \mathbb{R}} \zeta + \frac{1}{1-\alpha}\mathbb{E}\left[\left([\sum_{i=1}^{N} -p_i x_i + a_i + \frac{h_i x_i^2}{2\omega_i}] - \zeta\right)^+\right] \tag{9}$$

$$\min_{x \in S} \sum_{i=1}^{N} -p_i x_i + a_i + \mathbb{E}[\frac{h_i x_i^2}{2\omega_i}] + c(\mathbb{E}[|\sum_{i=1}^{N} \frac{h_i x_i^2}{2\omega_i} - \sum_{i=1}^{N} \mathbb{E}[\frac{h_i x_i^2}{2\omega_i}]|^p])^{1/p} \tag{10}$$

and

$$\min_{x \in S} \sum_{i=1}^{N} -p_i x_i + a_i + \mathbb{E}[\frac{h_i x_i^2}{2\omega_i}] + c(\mathbb{E}[(\sum_{i=1}^{N} \frac{h_i x_i^2}{2\omega_i} - \sum_{i=1}^{N} \mathbb{E}[\frac{h_i x_i^2}{2\omega_i}])_+^p])^{1/p} \tag{11}$$

Note that the objective functions in eqs. (9), (10) and (11) are PD. The convexity of the corresponding SP problems is studied in Borgonovo and Peccati (2008). Through eqs. (9), (10) and (11) are PD the decision-makers determine the optimal policy reflecting their (different) risk attitudes. Borgonovo and Peccati (2008) compare the policies via numerical experiments and test their sensitivity to the choice of the demand distribution.

The above review shows that PD functions are widely present in O.R. applications, in particular in all SP problems with coherent measures of risk. The review also shows that the optimization problems have been thoroughly studied from the convexity and monotonicity viewpoints. In particular, the works of Ruszczynski and Shapiro (2005) and Rockafellar et al (2006) provide a comprehensive approach to their theory. We address a different aspect. Investigation has not been carried out in the direction of uncovering what stochastic properties are of interest to decision-makers in the various SP problems. This knowledge is crucial in understanding the difference in the optimization problem results, as well as in providing decision-makers with insights on problem structure and on sensitivity to imprecision. It is our purpose to introduce a method for the derivation of these indications at the problem formulation stage, so as to avoid reliance on ad-hoc post-optimality numerical experiments. The development of the required theoretical framework is the first step in this direction.

# 3 Piecewise Defined Functions: Analytical Properties

This Section introduces the theoretical background necessary to study optimization problems characterized by PD objective functions.

Piecewise linear mappings are used in Withehead (1961). The concept of a piecewise smooth manifold is due to Hutchings et al (2002). Piecewise smooth curves are dealt with in Ekholm et al (2002). Piecewise affine Lipschitz mappings are utilized in Muller and Sverak (2003). However, PD functions have been mostly studied in one-dimension. Single-variable PD function integration is discussed in Jeffrey et al (1997) and Jeffrey and Rich (1998). A canonical form for one-dimensional PD functions on linearly ordered sets is provided for in Carette (2007). Howell (2008) (see Ch. 28) formalizes the Laplace transform of one-dimensional PD functions. Multivariate PD functions are formally introduced in Herrera (2007), Section 9, to formulate a "*systematic theory of finite element methods*", as follows.

Let $\Omega \subseteq \mathbb{R}^n$, with $\Omega$ connected and open. Let $\partial\Omega$ denote the frontier of $\Omega$. A finite partition of $\Omega$ ($\Pi_\Omega$) is a collection of subsets of $\Omega$, $\Pi_\Omega = \{\Omega_1, \Omega_2, ..., \Omega_N\}$, such that $\Omega_i \subseteq \Omega$, $\Omega_i \cap \Omega_j = \varnothing$ ($i \neq j$, and $i = 1, 2, ..., N$) and $\Omega = \cup\Omega_i$. The boundary and closure of $\Omega_i$ are here denoted by $\partial\Omega_i$ and $\overline{\Omega}_i$, respectively.

$$\Sigma = \cup(\partial\Omega_i \cap \partial\Omega_j) \tag{12}$$

denotes the internal boundary. We say that $\Omega_i$ and $\Omega_j$ are neighboring, if $\partial\Omega_i \cap \partial\Omega_j \neq \varnothing$. Herrera (2007) formulates the definition of PD functions as the piecewise representation of a function $f$ which is "*locally defined*" on $\Omega$. In particular, a PD function is thought of as the one-to-one correspondence between $f$ and the sequence of functions $f_i : \Omega_i \to \mathbb{R}$, $i = 1, 2, ..., N$, each of which represents the restriction of $f$ onto $\Omega_i$. In the remainder, we shall call $f_i$ a constituent of $f$. As stated in Herrera (2007), the definition is quite general and any function can be PD, in accordance with this definition. However, it is a shared opinion among several works [Howell (2008), Jeffrey et al (1997), Jeffrey and Rich (1998), Carette (2007)] that the PD character is present when some distinction among the constituents is there. In addition, Herrera (2007) states that "*The definition of $f$ on $\Sigma$ is immaterial because the Lebesgue measure of $\Sigma$ is zero, so, $f$ can be arbitrarily defined on $\Sigma$ [Herrera (2007); p.615].*" However, while this is true for measurability, the assignment of $f$ at $\omega \in \Sigma$ is necessary for continuity and differentiability.

To address these two issues, we refine Herrera's definition. To obtain a stronger characterization of the piecewise nature, we propose the notion of distinct constituents. Given $\Omega_i, \Omega_j \in \Pi_\Omega$, consider two subsets $\Omega_{i\to j}$, $\Omega_{j\to i}$, such that $\Omega_{i\to j} \subseteq \Omega_j$ and $\Omega_{j\to i} \subseteq \Omega_i$ ($\Omega_{i\to j}$, $\Omega_{j\to i}$ can possibly be null sets) and the sets $E_i = \Omega_i \cup \Omega_{i\to j}$ and $E_j = \Omega_j \cup \Omega_{j\to i}$ (Figure 1).

[Insert Figure 1 about here]

**Definition 1** *Two constituents* $f_i : E_i \to \mathbb{R}$ *and* $f_j : E_j \to \mathbb{R}$, $f_s$ *are distinct:*
*a) when* $\Omega_{i\to j} = \varnothing$ *and* $\Omega_{j\to i} = \varnothing$, *i.e.,* $E_i = \Omega_i$ *and* $E_j = \Omega_j$
*or*
*b) given* $\Omega_{i\to j} \neq \varnothing$ *or* $\Omega_{j\to i} \neq \varnothing$, *if they satisfy the condition*

$$f_i(\omega) \neq f_j(\omega) \ \ \mu\text{.a.e. on } \Omega_{i\to j} \cup \Omega_{j\to i} \tag{13}$$

Case $a$ in Definition 1 covers the situations in which the functional forms of $f_i$ and $f_j$ do not allow to extend them outside $\Omega_i$ or $\Omega_j$. The constituents are, then, "distinct" by default. Conversely, in case $b$ the functional forms of $f_i$ and $f_j$ allow them to exist also outside $\Omega_i$ and $\Omega_j$. The set $\Omega_{i\to j} \cup \Omega_{j\to i} = E_i \cap E_j$ is the subset of $\Omega$ on which $f_i$ and $f_j$ can, potentially, be simultaneously defined. Eq. (13) in Definition 1 states that they are distinct if they differ on $\Omega_{i\to j} \cup \Omega_{j\to i}$, with the exclusion of, at most, a set of null measure.

To insure the continuity and differentiability of $f$, Herrera's definition needs to be complemented by the assignment of $f$ on the internal boundary. In this respect, we denote the value of $f$ attained at any $\omega \in \Sigma$ by $f_\Sigma(\omega)$. We then formulate the following definition of PD functions, which is of interest in the remainder of this work.

**Definition 2** *Given a connected open set,* $\Omega$, *and an associated finite partition,* $\Pi_\Omega$, *consider a sequence of distinct constituents,* $f_s$, $s = 1, 2, ..., N$. *Then, the function* $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$*:*

$$f = \begin{cases} f_s(\omega) & \omega \in \Omega_s \\ f_\Sigma(\omega) & \omega \in \Sigma \end{cases} \tag{14}$$

*is a PD function.*

The following Example illustrates Definition 2.

**Example 4** *Let* $\Omega_1 = \{\omega : \omega_1^2 + \omega_2^2 < 1\}$, $\Sigma = \{\omega : \omega_1^2 + \omega_2^2 = 1\}$ *and* $\Omega_2 = \{\omega : \omega_1^2 + \omega_2^2 > 1\}$. *The function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f = \begin{cases} \sqrt{1 - \omega_1^2 - \omega_2^2} & \omega_1^2 + \omega_2^2 < 1 \\ \sqrt{\omega_1^2 + \omega_2^2 - 1} & \omega_1^2 + \omega_2^2 > 1 \\ 0 & \omega \in \Sigma \end{cases} \tag{15}$$

*is PD. In fact, the constituents are distinct by the first part of Definition 1.*

*2) Let $\Sigma$ be the frontier of the unitary rectangle in $\mathbb{R}^2$. The function $f : \mathbb{R}^2 \to \mathbb{R}$*

$$f = \begin{cases} \sin(\omega_1 + \omega_2) & \omega \in (0,1)^2 \\ 1 & \omega \in \Sigma \\ (\omega_1 + \omega_2)^2 & \textit{otherwise} \end{cases} \tag{16}$$

*is PD. In fact, the constituents are distinct by the second part of Definition 1.*
*3) Let again $\Sigma$ be the frontier of the unitary rectangle in $\mathbb{R}^2$. The function $f : \mathbb{R}^2 \to \mathbb{R}$*

$$f = \begin{cases} \sin(\omega_1 + \omega_2) & \omega \in (0,1)^2 \cup \Sigma \\ \sin(\omega_1 + \omega_2) & \textit{otherwise} \end{cases} \tag{17}$$

*is not PD in the sense of this work. In fact $E_i \cap E_j = \Omega_{i-j} \cup \Omega_{j-i} = \mathbb{R}^2$, and the set of points where $f_1$ equals $f_2$ is not a set of null measure.*

Definition 2 allows to formalize the continuity and differentiability of PD functions. We first recall the definition of contact between two functions.

**Definition 3** *Two functions $f_i$,$f_j \in C^m(E)$ have a contact of order $k$ ($k = 0, 1, 2, ..., m$) at $\omega_0 \in E$, if $\dfrac{\partial^s f_i(\omega_0)}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_s}} = \dfrac{\partial^s f_j(\omega_0)}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_s}}$ for all $s = 0, 1, 2, ..., k$.*

The following result holds.

**Proposition 1** *Let $f \in C^l(\Omega)$. Then, the constituents $f_i$ are at least $C^l(\Omega_i)$, for all $i = 1, 2, ..., N$. Conversely, let $f_i \in C^{k_i}(\Omega_i)$. Then, $f \in C^{k_{\min}}(\Omega_i)$ where $k_{\min}$ is the minimum between $k_i$ and the order of the contacts between any $f_i$ and $f_j$ ($i \neq j$).*

**Proof.** We prove the first assertion by contradiction. Consider any $\omega \in \Omega_i$, and allow $f_i \in C^{k_i}(\omega)$ with $k_i < l$. This implies that $f_i \notin C^l(\Omega_i)$. Now, $f_i \notin C^l(\Omega_i) \implies f \notin C^l(\Omega_i) \implies f \notin C^l(\Omega)$, which contradicts the hypothesis.
Second assertion. At any interior point $\omega \in \Omega_i$, $f$ is as regular as the corresponding constituent $f_i$. Thus, on the whole $\Omega$, $f$ is at most as regular as the least regular of the constituents: $f \in C^{s_i}(\omega)$, with $s_i = \min_i(k_i)$. Consider now $\omega \in \Sigma$. Suppose that, at $\omega \in \Sigma$, $\Omega_i$ and $\Omega_j$ (two sets for simplicity) are neighboring ($\partial\Omega_i \cap \partial\Omega_j \neq \varnothing$) and $f_\Sigma(\omega)$ is such that $f_i$ and $f_j$ have a contact of order $l$ at $\omega$. If $l > \min_i k_i$, then $f \in C^{z_i}(\omega)$, with $z_i = \min_{i,j} k_i, k_j$. If this happens for all $\omega \in \Sigma$, then $f \in C^{s_i}(\omega)$, still. Conversely, if $l < \min_i k_i$ at one point $\omega \in \Sigma$, then $f \notin C^{s_i}(\omega)$, and belongs to $f \in C^l(\Omega)$. ∎

Proposition 1 has the following interpretation. Suppose $f \in C^0(\Omega)$. Then, all the constituents are at least continuous. In addition, in order for $f \in C^0(\Omega)$ to hold, they must have a contact of order at least 0 on the internal boundary. Conversely, it is not guaranteed that $f \in C^0(\Omega_i)$, even if each of the $f_i \in C^\infty(\Omega_i)$ $\forall i = 1, 2, ..., N$. This happens if at just one point of $\Sigma$ one of the contacts between any two constituents is not of order 0. A final observation. Consider $\Omega \subset \mathbb{R}^n$, bounded. Then, $f_i \in C^r(\Omega_i)$ ($r = 0$ or more), guarantees $f \in \mathcal{Z}$.

A PD function can be either directly assigned through eq. (14), or the result of operations performed on other functions, as the next Examples show.

**Example 5** *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f = \begin{cases} 1 - \omega_1^2 - \omega_2^2 & \omega_1^2 + \omega_2^2 \leq 1 \\ 0 & \omega \in \Sigma \\ \omega_1^2 + \omega_2^2 - 1 & \omega_1^2 + \omega_2^2 > 1 \end{cases} \tag{18}$$

*where* $\Sigma = \{\omega : \omega_1^2 + \omega_2^2 = 1\}$. *In terms of Definition 2, is PD as the second part of Definition 1 is satisfied.*

**Example 6** *Consider now the function:*

$$g = |h| \tag{19}$$

*where* $h(\omega) = 1 - \omega_1^2 - \omega_2^2$.
*One has to study the sign of* $h$*, that leads to:* $h < 0$*, if* $\omega_1^2 + \omega_2^2 > 1$*,* $h = 0$*, if* $\omega_1^2 + \omega_2^2 = 1$*, and* $h > 0$*, if* $\omega_1^2 + \omega_2^2 < 1$.
*Thus, using the symbols of Definition (2), we have* $\Omega_1 = \{\omega : \omega_1^2 + \omega_2^2 < 1\}$, $\Sigma = \{\omega : \omega_1^2 + \omega_2^2 = 1\}$ *and* $\Omega_2 = \{\omega : \omega_1^2 + \omega_2^2 > 1\}$.
*The constituents of* $g$ *on* $\Omega_1$ *and* $\Omega_2$ *are* $g_1 = 1 - \omega_1^2 - \omega_2^2$ *and* $g_2 = \omega_1^2 + \omega_2^2 - 1$. *Thus, one can write:*

$$g = \begin{cases} 1 - \omega_1^2 - \omega_2^2 & \omega \in \Omega_1 \\ 0 & \omega \in \Sigma \\ \omega_1^2 + \omega_2^2 - 1 & \omega \in \Omega_2 \end{cases} \tag{20}$$

*Note that the function in eq. (18) is equivalent to the function in eq. (20), thus* $g = f$.

Examples 5 and 6 reveal two ways of assigning PD functions. A first way [Example 5, eq. (18)] consists of the direct assignment of both $\Omega_i$ and $f_i$. The subdomains $\Omega_i$ are assigned independently of the constituents. In Example 6 [eq. (19)], the PD function is the result of some operation performed on another function defined on $\Omega$. In this case, the functional form of the constituents and the partition of $\Omega$ are linked. As we are going to see, from an operational viewpoint, it is convenient to exhibit this difference. We therefore group PD functions into two types.

**Definition 4** *Type I: a PD function is directly assigned as eq. (14).*
*Type II: A PD function is the result of operations performed on other functions which link the partition of $\Omega$ to its assignment.*

In the next Section, results for the moments of PD functions are presented.

## 4 Moment Calculations for Piecewise-Defined Functions

This Section discusses the computation of the moments of PD functions.

Consider a decision-maker who is not certain about $\omega$, and let $(\Omega, \mathcal{B}(\Omega), \mu)$ denote the corresponding subjective probability space. Given a partition $\Pi_\Omega$ (defined in Section 3), we require $\mu(\Omega_i) > 0 \ \forall i$, and $\mu(\Omega) = \sum_{i=1}^{N} \mu(\Omega_i) = 1$ for non-triviality.

Before coming to the calculations of the moments of a PD function, we state a preliminary observation on the measurability of $f$ and of its constituents.

**Remark 1** *Let the image of $f$ be strictly included in $\mathbb{R}$ and $\Omega$ bounded. If $f \in \mathcal{L}^p(\Omega)$, then $f_i \in \mathcal{L}^{p_i}(\Omega_i)$, $i = 1, 2, ..., N$, where $p_i \geq p$. Viceversa, if $f_i \in \mathcal{L}^{p_i}(\Omega_i)$ then $f \in \mathcal{L}^{p_{min}}(\Omega)$, where $p_{min} = \min_i p_i$, $i = 1, 2, ..., N$.*

As discussed in Section 3, Remark 1 holds and $f \in \mathcal{L}^p(\Omega)$ by assuming $f_i \in C^r(\Omega_i)$, $r = 0$ or more.

Let $f \in \mathcal{L}^p(\Omega)$. We denote the $p^{th}$ moment of $f$ by (see Table 1 for notation):

$$\mathbb{E}^p[f] = \int_\Omega f^p d\mu \tag{21}$$

and the $p^{th}$ central moment of $f$ by

$$\mathbb{M}_f^p = \mathbb{E}^p[(f - \mathbb{E}[f])] = \int_\Omega (f - \mathbb{E}[f])^p d\mu$$

The following result relates the moment of order $p$ of a PD function to the integration of its constituents.

**Proposition 2** *Let $f \in \mathcal{L}^p(\Omega)$ be a PD function. Then*

$$\mathbb{E}^p[f] = \sum_{i=1}^{N} \int_{\Omega_i} f_i^p d\mu \tag{22}$$

**Proof.** By Remark 1, if $f \in \mathcal{L}^p(\Omega)$ and then all $f_i \in \mathcal{L}^p(\Omega_i)$. The right hand side then follows from the additivity property of integrals and from eq. (14). ■

Proposition 2 [see eq. (14)] states that the moment of order $p$ of a PD function is the sum of the integrals on $\Omega_i$ of the constituents $f_i$ elevated to the $p$. Observe that $\int_{\Omega_i} f_i^p$ is not the $p^{th}$-moment of $f_i$. Let us, however, extend the constituents to the whole $\Omega$, as follows.

**Definition 5 (Extended Constituent)** *Let $\varphi_i : \Omega \to \mathbb{R}$ with*

$$\varphi_i(\omega) = \begin{cases} f_i(\omega) & \omega \in \Omega_i \\ 0 & otherwise \end{cases} \tag{23}$$

*$\varphi_i$ is called an extended constituent of $f$.*

We then have the following result for the $p^{th}$ moment of $f$.

**Lemma 1**

$$\mathbb{E}^p[f] = \sum_{i=1}^{N} \mathbb{E}^p[\varphi_i] \tag{24}$$

**Proof.** Given Definition 5 and Proposition 2, we have:

$$\mathbb{E}^p[f] = \int_{\Omega} f^p d\mu = \sum_{i=1}^{N} \int_{\Omega_i} f_i^p d\mu = \sum_{i=1}^{N} \int_{\Omega} \varphi_i^p d\mu = \sum_{i=1}^{N} \mathbb{E}^p[\varphi_i] \tag{25}$$

■

**Remark 2** *Proposition 2 and Lemma 1 exploit the generalized additivity property of integrals over the integration range. In this respect, the role played by the disjointedness assumption proper of the partition of $\Omega$ is crucial.*

Lemma 1 [see eq. (24)] states that any moment of a PD function is the sum of the moments of its extended constituents. As we are going to see shortly, extended constituents play also a relevant role in the central moments of $f$.

**Theorem 1** *Let $f$ be a PD function (Definition 2). Let $f \in \mathcal{L}^p(\Omega)$. Then, the central moments of $f$ are given by:*

$$\mathbb{M}_f^p = \sum_{i=1}^{N} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} (\mathbb{E}[f])^{p-q} \mathbb{E}^q[\varphi_i] \tag{26}$$

**Proof.** Given Definitions 2 and 5, one has:

$$\sum_{i=1}^{N}\int_{\Omega_i}(f_i-\mathbb{E}[f])^p d\mu=\sum_{i=1}^{N}\int_{\Omega_i}\sum_{q=0}^{p}\binom{p}{q}f_i^q(-\mathbb{E}[f])^{p-q}\mathrm{d}\mu=\sum_{i=1}^{N}\sum_{q=0}^{p}\binom{p}{q}\int_{\Omega_i}f_i^q(-\mathbb{E}[f])^{p-q}\mathrm{d}\mu \tag{27}$$

$$=\sum_{i=1}^{N}\sum_{q=0}^{p}\binom{p}{q}(-\mathbb{E}[f])^{p-q}\int_{\Omega_i}f_i^q\mathrm{d}\mu \tag{28}$$

From Lemma 2, one has $\mathbb{E}^q[\varphi_i]=\displaystyle\int_{\Omega_i}f_i^q\mathrm{d}\mu$. Then,

$$=\sum_{i=1}^{N}\sum_{q=0}^{p}\binom{p}{q}(-\mathbb{E}[f])^{p-q}\mathbb{E}^q[\varphi_i] \tag{29}$$

q.e.d. ■

Eq. (26) states that the central moment of order $p$ of a PD function involves the linear combination of the moments of all orders up to $p$ of its extended constituents.

In the next Section, we utilize the findings of this Section and Section 3 to derive the analytic and stochastic properties of $\max(\cdot)$ and $|\cdot|$ PD functions.

## 5 The max and absolute value PD Functions

The literature review of Section 2 shows that PD functions induced by $\max(\cdot)$ and $|\cdot|$ operations appear frequently in O.R. applications. $\max(\cdot)$ and $|\cdot|$ functions are PD functions of Type II. Thus, the functional forms of the constituents is linked to the partition of the domain. Section 5.1 derives results for the $\max(\cdot)$ function, Section 5.2 for the $|\cdot|$ function.

### 5.1 Multivariate $\max(\cdot)$ PD functions

In this Section, we analyze the analytic and stochastic properties of PD functions that result as application of the $\max(\cdot)$ operation. For clarity of presentation and following the results of Sections 3 and 4, we present the analysis in three phases:

1. Function assignment (Determination the partition of $\Omega$, $\{\Omega_1,\Omega_2,...,\Omega_N\}$, identification of the boundaries; determination of the constituents of $f$);

2. Analysis of continuity and differentiability properties;

3. Moment Calculation.

Let

$$f = \max(t, h) : \Omega \subseteq \mathbb{R}^n \to \mathbb{R} \tag{30}$$

with $t, h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$.

Phase 1. The partition of $\Omega$ is determined by the solution of the inequality:

$$t(\omega) > h(\omega) \tag{31}$$

Then, one obtains the partition:

$$\Pi_\Omega = \{\Omega_t, \Omega_h\} \tag{32}$$

with

$$\Omega_t = \{\omega : t(\omega) > h(\omega)\} \text{ and } \Omega_h = \{\omega : h(\omega) < t(\omega)\} \tag{33}$$

The internal boundary is defined by

$$\Sigma = \{\omega : t(\omega) - h(\omega) = 0\} \tag{34}$$

The properties of $\Sigma$ are determined by a classical Theorem of differential geometry (see Spivak (2005), Vol I).

**Theorem 2** *Consider $g = t(\omega) - h(\omega) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $\Omega$ open. Assume that $g \in C^r(E)$, $r \geq 1$ and $0 \in \mathbb{R}$ is a regular point of $g$. Then, $\Sigma$ is either empty or a $n-1-$dimensional $C^r$ manifold.*

Theorem 2 states that if $t, h \in C^r(\Omega)$, and a solution of eq. (22) exists, then $\Sigma$ is a $C^r$, $n-1$ dimensional manifold. In other words, if $t$ and $h$ are smooth, the internal boundary is an hypersurface of dimensions $n-1$ that shares the same regularity properties as $t$ and $h$. Theorem 2 is also readily extended to the case in which $t \in C^{r_t}(\Omega)$ and $h \in C^{r_h}(\Omega)$. Then, $g = t - h \in C^{\min(r_t, r_h)}(\Omega)$. In other words, $r$ in Theorem 2 is, in general, $\min(r_t, r_h)$.

Note that, since $t(\omega) = h(\omega)$ for $\omega \in \Sigma$, $f_\Sigma(\omega)$ can be equivalently assigned equal to $t$ or to $h$ at $\omega \in \Sigma$. Then, we have the resulting PD function:

$$f(\omega) = \max(t, h) = \begin{cases} t(\omega) & \omega \in \Omega_t \\ t(\omega) & \omega \in \Sigma \\ h(\omega) & \omega \in \Omega_h \end{cases} \tag{35}$$

where the partition of $\Omega$ is in eq. (33). The extended constituents are:

$$\varphi_t = \begin{cases} t(\omega) & \omega \in \Omega_t \cup \Sigma \\ 0 & otherwise \end{cases} \text{ and } \varphi_h = \begin{cases} h(\omega) & \omega \in \Omega_h \\ 0 & otherwise \end{cases} \tag{36}$$

Alternatively, by assigning $f_\Sigma(\omega) = h(\omega)$, one would obtain the extended constituents:

$$\varphi_t = \begin{cases} t(\omega) & \omega \in \Omega_t \\ 0 & otherwise \end{cases} \text{ and } \varphi_h = \begin{cases} h(\omega) & \omega \in \Omega_h \cup \Sigma \\ 0 & otherwise \end{cases} \tag{37}$$

Phase 2. Concerning the continuity and differentiability properties of $f = \max(t, h)$, the following Remark is a consequence of Proposition 1.

**Remark 3** *Let $t, h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, with $t$ and $h$ belonging to $C^{r_t}(\Omega)$ and $C^{r_h}(\Omega)$ respectively. Let $t$ and $h$ have a contact of order $k$ at $\omega \in \Sigma$. Then, by Proposition 1, $f \in C^{r_{\max}}(\Omega)$, where $r_{\max} = \min(r_t, r_h, k)$.*

Remark 3 implies that the degree of smoothness of $f = \max(g, h)$ is determined not only by the regularity of $t$ and $h$, but also by the order of their contact.

Phase 3. The following results hold for the moments of $f = \max(t, h)$.

**Lemma 2** *Let $f = \max(t, h)$, $t, h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, with $t, h \in \mathcal{L}^p(\Omega)$. Then, we have:*

$$\mathbb{E}^p_{\max(t,h)} = \mathbb{E}^p[\varphi_t] + \mathbb{E}^p[\varphi_h] \tag{38}$$

*with $p \geq 1$, and*

$$\mathbb{M}^p_{\max(t,h)} = \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} (\mathbb{E}[\varphi_t] + \mathbb{E}[\varphi_h])^{p-q} (\mathbb{E}^q[\varphi_t] + \mathbb{E}^q[\varphi_h]) \tag{39}$$

*with $p > 2$.*

**Proof.** Eq. (38) is obtained by combining eq. (36) [or eq. (37)] with Lemma 1. Eq. (39): note that from eq. (35), in eq. (24), it is $N = 2$.

$$\begin{aligned} \mathbb{M}^p_f &= \textstyle\sum_{i=1}^2 \sum_{q=0}^p \binom{p}{q} (-1)^{p-q} (\mathbb{E}[\max(t,h)])^{p-q} \mathbb{E}^q[\varphi_i] = \\ &= \textstyle\sum_{q=0}^p \binom{p}{q} (-1)^{p-q} (\mathbb{E}[\max(t,h)])^{p-q} \mathbb{E}^q[\varphi_t] + \sum_{q=0}^p \binom{p}{q} (\mathbb{E}[\max(t,h)])^{p-q} \mathbb{E}^q[\varphi_h] = \\ &= \textstyle\sum_{q=0}^p \binom{p}{q} (-1)^{p-q} (\mathbb{E}[\varphi_t] + \mathbb{E}[\varphi_h])^{p-q} (\mathbb{E}^q[\varphi_t] + \mathbb{E}^q[\varphi_h]) \end{aligned} \tag{40}$$

q.e.d. ■

In eq. (39), the summands contain the products $(\mathbb{E}[\varphi_t]+\mathbb{E}[\varphi_h])^{p-q}$ times $(\mathbb{E}^q[\varphi_t]+\mathbb{E}^q[\varphi_h])$, in which the moments of $\varphi_t$ and $\varphi_h$ are multiplied by the powers of $\mathbb{E}[\varphi_t]$ and $\mathbb{E}[\varphi_h]$.

Eq. (39) is generalized in Appendix A to the case of $f = \max(g_1, g_2, ..., g_n)$.

In the next Section, results for $|\cdot|$ PD functions are derived.

## 5.2 Multivariate $|\cdot|$ PD functions

Consider the function $f = |g|$, with $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$.

Phase 1. The partition of $\Omega$ is determined by studying the inequality:

$$g(\omega) > 0 \tag{41}$$

One obtains:

$$\Pi_\Omega = \{\Omega_-, \Omega_+\} \tag{42}$$

with

$$\Omega_- = \{\omega : g(\omega) < 0\} \text{ and } \Omega_+ = \{\omega : g(\omega) > 0\} \tag{43}$$

The internal boundary is

$$\Sigma = \{\omega : g(\omega) = 0\} \tag{44}$$

The properties of $\Sigma$ are determined by Theorem 2, which, in this case, applies directly to $g$.

Recalling eq. (43), the expression of $f$ is equal to:

$$f(\omega) = \begin{cases} g(\omega) & \omega \in \Omega_+ \\ 0 & \omega \in \Sigma \\ -g(\omega) & \omega \in \Omega_- \end{cases} \tag{45}$$

The extended constituents are:

$$\varphi_+ = \begin{cases} g(\omega) & \omega \in \Omega_+ \\ 0 & otherwise \end{cases} \text{ and } \varphi_- = \begin{cases} -g(\omega) & \omega \in \Omega_- \\ 0 & otherwise \end{cases} \tag{46}$$

Recalling that $|g| = \max(g, -g)$, eq. (46) can be seen as a special case of eq. (37), with $t = g$ and $h = -g$.

Phase 2. Concerning on the continuity and differentiability properties of $f = |g|$, the following holds.

**Proposition 3** *Let $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ belong to $C^r(\Omega)$. If $r = 0$, then $g \in C^0(\Omega)$. If $r > 0$, let $\varphi_+$ and $\varphi_-$ have a contact of order $k$ at $\omega \in \Sigma$. Then, $|g| \in C^k(\Omega)$.*

**Proof.** For $r = 0$, the result is immediate. For $r > 0$, let $u = \min(r, k)$. By theorem 1, then, $g \in C^u(\Omega)$. Now, since $g \in C^r(\Omega)$, derivatives of orders higher than $r$ cannot be defined. Therefore, $\varphi_+$ and $\varphi_-$ cannot have a contact of order greater than $r$. Hence, $k \leq r$, from which $u = k$. ■

**Remark 4** *In order $\varphi_+$ and $\varphi_-$ to have a contact of order $k$ at $\omega \in \Sigma$, all the derivatives of $g$ up to order $k$ must be null at $\omega \in \Sigma$. Conversely, if the derivatives of $g$ are null at $\omega \in \Sigma$ up to order $k$, then $\varphi_+$ and $\varphi_-$ have a contact of order $k$. Thus, the value of $k$ in Proposition 3 is determined by the order of null derivatives of $g$ on $\Sigma$.*

Phase 3. Applying the results of Sections 4 and 5.1, we have the following statement concerning the moments of $f = |g|$.

**Lemma 3**

$$\mathbb{E}^p[|g|] = \mathbb{E}^p[\varphi_+] + \mathbb{E}^p[\varphi_-] \tag{47}$$

*and*

$$\mathbb{M}^p_{|g|} = \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} (\mathbb{E}[\varphi_+] + \mathbb{E}[\varphi_-])^{p-q} (\mathbb{E}^q[\varphi_+] + \mathbb{E}^q[\varphi_-]) \tag{48}$$

**Proof.** Recalling that $|g| = \max(g, -g)$, then eq. (48) follows from eq. (39). ■

We note that the summands in eq. (48) contain the product of $(\mathbb{E}[\varphi_+] + \mathbb{E}[\varphi_-])^{p-q} \cdot (\mathbb{E}^q[\varphi_+] + \mathbb{E}^q[\varphi_-])$, in which the moments of $\varphi_+$ and $\varphi_-$ are multiplied by the powers of $\mathbb{E}[\varphi_+]$ and $\mathbb{E}[\varphi_-]$. This is equivalent to replace $\varphi_t$ and $\varphi_h$ in eq. (39) with $\varphi_+$ and $\varphi_-$, respectively.

In the next Section, we apply the findings obtained sofar to the study of mean-deviation, mean-upper-semideviation and CVAR optimization problems.

# 6 Stochastic Optimization with Coherent Risk Measures

In this Section, we show how the results of the previous Sections can be implemented to derive decision-making insights at the formulation stage of risk coherent SP problems.

## 6.1 Mean-Deviation SP Problems

A mean-deviation decision-maker solves the SP problem of eq. (6) (Section 2). Eq. (6) involves the moments of the PD function $|Z - \mathbb{E}[Z]|$. By the findings of Section 5.2, we have the following result on mean-deviation PD problems.

**Theorem 3** *Consider a p-mean-deviation decision-maker. Then, if $Z(x, \omega) \in \mathcal{L}^p(\Omega)$, the optimal decision-maker's choice $(x)$ solves the SP problem characterized by the objective function*[7]

$$\rho_{md} = \mathbb{E}[Z] + c[\sum_{q=0}^{p} \binom{p}{q} (-\mathbb{E}[Z])^{p-q} (\mathbb{E}^p[Z_+] + \mathbb{E}^p[Z_-])]^{1/p} \tag{49}$$

[7] In eq. (51), the dependence on $x$ is hidden for notational simplicity.

*where*

$$Z_+ := \begin{cases} Z(x,\omega) & \omega \in \Omega_+ \cup \Sigma_{md} \\ 0 & otherwise \end{cases} \text{ and } Z_- := \begin{cases} -Z(x,\omega) & \omega \in \Omega_- \\ 0 & otherwise \end{cases} \tag{50}$$

*and*

$$\begin{cases} \Omega_+ = \{\omega : Z(x,\omega) > \mathbb{E}[Z(x,\omega)]]\} \\ \Sigma_{md} = \{\omega : Z(x,\omega) = \mathbb{E}[Z(x,\omega)]]\} \\ \Omega_- = \{\omega : Z(x,\omega) < \mathbb{E}[Z(x,\omega)]]\} \end{cases} \tag{51}$$

**Proof.** In eq. (6), for any given $(x)$, let $w(\omega) : \Omega \to \mathbb{R}$ such that $w(\omega) = Z(x,\omega) - \mathbb{E}[Z(x,\omega)]$. Then, eq. (6) is written as:

$$\rho_{\mathrm{md}} = \mathbb{E}[Z] + c(\mathbb{E}[|w|^p])^{1/p} \tag{52}$$

Then, applying Lemma 3, one gets:

$$\rho_{\mathrm{md}} = \mathbb{E}[Z] + c(\mathbb{E}^p[\varphi_+] + \mathbb{E}^p[\varphi_-])^{1/p} \tag{53}$$

where the functions $\varphi_+$ and $\varphi_-$ in eq. (53) are the extended constituents of the PD function:

$$f = \begin{cases} Z(x,\omega) - \mathbb{E}[Z(x,\omega)] & \omega \in \Omega_+ \\ 0 & \omega \in \Sigma_{md} \\ -Z(x,\omega) - \mathbb{E}[Z(x,\omega)] & \omega \in \Omega_- \end{cases} \tag{54}$$

i.e.,

$$\varphi_+ := \begin{cases} Z(x,\omega) - \mathbb{E}[Z(x,\omega] & \omega \in \Omega_+ \cup \Sigma_{md} \\ 0 & otherwise \end{cases} \tag{55}$$

and

$$\varphi_- := \begin{cases} -Z(x,\omega) - \mathbb{E}[Z(x,\omega)] & \omega \in \Omega_- \\ 0 & otherwise \end{cases} \tag{56}$$

At this stage, introduce the functions $Z_+$ and $Z_-$ as defined in eq. (50). Then:

$$\begin{aligned} \mathbb{E}^p[\varphi_+] = \int\limits_{\Omega_+} (Z - \mathbb{E}[Z])^p \,\mathrm{d}\mu = \int\limits_{\Omega_+} \left( \sum_{q=0}^{p} \binom{p}{q} Z^q (-\mathbb{E}[Z])^{p-q} \right) \mathrm{d}\mu = \\ = \sum_{q=0}^{p} \binom{p}{q} (-\mathbb{E}[Z])^{p-q} \int\limits_{\Omega_+} Z(x,\omega)^q \mathrm{d}\mu \end{aligned} \tag{57}$$

Since $\int\limits_{\Omega_+} Z^q d\mu = \int\limits_{\Omega} Z_+^q d\mu = \mathbb{E}^q[Z_+]$, one gets:

$$\mathbb{E}^p[\varphi_+^p] = \sum_{q=0}^{p} \binom{p}{q} (-\mathbb{E}[Z])^{p-q} \mathbb{E}^q[Z_+] \tag{58}$$

A similar argument leads to $\mathbb{E}^p[\varphi_-] = \sum_{q=0}^{p} \binom{p}{q} (-\mathbb{E}[Z])^{p-q} \mathbb{E}^q[Z_-]$, which ends the proof. ■

Concerning eqs. (49) - (51), we note that:

1. In eq. (51), given $x$, $\Sigma_{\mathrm{md}}$ is the set of points where the expected loss equals the actual loss incurred by the decision-maker;

2. $\Omega_+$ and $\Omega_-$ are the subdomains where $Z$ is greater (respectively) lower than the expected loss. Correspondingly, the functions $Z_+$ and $Z_-$ [eq. (50)] are greater (respectively) lower than the expected loss at the points of $\Omega$ where they are different from 0. We refer to $Z_+$ as to the "excess loss function" and to $Z_-$ as to the "defect loss function".

Let us now deal with the use of Theorem 3 in problem formulation. Theorem 3 [eq. (49)] states that the optimal choice of a decision-maker using a mean-deviation objective function of order $p$ is determined by the moments of order 1 to $p$ of the restrictions of the loss functions on $\Omega_+$ and $\Omega_-$. Thus, the stochastic properties of both the excess and defect loss functions are of interest to the decision-maker. The next Corollary enables us to identify the families of distributions that leave the decision-maker indifferent to imprecision.

**Corollary 1** *The solution of a mean-deviation SP problem of order p is invariant for families of distributions leading to the same values of* $\mathbb{E}[Z]$, $\mathbb{E}^s[Z_+]$, $\mathbb{E}^s[Z_-]$ *with* $s = 1, 2, .., p$.

The special case of $p = 1$ deserves further attention. For $p = 1$, in fact, one obtains the mean-absolute-deviation (MAD) risk function [Ruszczynski and Shapiro (2005)], which has been widely applied in portfolio optimization and inventory management [Siegmann and Lucas (2005), Ahmed et al (2007)]. Corollary 1 implies that risk averse decision makers utilizing MAD are insensitive to imprecision for families of distributions leading to the same $\mathbb{E}[Z]$, $\mathbb{E}[Z_+]$, $\mathbb{E}[Z_-]$. This result generalizes previous literature findings (see Borgonovo and Peccati (2008)). In Borgonovo and Peccati (2008), sensitivity to imprecision in the demand distribution for the multi-item inventory problems of eqs. (9), (10) and (11) is discussed. A case study is presented involving two families of distributions, multivariate Beta and

Gamma, respectively. The parameters of the distributions are such that they produce the same values of $\mathbb{E}[Z]$, $\mathbb{E}[Z_+]$, $\mathbb{E}[Z_-]$. It is shown numerically that the optimal choice of the MAD risk coherent decision-maker is the same independently of whether she/he selects the Beta or the Gamma distribution to represent her/his state of knowledge on demand. Note that, in this case, the decision-maker is insensitive to imprecision even if a change in the support of the random variable (from finite with Beta, to infinite with Gamma) is registered.

Consider now the case $p = 2$. Then, two distributions lead to the same optimal policy if they imply the same $\mathbb{E}[Z]$, $\mathbb{E}[Z_+]$, $\mathbb{E}[Z_-]$, $\mathbb{E}^2[Z_+]$, and $\mathbb{E}^2[Z_-]$. Note that the set of distributions that lead to the same $\mathbb{E}[Z]$, $\mathbb{E}[Z_+]$, $\mathbb{E}[Z_-]$ strictly includes the set of distributions which lead also to the same $\mathbb{E}^2[Z_+]$, $\mathbb{E}^2[Z_-]$. The same happens for any increase in $p$. Corollary 1 implies that sensitivity to imprecision increases with the order of the deviation measure ($p$).

In the next Section, we consider the optimization problem of a decision-maker utilizing the same loss function but a mean-upper-semideviation risk measure of order $p$.

## 6.2 Mean-upper-semideviation SP Problems

Mean-upper-semideviation SP problems are characterized by the objective function in eq. (7) (Section 2). Eq. (7) involves the PD function $\max(0, Z - \mathbb{E}[Z])$. By Section 5.1, one obtains the following result.

**Theorem 4** *Consider a $p$ mean-upper-semideviation decision-maker. Then, if $Z(x, \omega) \in \mathcal{L}^p(\Omega)$, the optimal decision-maker's choice $(x)$ solves the SP problem with objective function*

$$\rho_{musd} = \mathbb{E}[Z] + c\left[\sum_{q=0}^{p} \binom{p}{q} (-\mathbb{E}[Z])^{p-q} \mathbb{E}^q[Z_+]\right]^{1/p} \tag{59}$$

*where $Z_+(x, \omega)$, $\Omega_+$ and $\Sigma_{musd}$ are the same as for a mean-deviation decision-maker (Theorem 3).*

**Proof.** In eq. (7), for any given $x$, let $h(\omega) : \Omega \to \mathbb{R}$ such that $h = Z - \mathbb{E}[Z]$. Then, eq. (7) is written as:

$$\rho_{\text{musd}} = \mathbb{E}[Z] + c(\mathbb{E}\left[\max(0, h)^p\right])^{1/p} \tag{60}$$

Applying Lemma 2, one gets

$$\mathbb{E}^p[\max(0, h)] = \mathbb{E}^p[\varphi_h] \tag{61}$$

where

$$\varphi_h := \begin{cases} Z(x, \omega) - \mathbb{E}[Z(x, \omega] & \omega \in \Omega_h \cup \Sigma_{\text{musd}} \\ 0 & \textit{otherwise} \end{cases} \tag{62}$$

and

$$\begin{aligned}\Omega_h &= \{\omega : Z(x,\omega) > \mathbb{E}[Z(x,\omega)]]\} \\ \Sigma_{\text{musd}} &= \{\omega : Z(x,\omega) = \mathbb{E}[Z(x,\omega)]]\}\end{aligned} \tag{63}$$

By comparing eqs. (62), (63) to eqs. (51) and (55), one obtains that $\Omega_h = \Omega_+$, $\Sigma_{\text{musd}} = \Sigma_{\text{md}}$ and $\varphi_h = \varphi_+$. Hence, eq. (7) becomes:

$$\rho_{\text{musd}} = \mathbb{E}[Z] + c(\mathbb{E}^p[\varphi_+])^{1/p} \tag{64}$$

Let $Z_+$ as in eq. (50). Then, $\mathbb{E}^p[\varphi_+]$ is given by eq. (58). The proof is completed by inserting eq. (58) in eq. (61), and eq. (61) in eq. (60). ■

Theorem 4 [eq. (49)] states that the optimal choice of decision-maker using a mean-upper-semideviation of order $p$ is determined by the moments of the excess loss function and by the expected loss. In terms of sensitivity to imprecision, this result implies the following.

**Corollary 2** *The solution of a mean-upper-semideviation SP of order p is invariant for families of distributions leading to the same values of* $\mathbb{E}[Z]$, $\mathbb{E}^q[Z_+]$ *with* $q = 1, 2, .., p$.

By comparing Theorems 4 and 3, one notes that a mean-deviation SP problem involves the moments of the defect loss function in addition to the moments of the excess loss function (eqs. (49) and (59) differ for the presence of the terms containing $\mathbb{E}^q[Z_-]$). This implies that, if two (or more) distributions satisfy Corollary 1, then they simultaneously satisfy Corollary 2. In other words,

**Corollary 3** *All distribution families that leave optimal mean-deviation policies unaltered, also leave the corresponding mean-upper-semideviation policies unaltered.*

We note that the converse is not true.

## 6.3 CVAR SP Problems

In this Section, we discuss CVAR optimization problems. The objective function is reported in Example 1, eq. (4). It entails the computation of $\mathbb{E}[\max(0, Z)]$. Therefore the findings of Section 5.1 apply. We obtain the following result.

**Theorem 5** *Consider a CVAR decision-maker. Then, if* $Z(x,\omega) \in \mathcal{L}^p(\Omega)$, *the optimal decision-maker's choice* $(x)$ *solves the stochastic program determined by the objective function*

$$H = \zeta\left(1 - \frac{\mu(\Omega_\zeta)}{1-\alpha}\right) + \frac{\mathbb{E}[Z_\zeta]}{1-\alpha} \tag{65}$$

*where*

$$Z_\zeta(x,\omega) = \begin{cases} Z(x,\omega) & \omega \in \Omega_\zeta (\cup \Sigma_{CVAR}) \\ 0 & otherwise \end{cases} \tag{66}$$

*and*

$$\begin{cases} \Omega_\zeta = \{\omega : Z(x,\omega) > \zeta\} \\ \Sigma_{CVAR} = \{\omega : Z(x,\omega) = \zeta\} \\ \Omega_0 = \{\omega : Z(x,\omega) < \zeta\} \end{cases} \tag{67}$$

**Proof.** In eq. (4), for any given $x$, let $h(\omega) : \Omega \to \mathbb{R}$ such that $h = Z - \zeta$. Then, eq. (4) is written as:

$$H = \zeta + \frac{1}{1-\alpha}\mathbb{E}[\max(0,h)] \tag{68}$$

Applying Lemma 2, one gets

$$\mathbb{E}[\max(0,h)] = \mathbb{E}[\varphi_h] \tag{69}$$

with

$$\varphi_h(x,\omega) = \begin{cases} Z(x,\omega) - \zeta \ \text{ if } & \omega \in \Omega_\zeta \\ 0 & otherwise \end{cases} \tag{70}$$

Hence, eq. (7) becomes:

$$H = \zeta + \frac{1}{1-\alpha}\mathbb{E}[\varphi_h] \tag{71}$$

Now, observe that

$$\mathbb{E}[\varphi_h] = \int_{\Omega_\zeta} [Z(x,\omega) - \zeta]\,\mathrm{d}\mu = \int_{\Omega_\zeta} Z(x,\omega)\mathrm{d}\mu - \int_{\Omega_\zeta} \zeta\mathrm{d}\mu \tag{72}$$

Let $Z_\zeta$ as in eq. (66). Then, $\int_{\Omega_\zeta} Z(x,\omega)\mathrm{d}\mu = \int_\Omega Z_\zeta(x,\omega)\mathrm{d}\mu = \mathbb{E}[Z_\zeta]$. Hence:

$$\mathbb{E}[\varphi_h] = \mathbb{E}[Z_\zeta] - \zeta\int_{\Omega_\zeta}\mathrm{d}\mu = \mathbb{E}[Z_\zeta] - \zeta\mu(\Omega_\zeta) \tag{73}$$

Substituting into eq. (68), one gets eq. (65). ■

In eq. (65), $\mu(\Omega_\zeta)$ is the measure of $\Omega_\zeta$. As eq. 67 states, $\Omega_\zeta$ is the subset of $\Omega$ in which $Z$ exceeds $\zeta$. Thus, $\mu(\Omega_\zeta)$ is the probability that the loss exceeds $\zeta$. At the optimum, i.e., after solving the optimization problem, one has $\zeta^* = VAR^*_\alpha$. Then, $\mu(\Omega_{\zeta^*}) = 1 - \alpha$, and $H_\alpha{}^* = \dfrac{\mathbb{E}[Z_{\zeta^*}]}{1-\alpha} = CVAR_\alpha$. Note that $\mathbb{E}[Z_{\zeta^*}]$ is the expectation of $Z_{\zeta^*}$ under the unconditional distribution. The term $\dfrac{1}{1-\alpha}$ is the correction factor necessary to include the conditional expectation (= change in distribution) implied by the definition of CVAR[8].

As far as the sensitivity to imprecision in the distribution is concerned, Theorem 5 [eq. (65)] allows one to establish the following result.

**Corollary 4** *The solution of a CVAR problem is invariant for families of distributions leading to the same values of* $\mu(\Omega_\zeta)$ *and* $\mathbb{E}[Z_\zeta]$.

Theorems (3), (4) and (5) allow one to compare CVAR problems to mean-deviation and mean-upper deviation problems. Eq. (65) indicates that the stochastic properties that influence a CVAR decision-maker are $\mathbb{E}[Z_\zeta]$ and $\mu(\Omega_\zeta)$, while eqs. (49) and (59) show that mean-deviation and mean-upper-semideviation decision-makers are interested in $\mathbb{E}[Z]$, $\mathbb{E}^r[Z_-]$, and $\mathbb{E}^r[Z_+]$, $r = 1, 2, ..., p$. Thus, different stochastic properties are of interest to the decision-makers. Let us, however, examine the stochastic properties more closely. In eq. (65), for any given $x$, $\Sigma_{CVAR}$ is the set of points where the loss equals VAR. Thus, $\Sigma_{CVAR} \neq (\Sigma_{md} = \Sigma_{musd})$, in general. However, if $\alpha$ is selected in such a way that $\zeta^*(\alpha) = \mathbb{E}[Z(x,\omega)]$, then by comparison of eqs. (51) and (67) one obtains that $\Sigma_{CVAR} = \Sigma_{md} = \Sigma_{musd}$. In this case, it holds that $\Omega_\zeta = \Omega_+$, and $\mathbb{E}[Z_\zeta] = \mathbb{E}[Z_+]$. However, none of the deviation SP problems is characterized by $\mu(\Omega_\zeta)$. Thus, in contrast to the results obtained for the deviation problems, no general conditions can be stated to identify families of distributions that simultaneously leave a CVAR decision-maker and a deviation decision-maker insensitive to imprecision.

# 7 Conclusions

In this work, we have formalized a new analytical approach to support the formulation of optimization problems involving PD functions.

The first part of this work has developed the required technical background. We have introduced a new definition of multivariate PD functions that complements Herrera (2007)'s definition. We have obtained results for the continuity and differentiability of PD functions. We have then addressed the calculation of the moments of PD functions. We have derived formulas for the decomposition of the moments. Findings show that extended constituents play a central role. The moments of a PD function are the sum of the moments of the same order ($p$) of its extended constituents and the $p^{th}$ central moment is given by the linear combination of the moments of all orders from 1 to $p$ of the extended constituents.

[8] In fact, restricting the loss function over $\Omega_\zeta$ and taking the expectation without reshaping the distribution is equivalent to take the expectation of $Z$ over the conditional distribution defined in Rockafellar and Uryasev (2002), eq. (8), p. 1449 and "*obtained by rescaling the portion of the graph of the original distribution between the horizontal lines at levels* $1-\alpha$ *and* 1".

We have then focused on $|\cdot|$ and $\max(\cdot)$ PD functions, which appear frequently in O.R. applications. We have studied their continuity and differentiability properties. and derived closed form expressions for the calculation of their moments.

We have applied the findings to mean-deviations, mean-upper-semideviation and CVAR optimization problems. The main results can be summarized as follows: *i*) for any $p$-measurable loss function, mean-deviation problems of order $p$ are characterized by all moments up to $p$ of both the excess and defect loss functions; *ii*) $p$-mean-semideviation problems involve only moments of the excess loss function; *iii*) families of distributions leaving the optimal policy of a mean-deviation decision-maker unchanged, also leave the corresponding $p$-mean-upper-semideviation optimal policy unchanged; *iv*) sensitivity to imprecision increases with the order ($p$) of the deviation measures for both mean-deviation and mean-semideviation problems; *v*) CVAR decision-makers are insensitive to imprecision for families of distributions leading to the same $\mathbb{E}[Z_\zeta]$ and $\mu(\Omega_\zeta)$. The method allows to obtain insights at the problem formulation stage, thus eliminating reliance on ad-hoc post-optimality numerical calculations.

## References


[Antunes and Dias (2007)] ANTUNES C.H. AND DIAS L.C., 2007: "Managing Uncertainty in Decision Support Models", Editorial, *European Journal of Operational Research*, 181 (3), pp. 1425-1426.

[Ahmed et al (2007)] AHMED S., CAKMAK U. AND SHAPIRO A., 2007: "Coherent Risk Measures in Inventory Problems," *European Journal of Operational Research*, 182 (1), pp. 226-238.

[Artzner et al (1999)] ARTZNER P., DELBAEN F., EBER J.M. AND HEATH D., 1999: "Coherent measures of risk," *Mathematical Finance*, 9, pp. 203-228.

[Arcelus et al (2006)] ARCELUS F.J., PAKKALA T.P.M. AND SRINIVASAN G., 2006: "On the interaction between retailers inventory policies and manufacturer trade deals in response to supply-uncertainty occurrences", *Annals of Operations Research*, 143 (1), pp. 45-58.

[Beale (1955)] BEALE E., 1955: "On minimizing a convex function subject to linear inequalities", *Journal of the Royal Statistical Society B*, 17, pp. 173–184.

[Benati (2004)] BENATI S., 2004: "The computation of the worst conditional expectation", *European Journal of Operational Research*, 155, 414–425.

[Bertsimas and Shioda (2003)] BERTSIMAS D. AND SHIODA R., 2003: "Restaurant Revenue Management", *Operations Research*,51 (3), pp. 472–486.

[Borgonovo and Peccati (2008)] BORGONOVO E. AND PECCATI L., 2008: "Financial Management in Inventory Problems: Risk Averse vs Risk Neutral Policies," *International Journal of Production Economics*, forthcoming.

[Carette (2007)] CARETTE J., 2007: "A canonical form for some piecewise defined functions", in Proceedings of ISSAC 2007.

[Chaton and Doucet (2003)] CHATON C. AND DOUCET J., 2003: "Uncertainty and Investment in Electricity Generation with an Application to the Case of Hydro-Québec", *Annals of Operations Research*, 120, 59–80, 2003.

[Cooper and Gupta (2006)] COOPER W.L. AND GUPTA D., 2006: "Stochastic Comparisons in Airline Revenue Management", *Manufacturing&Service Operations Management*, 8 (3), pp. 221–234.

[Dantzig (1955)] DANTZIG G.B., 1955: "Linear programming under uncertainty", *Management Science*, 1, pp. 197-206.

[Dantzig (1999)] DANTZIG G.B., 1999: "Planning under uncertainty", *Annals of Operations Research*, 85, pp. 1-5.

[Dupačova (2002)] DUPAČOVA J., 2002: "Applications of stochastic programming: Achievements and questions", *European Journal of Operational Research*, 140, pp. 281–290.

[Ekholm et al (2002)] EKHOLM T., BRIAN WHITE B., AND WIENHOLTZ D., 2002: "Embeddedness of minimal surfaces with total boundary curvature at most $4\pi$", *Annals of Mathematics*, 155 (2002), pp. 209–234.

[Gotoh and Takano (2007)] GOTOH J. AND TAKANO Y., 2007: "Newsvendor Solutions via Conditional Value-at-Risk Minimization," *European Journal of Operational Research*, 179 (1), pp. 80-96.

[Herrera (2007)] HERRERA I., 2007: "Theory of Differential Equations in Discontinuous Piecewise-Defined Functions", *Numerical Methods for Partial Differential Equations*, 23, pp. 597–639.

[Howell (2008)] HOWELL K., 2008: "Piecewise-Defined Functions and the Laplace Transform", Ch. 28 of the Lecture notes for the course in Applied Differential Equations, http://www.uah.edu/howell/DE/Alt_Text/Part4/PW_Def_Fcts.pdf

[Hutchings et al (2002)] HUTCHINGS M., MORGAN F., RITORÈ M., AND ROS A., 2002: "Proof of the Double Bubble Conjecture", *Annals of Mathematics*, 155, 459–489.

[Jeffrey et al (1997)] JEFFREY D.J., LABAHN G., VON MOHRENSCHILDT M. AND RICH A.D., 1997: "Integration of the Signum, Piecewise and Related Functions", *Proceedings of ISSAC97*.

[Jeffrey and Rich (1998)] JEFFREY D.J. AND RICH A.D., 1998: "Recursive Integration of Piecewise-Continuous Functions", *Proceedings of ISSAC98*.

[Muller and Sverak (2003)] MULLER S. AND SVERAK V., 2003: "Convex integration for Lipschitz mappings and counterexamples to regularity", *Annals of Mathematics*, 157, pp. 715–742.

[Nozick et al (2004)] NOZICK L., TURNQUIST M. A. AND NINGXIONG X., 2004: "Managing Portfolios of Projects under Uncertainty", *Annals of Operations Research*, 132, pp. 243-256.

[Rockafellar and Uryasev (2002)] ROCKAFELLAR R.T. AND S. URYASEV, 2002: "Conditional Value-at-Risk for general loss distributions," *Journal of Banking and Finance*, 26, pp. 1443-1471.

[Rockafellar et al (2003)] ROCKAFELLAR R.T., URYASEV S. AND ZABARANKIN M., 2003: "Deviation Measures in Risk Analysis and Optimization", Research Report # 2002-7, Risk Management and Financial Engineering Lab, University of Florida, Gainesville, FL.

[Rockafellar et al (2006)] ROCKAFELLAR R.T., URYASEV S. AND ZABARANKIN M., 2003: "Master funds in portfolio analysis with general deviation measures," *Journal of Banking and Finance*, 30, pp. 743–778.

[Ruszczynski and Shapiro (2005)] RUSZCZYNSKI A. AND SHAPIRO A., 2005: "Optimization of risk measures." in *Probabilistic and Randomized Methods for Design under Uncertainty*, G. Calafiore and F. Dabene (Eds.), Springer Verlag, London (UK).

[Schaefer and Schaefer (2004)] SCHAEFER L.A. AND SCHAEFER A.J., 2004: "Locating

Hybrid Fuel Cell–Turbine Power Generation", *Annals of Operations Research*, 132, 301–322.

[Shah and Madhavan (2004)] SHAH S. AND MADHAVAN K.P., 2004: "Design of Controllable Batch Processes in the Presence of Uncertainty", *Annals of Operations Research*, 132, pp. 223–241.

[Siegmann and Lucas (2005)] SIEGMANN A. AND LUCAS A., 2005: "Discrete-Time Financial Planning Models Under Loss-Averse Preferences", *Operations Research*, 53 (3), pp. 403–414.

[Spivak (2005)] SPIVAK M, 2005: "A Comprehensive Introduction to Differential Geometry", Third Edition, Publish or Perish INC., Huston.

[Tomazi (2004)] TOMAZI K.G., 2004: "Optimization of Batch Reactions in Series with Uncertainty", *Annals of Operations Research*, 132, pp. 189–206.

[Withehead (1961)] WITHEHEAD J.H.C., 1961: "Manifolds with Transverse fields in Euclidean Space", *Annals of Mathematics*, 73 (1), pp. 154-212.

# 8 Appendices

## 8.1 Appendix A: moment calculations for $f = max(h_1, h_2, ..., h_N)$

In this appendix, we present the extension of the results of Section 5.1 to PD functions of the type

$$f = max(h_1, h_2, ..., h_N) \tag{74}$$

In this case, the partition is:

$$\Pi_\Omega = \{\Omega_1, \Omega_2, ..., \Omega_N\} \tag{75}$$

with

$$\Omega_i = \{\omega : h_i(\omega) > h_j(\omega)\ \forall j = 1, 2, ..., N;\ i \neq j\} \tag{76}$$

and

$$\Sigma = \{\omega : h_i(\omega) - h_j(\omega) = 0; \quad i, j = 1, 2, ..., N; \quad j \neq i\} \tag{77}$$

The functional form of $f$ and the extended constituents ($\varphi_i$, $i = 1, 2, ..., N$) then follow straightforwardly. Letting $h_i \in C^{r_i}(\Omega)$, and $k_{i,j}$ the order of the contact between $h_i$ and $h_j$

at $\omega \in \Sigma \cap (\partial\Omega_i \cap \Omega_j)$, Theorem 2 follows with $r = min(r_i, k_{i,j})$. Eqs. (38) and (39) become:

$$\mathbb{E}^p_{\max(t,h)} = \sum_{i=1}^{N} \mathbb{E}^p[\varphi_i] \tag{78}$$

and

$$\mathbb{M}^p_{\max(h_1,,h_2,\dots,h_N)} = \sum_{j=1}^{N}\sum_{q=0}^{p} \binom{p}{q} (-\sum_{i=1}^{N} \mathbb{E}[\varphi_i])^{p-q} (\mathbb{E}^q[\varphi_j]) \tag{79}$$

respectively.

## 8.2 Appendix B: moment calculation for $f = \sum_{i=1}^{N} |g_i|$

In this Appendix, we report the extension of the results of Section 5.2 to functions of the type:

$$f = |g_1| + |g_2| + \dots + |g_N| = \sum_{i=1}^{N} |g_i| \tag{80}$$

In this case, $\Omega$ is partitioned in $2^N$ subsets, possibly null, depending on the signs of $g_i$, $i = 1, 2, .., N$. Let $\Omega_1 = \Omega_{++,\dots,+} = \{\omega : g_i(\omega) > 0, \forall i = 1, 2, \dots, N\}$ and $\Omega_{2^N} = \Omega_{-,-,\dots,-} = \{\omega : g_i(\omega) < 0, \forall i = 1, 2, \dots, N\}$. Similarly, $\Omega_i = \Omega_{+,-,\dots,+}$ shall denote the generic subset in which $g_1 > 0$, $g_2 < 0$, etc. The internal boundary is $\Sigma = \bigcup_{i=1}^{N} \Sigma_i$, where $\Sigma_i = \{\omega : g_i(\omega) = 0, \forall i = 1, 2, \dots, N\}$. For simplicity, let us temporarily set $N = 2$. We have 4 possible regions:

$$\begin{aligned} \Omega_1 &= \Omega_{+,+} \\ \Omega_2 &= \Omega_{-,+} \\ \Omega_3 &= \Omega_{-,-} \\ \Omega_4 &= \Omega_{-,-} \end{aligned} \tag{81}$$

Then, the PD expression of eq. (80) is:

$$f = \begin{cases} f_{+,+} = g_1 + g_2 & \omega \in \Omega_{+,+} \cup (\partial\Omega_{+,+} \cap \Sigma) \\ f_{-,+} = -g_1 + g_2 & \omega \in \Omega_{-,+} \cup (\partial\Omega_{-,+} \cap \Sigma \backslash (\partial\Omega_{+,+} \cap \partial\Omega_{-,+})) \\ f_{+,-} = g_1 - g_2 & \omega \in \Omega_{+,-} \cup (\partial\Omega_{+,-} \cap \Sigma \backslash (\partial\Omega_{+,-} \cap \partial\Omega_{+,+}) \backslash (\partial\Omega_{+,-} \cap \partial\Omega_{-,+})) \\ f_{-,-} = -g_1 - g_2 & \omega \in \Omega_{-,-} \end{cases} \tag{82}$$

The extended constituents are, then, denoted by $\varphi_{+,+}, \varphi_{-,+}, \varphi_{+,-}, \varphi_{+,+}$ and obtained by extending the functions in eq. (82) to the whole $\Omega$. For example:

$$\varphi_{+,+} = \begin{cases} f_{+,+} & \omega \in \Omega_{+,+} \cup (\partial\Omega_{+,+} \cap \Sigma) \\ 0 & otherwise \end{cases} \tag{83}$$

and similarly $\varphi_{-,+}, \varphi_{+,-}$ and $\varphi_{+,+}$.

One then obtains the following generalization of Lemma 2:

$$\mathbb{E}^p[|g|] = \mathbb{E}^p[\varphi_{+,+}] + \mathbb{E}^p[\varphi_{-,+}] + \mathbb{E}^p[\varphi_{+,-}] + \mathbb{E}^p[\varphi_{-,-}] \tag{84}$$

and

$$\begin{aligned}\mathbb{M}^p_{|g|} = \textstyle\sum_{q=0}^{p}\{\binom{p}{q}(-1)^{p-q}(\mathbb{E}[\varphi_{+,+}] + \mathbb{E}[\varphi_{-,+}] + \mathbb{E}[\varphi_{+,-}] + \mathbb{E}[\varphi_{-,-}])^{p-q}\cdot \\ \cdot(\mathbb{E}^q[\varphi_{+,+}] + \mathbb{E}^q[\varphi_{-,+}] + \mathbb{E}^q[\varphi_{+,-}] + \mathbb{E}^q[\varphi_{-,-}])\}\end{aligned} \tag{85}$$

By numbering the extended constituents as follows $\varphi_{+,+} = \varphi_1$, $\varphi_{-,+} = \varphi_2$, $\varphi_{+,-} = \varphi_3$, and $\varphi_{-,-} = \varphi_4$, one can rewrite eqs. (84) and (85) more synthetically as:

$$\mathbb{E}^p[|g|] = \sum_{i=1}^{4} \mathbb{E}^p[\varphi_i] \tag{86}$$

and

$$\mathbb{M}^p_{|g_1+g_2|} = \sum_{q=0}^{p}\{\binom{p}{q}(-1)^{p-q}(\sum_{i=1}^{4}\mathbb{E}[\varphi_i])^{p-q}\cdot(\sum_{i=1}^{4}\mathbb{E}^q[\varphi_i]) \tag{87}$$

For $N = 3$, one has to consider 8 regions, denoted as $\Omega_{+,+,+}$, $\Omega_{-,+,+}$,$\Omega_{+,-,+}, ..., \Omega_{-,-,-}$. For $N = 4$, one has to consider 16 regions, etc.. By applying the same approach as used for $N = 2$, one can then extend eqs. (84) and (85) to the case $N > 2$. One obtains:

$$\mathbb{E}^p[|g|] = \sum_{i=1}^{2^N} \mathbb{E}^p[\varphi_i] \tag{88}$$

and

$$\mathbb{M}^p_{\sum_{i=1}^{N}|g_i|} = \sum_{q=0}^{p}\{\binom{p}{q}(-1)^{p-q}(\sum_{i=1}^{2^N}\mathbb{E}[\varphi_i])^{p-q}(\sum_{i=1}^{2^N}\mathbb{E}^q[\varphi_i]) \tag{89}$$

Table 1: Notation used in this work

| **Symbol** | **Meaning** |
|---|---|
| $f$ | Piecewise defined function |
| $N$ | Number of constituents of $f$ |
| $\Omega$ | Domain of $f$ |
| $\Pi$ | Partition of $\Omega$ |
| $\Omega_i$ | An element of $\Pi$ |
| $\mathbf{1}_{\Omega_i}$ | Indicator function of $\Omega_i$ |
| $n$ | Dimension of $\Omega$ and $\Omega_i$ |
| $\partial\Omega_i$ | Frontier of $\Omega_i$ |
| $\Sigma = \cup(\partial\Omega_i \cap \partial\Omega_j)$ | Internal Boundary |
| $N$ | Number of subdomains and of constituents |
| $f_i$ | Constituent of $f$ |
| $\varphi_i$ | Extended Constituent |
| $(\Omega, \mathcal{B}(\Omega), \mu)$ | Probability space on $\Omega$ |
| $\mathbb{E}^p[f]$ | Moment of order $p$ of $f$ |
| $\mathbb{M}_f^p$ | Central Moment of order $p$ of $f$ |
| $\rho(\cdot)$ | Coherent Measure of Risk |
| $Z(x, \omega)$ | Loss Function |
| $S$ | Feasible Set |
| $\rho_{\text{md}}$ | mean-deviation risk-function |
| $\rho_{\text{musd}}$ | mean-upper-semideviation risk-function |
| $CVAR_\alpha$ | Conditional value at risk |
| $VAR_\alpha$ | Value at Risk |
| $H$ | Auxiliary function for CVAR optimization |
| $\zeta$ | Parameter in CVAR optimization |