Composite Multilinearity, Epistemic Uncertainty and Risk Achievement Worth

E. Borgonovo[1] and C.L. Smith[2]
[1]ELEUSI Research Center, Bocconi University, Milan, Italy
[2]Idaho National Laboratory, Idaho Falls, USA

Abstract

Risk Achievement Worth is one of the most widely utilized importance measures. RAW is defined as the ratio of the risk metric value attained when a component has failed over the base case value of the risk metric. Traditionally, both the numerator and denominator are point estimates. Relevant literature has shown that inclusion of epistemic uncertainty i) induces notable variability in the point estimate ranking and ii) causes the expected value of the risk metric to differ from its nominal value. We investigate the conditions under which the equality of the nominal and expected values of a reliability risk metric holds. We then study how the presence of epistemic uncertainty affects RAW and the associated ranking. We propose an extension of RAW (called ERAW) which allows one to obtain a ranking robust to epistemic uncertainty. We discuss the properties of ERAW and the conditions under which it coincides with RAW. We apply our findings to a probabilistic risk assessment model developed for the safety analysis of NASA lunar space missions.

Keywords: Reliability Analysis; Risk Analysis; Importance Measures; Uncertainty Analysis.

1 Introduction

In the risk analysis of complex operational systems, risk analysts often make use of quantitative models to support decision-making [Dillon et al. (2003)]. Decision-support models help them in reproducing system behavior and in assessing several measures of system performance and safety. Importance measures play a central role in informing analysts about how system risk is contributed by systems, structures and components (SSCs). Importance measure ranking is then used by decision-makers in applications such as the prioritization of maintenance activities, the assignment of SSCs to graded quality assurance programs, the safety categorization of SSCs, etc. [Borgonovo and Smith (2011)]. One of the most widely used importance measures is risk achievement worth (RAW). Among the reasons of RAW's widespread diffusion is its ability to produce clear and essential insights to risk analysts. RAW informs about the potential increase in risk associated with the occurrence of an event. It is defined as follows. Let U^{nom} be the nominal value of the risk metric[1]. Let E denote the event of interest. Typically, E is the failure of a given component or the

[1]In reliability applications, U is usually a reliability or unreliability function, in probabilistic risk assessment a core damage frequency or large early release frequency, in space applications it can be a loss of crew or loss of mission.

occurrence of a basic event. Then, letting U^+ denote the conditional value of the risk metric given E, one defines[Vesely et al. (1986)][2]

$$RAW_E := \frac{U_E^+}{U^{nom}}. \tag{1}$$

One of the main assumptions underlying the definition of RAW [eq. (1)], is that exogenous variables (e.g., failure probabilities) are known. Both U_E^+ and U^{nom}, in fact, are obtained with the exogenous variables fixed at a certain value. This assumption is common across importance measures: the Birnbaum [Birnbaum (1969)], Fussell-Vesely [Fussell (1975)], Risk Reduction Worth [Vesely et al. (1986)], Criticality [Cheok et al. (1998)], Differential [Borgonovo and Apostolakis (2001)], Composite [Ramirez-Marquez and Coit (2005)], Joint [Gao et al. (2007)] and Total-order [Borgonovo (2010), Do Van et al. (2010)] are all defined with the reliability model calculated at the nominal value of the parameters.

We recall that the nominal value of the risk metric captures aleatory uncertainty [Apostolakis (1990), Aven (2010)], namely, our lack of knowledge in the final result of a the statistical experiment under scrutiny (i.e., system or component failures in our case). However, a second type of uncertainty characterizes most practical applications, i.e., epistemic (or parametric) uncertainty. Epistemic uncertainty refers to our lack-of-knowledge about the values of the parameters of the model through which the risk metric is evaluated [for a recent discussion of aleatory vs epistemic uncertainty in operations research modelling, see Kleijnen et al. (2011)]. Systematic approaches for including epistemic uncertainty in reliability importance measures are offered by Modarres and Agarwal (1996) and Borgonovo (2008). These works propose to investigate variability in importance measure by simple Monte Carlo simulation. The values of the importance measures are computed at each Monte Carlo run and the corresponding variation ranges are registered. The authors of the present work applied this type of approach in a realistic application. Results are produced through a standard reliability analysis software package [SAPHIRE, Smith et al. (2008)]. Figure 1 presents numerical findings for uncertainty in RAW.

Figure 1 allows the decision-maker to appreciate her confidence in the importance measure (RAW in this case) results. However, the overlapping between the distributions of the RAWs is such that she cannot separate with certainty important from non-important elements. Thus, the problem emerges of confidently entrusting the ranking obtained through importance measures computed with the parameters at the nominal values, when one is in the presence of epistemic uncertainty. This issue is also encountered in Borgonovo (2008), where the assignment of components to safety categories becomes probabilistic.

In this work, we address this issue proposing an extension of the definition of RAW that makes it robust to epistemic uncertainty. Our goal is to utilize the same (no more and no less) information available to the decision-maker for performing a Monte Carlo simulation. Thus, the decision-maker's degree-of-belief about U is represented by its state-of-knowledge distribution (see also Kaplan and Garrick (1981), Apostolakis (1990), Aven (2010), Kleijnen et al. (2011)). Knowing

[2] Vesely et al. (1986) discuss that an alternative definition is represented by $RAW_E^{Difference} = U^+ - U^0$. However, as they already recognize, the ratio definition in eq. (1) is the most widely applied [see also Cheok et al. (1998)]

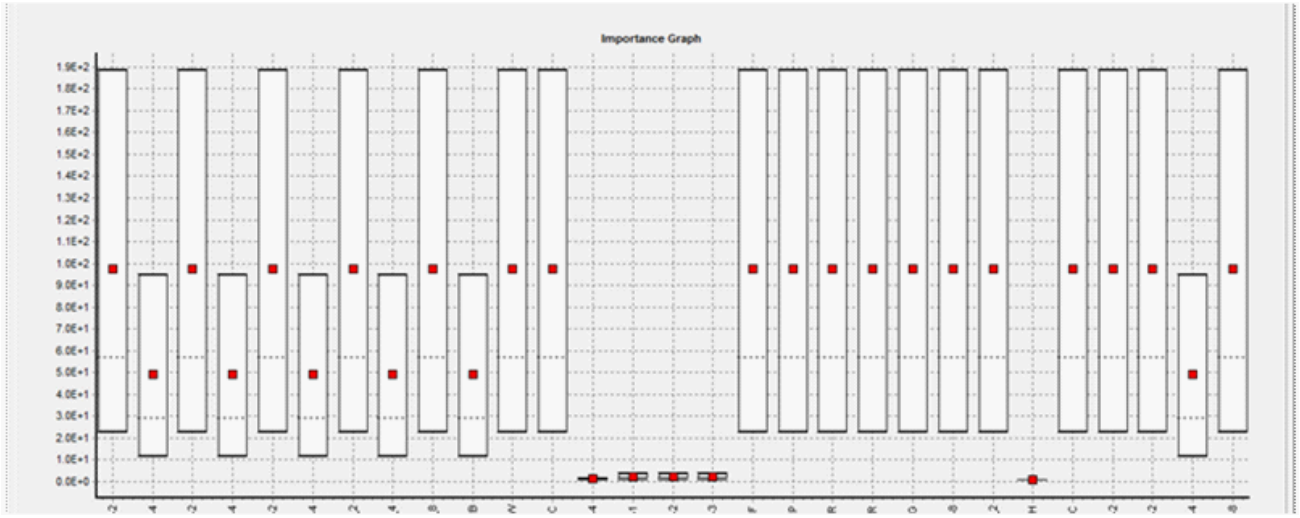

Figure 1: Uncertainty in RAW as displayed by the SAPHIRE computer code for the NASA space mission PSA model used in this work. The horizontal axis displays basic events. The vertical axis the corresponding distribution of RAW provoked by epistemic uncertainty. The bars extend from the 5^{th} to the 95^{th} quantiles of the RAW distributions.

that "E has happened" [eq. (1)] alters more than the sole nominal value of U, because it changes the distribution of U. We then propose an extension of the definition of RAW that considers the effect of E on the distribution of U. We call this extension epistemic RAW (ERAW). We discuss its properties and study the conditions under which RAW and ERAW coincide. This leads us to investigate the assumptions under which the nominal value of a reliability function coincides with its state-of-knowledge expected value. As we are to see, separability and composite multilinearity of the risk metric play an essential role in the investigation [on the relevance of multilinearity in operational research problems, we refer to Grabisch et al. (2003)]. We address the properties of ERAW for series and parallel systems. For systems with single points of failure, results show that information on component failure resolves both the aleatory and epistemic uncertainty about the system state. This leads us to address in greater detail an early problem in the computation of RAW: the presence of dependencies. Stochastic dependencies and state-of-knowledge dependencies[3] in the computation of RAW and ERAW are discussed in detail by means of a three-components in parallel system example.

We then investigate the utilization of ERAW by application to the Probabilistic Safety Assessment (PSA) model developed by the US Idaho National Laboratory for the risk analysis of NASA space missions [Borgonovo and Smith (2011)]. We discuss an algorithm for the numerical estimation of ERAW. Several numerical experiments are performed. As expected from the theoretical investigation, RAW and ERAW assume different values if state-of-knowledge dependences are factored into the analysis. The level of agreement between the ranking induced by ERAW and

[3]The terms stochastic and state-of-knowledge dependece are defined in Apostolakis and Moieni (1987). See also Section 2.

RAW is examined through Savage scores [Iman and Conover (1987)] and decision-making insights are discussed.

The remainder of the work is organized as follows. Section 2 discusses stochastic dependence and the definition of RAW, as well as its properties for parallel and series systems. Section 3 addresses the general framework of epistemic uncertainty in reliability functions, with focus on composite multilinearity and separability. Section 4 defines ERAW. Section 5 proposes an illustrative example. Section 6 presents the application to a NASA space mission model. Conclusions are offered in Section 7.

2 Risk Achievement Worth and Multilinearity: Some Properties

In this section, we briefly review the traditional definition of RAW and some of its properties.

We consider a system whose state is determined by the state of n components or, more generally, basic events. Let $\boldsymbol{\varphi}$ be the vector of basic event Boolean indicator variables. $\varphi_j = 1$ denotes the occurrence of basic event j. Ψ is the system indicator variable. $\Psi = 1$ is called top event and can denote system failure or success. The Boolean logic expression that maps $\boldsymbol{\varphi}$ onto Ψ is called system structure function. We denote it by $\Psi(\boldsymbol{\varphi})$. For instance, for a system of n components in series, it is $\Psi(\boldsymbol{\varphi}) = 1 - \prod_{i=1}^{n}(1-\varphi_i)$. Let $U = P(\Psi = 1)$ denote the probability of the top event occurring within mission time T. U is called risk metric. Let Z_i denote the indicator variable of a generic prime implicant (minimal cut set or minimal path set) and let M be the number of prime implicants. A prime implicant is a minimal collection of basic events whose occurrence causes the top event. Formally, $\Psi = Z_1 \vee Z_2 \vee ... \vee Z_n$. Then,

$$U = P(\bigcup_{i=1}^{M}\{Z_i = 1\}) = \sum_{i=1}^{M} P(Z_i = 1) - \sum_{i,j=1, i\neq j}^{M} P(\{Z_i = 1\} \cap \{Z_j = 1\}) + ... + (-1)^{n+1} P(\bigcap_{r=1}^{M}\{Z_r = 1\}) \tag{2}$$

In turn, Z_i is realized when all the indicator variables of the basic events contained in Z_i are unity. Let φ_{i_k} denote that indicator variable of the k^{th} basic event in Z_i and let m_i be the number of basic events in Z_i. Then, it is $Z_i = \varphi_{i_1} \wedge \varphi_{i_2} \wedge ... \wedge \varphi_{i_{m_i}}$. Hence,

$$P(Z_i = 1) = P(\cap_{k=1}^{m_i}\left\{\varphi_{i_k} = 1\right\}) = P(\varphi_{i_{m_i}} | \varphi_{i_{m_i-1}}, \varphi_{i_{m_i-2}}, ..., \varphi_{i_1}) \cdot P(\varphi_{i_{m_i}-1} | \varphi_{i_{m_i-2}}, \varphi_{i_{m-3}}, ..., \varphi_{i_1}) \cdot .... \cdot P(\varphi_{i_1}) \tag{3}$$

Eq. (3) suggests that $P(Z_i = 1)$ is the product of m_i conditional probabilities. Denoting the k^{th} one by $q_{i_k} = P(\varphi_{i_k} | \varphi_{i_{k-1}}, \varphi_{i_{k-2}}, ..., \varphi_{i_1})$, we come to

$$P(Z_i = 1) = \prod_{k=1}^{m_i} q_{i_k} \tag{4}$$

For second order terms, a similar notation applies, but with a caveat. In fact, by definition

$$P(Z_i = 1 \cap Z_r = 1) = P(\left\{\cap_{k=1}^{m_i}\left(\varphi_{i_k} = 1\right)\right\} \cap \left\{\cap_{c=1}^{m_r}\left(\varphi_{i_c} = 1\right)\right\}) \tag{5}$$

Note that some basic events in $\cap_{k=1}^{m_i}\left\{\varphi_{i_k}=1\right\}$ and $\cap_{c=1}^{m_r}\left(\varphi_{i_c}=1\right)$ can be identical. By idempotency, identical basic events simplify in the Boolean expression. Then, the number of basic event probabilities required to assess $P(Z_i=1\cap Z_r=1)$ is $m_i+m_j-m_{ir}$, where m_{ir} is the number of basic events shared by Z_i and Z_r. Also, the generic basic event probability is now conditional not only on the basic events in Z_i but also on the basic events in Z_r. To denote this fact, we use the notation q_{ir_s}, where $s=1,2,...,\ m_i+m_j-m_{ir}$. Then, we have

$$P(\{Z_i=1\}\cap\{Z_j=1\})=\prod_{s=1}^{m_j+m_j-m_{ir}}q_{ir_s} \tag{6}$$

A similar notation can be used for the higher order terms of eq. (2), obtaining:

$$U=U(\mathbf{q})=\sum_{k=1}^{M}\prod_{l=1}^{m_k}q_{k_l}-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q_{ir_s}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q_{irl_t}-... \tag{7}$$

where $\mathbf{q}$ denotes the vector of (all) conditional basic event probabilities. Eq. (7) states that U is a multilinear polynomial in $\mathbf{q}$. As proven in Borgonovo (2010), this statement holds for both coherent and non-coherent systems.

We can now come to the computation of RAW. This involves a two steps calculation. First, one evaluates $U^{nom}=U(\mathbf{q}^{nom})$. Then, let $E=\left\{\varphi_j=1\right\}$. To get U_j^+, one first sets $\varphi_j=1$ in $\Psi(\boldsymbol{\varphi})$, and next re-evaluates the risk metric utilizing the conditional probabilities given that $\varphi_j=1$ [for further discussion about the calculations of conditional risk metrics with stochastic dependences, we refer to Smith (1998)]. Thus, $q_{i_k}^{nom}$ needs to be reassessed becoming $q_{i_k}^+=P(\varphi_{i_k}|\varphi_{i_{k-1}}=1,\varphi_{i_{k-2}}=1,...,\varphi_{i_1}=1,\varphi_j=1)$, in general. The corresponding new vector of conditional probabilities is denoted by $\mathbf{q}^+$. Substituting into eq. (1), we obtain

$$RAW_j=\frac{U_j^+}{U^{nom}}=\frac{\sum_{k=1}^{M}\prod_{l=1}^{m_k}q_{k_l}^+-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q_{ir_s}^++\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q_{irl_t}^+-...}{\sum_{k=1}^{M}\prod_{l=1}^{m_k}q_{k_l}^{nom}-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q_{ir_s}^{nom}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q_{irl_t}^{nom}-...} \tag{8}$$

A note. To obtain RAW_j a computational shortcut is often followed by standard software. It consists in directly setting $q_i=1$ in eq. (7), leaving the remaining probabilities unchanged. This procedure is correct, provided that the rare event approximation is not used and the independence assumption is stated. Conversely, if stochastic dependences are present, one needs to adjust the basic event probabilities substituting $\mathbf{q}^+$ for $\mathbf{q}^{nom}$.

In the remainder, the following two properties of RAW for single points of failure and parallel systems are useful.

1. If basic event j is in series with the top event (single point of failure), then

$$RAW_j = 1/U^{nom} \tag{9}$$

2. If a system is made of n parallel components, then

$$RAW_j = 1/q_j^{nom} \tag{10}$$

Item 1 implies that, if basic event j is in series with basic event i, then $RAW_j = RAW_i$. Thus, all single point of failures have the same RAW. Item 2 states that, in a parallel system, the most unreliable component is the one associated with the highest RAW.

The discussion carried out in this section has detailed the traditional definition of RAW, which is computed with all basic event probabilities fixed at their nominal values. In order to address the definition of RAW in the presence of epistemic uncertainty, we need first to analyze the consequences of epistemic uncertainty on the risk metric. This is the subject of the next section.

3 Epistemic Uncertainty and Composite Multilinearity

In this section, we study generic multilinear functions in the presence of epistemic uncertainty. The reason why we address generic functions is that multilinearity characterizes several operations research models, besides reliability polynomials. Belief-network polynomials are multilinear in the probabilities [Park and Darwiche (2004)]. An important role is played by multilinearity in game theory [see Grabisch et al. (2003), Alonso-Meijide et al. (2008), Marichal and Mathonet (2011)], multiattribute utility theory [Reeves and J.J. (1989), Saaty (1994), Bordley and Kirkwood (2004)], global optimization (Floudas and Gounaris (2009)) (for a further review, see Borgonovo and Smith (2011)).

We write a multilinear function as

$$y = g(\mathbf{x}) = \sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k} \tag{11}$$

By eq. (11), if $\mathbf{x}$ is fixed at $\mathbf{x}^{nom}$, we obtain $y^{nom} = g(\mathbf{x}^{nom})$.

In several applications, risk analysts utilize more elaborate expressions for the x_i's and they become functions of one or more parameters. We write $\mathbf{x} = h(\boldsymbol{\lambda})$, $h : R^m \rightarrow R^n$. Eq. (11) then becomes composite multilinear:

$$y = g(\mathbf{x}(\boldsymbol{\lambda})) = \sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1}(\boldsymbol{\lambda}) \cdot x_{i_2}(\boldsymbol{\lambda}) \cdot \ldots \cdot x_{i_k}(\boldsymbol{\lambda}) \tag{12}$$

In eq. (12), for generality, we have allowed each exogenous variable x_i to depend on all $\lambda' s$. However,

in some instances x_{i_r} might depend upon just one specific parameter λ_{i_r}:

$$y = g_1(\mathbf{x}(\boldsymbol{\lambda})) = \sum_{k=1}^{n} \sum_{i_1<i_2<...<i_k}^{k} \delta_{i_1,i_2,...,i_k} \cdot x_{i_1}(\lambda_{i_1}) \cdot x_{i_2}(\lambda_{i_2}) \cdot ... \cdot x_{i_k}(\lambda_{i_k}) \tag{13}$$

If eq. (13) holds, we say that y is a separable composite multilinear function of the exogenous variables.

Using eq. (12), the nominal value of the risk metric remains y^{nom}, if $\boldsymbol{\lambda}$ is set at value(s) $\boldsymbol{\lambda}^{nom}$ that satisfy $\mathbf{x}^{nom} = h(\boldsymbol{\lambda}^{nom})$. However, in most practical situations, the decision-maker is unable to assign $\boldsymbol{\lambda}$ a certain value. Uncertainty in $\boldsymbol{\lambda}$ is called parametric or epistemic [Kleijnen et al. (2011)]. The distinction between aleatory and epistemic uncertainty has become an integral part of risk and reliability analysis with the works of Kaplan and Garrick (1981), Apostolakis (1990), Paté-Cornell (1996). Several approaches are available to address epistemic uncertainty, ranging from probability to possibility theory and the most appropriate depends on the decision-maker's state-of-information as well as on the final purposes of the analysis. A wide literature describing these approaches is available and its review is outside the scope of this paper. We refer to Aven (2010), Dubois et al. (2001), Dubois (2010) and Garrick (2010) for a thorough discussion. Throughout, we assume that available data allow the risk analyst to assess probability distributions on the parameters. $\boldsymbol{\lambda}$, then, becomes a random vector denoted by $\boldsymbol{\Lambda}$. We let $(\Omega_{\boldsymbol{\Lambda}},\mathcal{B}(\Omega_{\boldsymbol{\Lambda}}),P_{\boldsymbol{\Lambda}})$, $F_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) = P_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda} < \boldsymbol{\lambda})$, $f_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})$ denote the associated probability space, cumulative and density functions, respectively. We note that $F_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})$ is, in general, a joint distribution function. In the case an independence assumption is stated, then $F_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{m} F_i(\lambda_i)$, and one talks about state-of-knowledge independence [Apostolakis and Moieni (1987)].

By eq. (12), y cannot be assigned a certain value and its value in a given state of the world depends on the realization of $\boldsymbol{\Lambda}$ in such state. We use the notation $y = g(\mathbf{x}|\boldsymbol{\Lambda} = \boldsymbol{\lambda})$ for evidencing this dependence. In the broader framework of model uncertainty, eq. (12) is called model of the world [Apostolakis (1990)]. By world, one means the object of interest to the analyst in her investigation. The *unconditional solution of the model of the world* is obtained *by finding the weighted average of the solution of conditional models where the weights are the probabilities of the parameters and the assumptions* [see Apostolakis (1990), eq. (3), p. 1360]. We write:

$$\mathcal{Y} = \int g(\mathbf{x}|\boldsymbol{\Lambda} = \boldsymbol{\lambda}) dP_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) = \int \int ... \int g(\mathbf{x}|\boldsymbol{\Lambda} = \boldsymbol{\lambda}) f_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) \prod_{i=1}^{N} d\lambda_i = \mathbb{E}_{\boldsymbol{\Lambda}}[Y] \tag{14}$$

If y has the form in eq. (12), one obtains

$$\mathcal{Y} = \mathbb{E}_{\boldsymbol{\Lambda}}[\sum_{k=1}^{n} \sum_{i_1<i_2<...<i_k}^{k} \delta_{i_1,i_2,...,i_k} \cdot x_{i_1}(\boldsymbol{\lambda}) \cdot x_{i_2}(\boldsymbol{\lambda}) \cdot ... \cdot x_{i_k}(\boldsymbol{\lambda})] \tag{15}$$

By comparing eqs. (12) and (11), it is clear that, generally, $\mathcal{Y}$ will be different from y^{nom}, for two

reasons: *i)* eq. (12) is more general than eq. (11) from a structural viewpoint; and *ii)*, in eq. (15), we are also considering a type of uncertainty (epistemic) not included in eq. (12).

We then investigate the conditions under which $\mathcal{Y}$ and y^{nom} coincide. It is convenient to write $y^{nom} = \sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1}^{nom} \cdot x_{i_2}^{nom} \cdot \ldots \cdot x_{i_k}^{nom}$. We have the following result.

Lemma 1 *If*
1) y is separable [eq. (13)]
2) state-of-knowledge independence holds [$f_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) = \prod_k f_{i_k}(\lambda_{i_k})$]
3) the equality $\mathbf{x}^{nom} = \mathbb{E}_{\boldsymbol{\Lambda}}[\mathbf{x}(\boldsymbol{\lambda})]$ holds
then $y^{nom} = \mathcal{Y}$, that is

$$\mathbb{E}_{\boldsymbol{\Lambda}}[\sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1}(\boldsymbol{\lambda}) \cdot x_{i_2}(\boldsymbol{\lambda}) \cdot \ldots \cdot x_{i_k}(\boldsymbol{\lambda})] = \sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1}^{nom} \cdot x_{i_2}^{nom} \cdot \ldots \cdot x_{i_k}^{nom} \tag{16}$$

Proof. Assumption 1 allows us to write

$$\mathcal{Y} = \mathbb{E}_{\boldsymbol{\Lambda}}[\sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1}(\lambda_{i_1}) \cdot x_{i_2}(\lambda_{i_2}) \cdot \ldots \cdot x_{i_k}(\lambda_{i_k})] \tag{17}$$

If assumption 2 holds, by the linearity of the expectation operator and by the fact that, under independence, the expected value of a product is the product of the expected values, one obtains

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{\Lambda}}[\sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot x_{i_1}(\lambda_{i_1}) \cdot x_{i_2}(\lambda_{i_2}) \cdot \ldots \cdot x_{i_k}(\lambda_{i_k})] \\ &= \sum_{k=1}^{n} \sum_{i_1<i_2<\ldots<i_k}^{k} \delta_{i_1,i_2,\ldots,i_k} \cdot \mathbb{E}[x_{i_1}(\lambda_{i_1})] \cdot \mathbb{E}[x_{i_2}(\lambda_{i_2})] \cdot \ldots \cdot \mathbb{E}[x_{i_k}(\lambda_{i_k})] \end{aligned} \tag{18}$$

If assumption 3 holds, then $\mathbb{E}[x_{i_k}(\lambda_{i_k})] = x_{i_k}^{nom}$, which completes the proof. ■

Lemma 1 states that $\mathcal{Y}$ equals y^{nom} if: 1) y is a separable composite multilinear function of the parameters; 2) state-of-knowledge independence is assumed; and 3) the expected value of the functions $x_i(\lambda_i)$ coincides with the nominal value x_i^{nom}. These three conditions may not be actually encountered in practical applications, especially the ones concerning separability and independence. However, the assignment of nominal values equal to the expected value of a failure probability is part of common practice. Especially in the early stages of model building, analysts start with the nominal of the parameters and assess variation ranges around these values [Felli and Hazen (2004)]. Then, the mean value of x_i coincides with its nominal value.

In the next section, we discuss the case in which y is a reliability polynomial.

4 Epistemic Uncertainty and Risk Achievement Worth

In this section, we focus on the case in which y is the output of a reliability model. In this context, aleatory uncertainty expresses the fact that we are uncertain about the top event. Epistemic uncertainty refers to our uncertainty in the parameters of the model utilized to characterize U.

Thus, $U(\mathbf{q}|\mathbf{\Lambda})$ becomes a random variable and $u = U(\mathbf{q}|\mathbf{\Lambda} = \boldsymbol{\lambda})$ is a possible realization of the value of the risk metric. To get the unconditional model solution, one averages over epistemic uncertainty obtaining

$$\mathcal{U} = \int U(\mathbf{q}|\mathbf{\Lambda} = \boldsymbol{\lambda}) dP_{\mathbf{\Lambda}}(\boldsymbol{\lambda}) = \int \int ... \int U(\mathbf{q}|\mathbf{\Lambda} = \boldsymbol{\lambda}) f_{\mathbf{\Lambda}}(\boldsymbol{\lambda}) \prod_{i=1}^{N} d\lambda_i = \mathbb{E}_{\mathbf{\Lambda}}[U] \tag{19}$$

Suppose now that the decision-maker is informed that basic event j has happened, i.e., $\varphi_j = 1$. The new decision-maker's degree-of-belief about U is portrayed by the conditional value of the risk metric, namely $U(\mathbf{q}|\mathbf{\Lambda} = \boldsymbol{\lambda}, \varphi_j = 1)$. Then, let $(\Omega_{\mathbf{\Lambda}|\varphi_j}, \mathcal{B}(\Omega_{\mathbf{\Lambda}|\varphi_j}), P_{\mathbf{\Lambda}|\varphi_j=1})$ the new probability space associated with the uncertain factors, after the information $\varphi_j = 1$ has been gathered. The conditional expected value of the risk metric is given by

$$\mathcal{U}_j^+ := \mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[U|\varphi_j = 1] = \int \int ... \int U(\mathbf{q}|\mathbf{\Lambda} = \boldsymbol{\lambda}, \varphi_j = 1) f_{\mathbf{\Lambda}|\varphi_j=1}(\boldsymbol{\lambda}|\varphi_j = 1) \prod_{i=1}^{N} d\lambda_i \tag{20}$$

Eqs. (19) and (20) allow us to extend the definition of RAW.

Definition 1 *We call epistemic risk achievement worth the quantity*

$$RAW_j := \frac{\mathcal{U}_j^+}{\mathcal{U}} \tag{21}$$

Eq. (21) defines the importance of basic event j as the ratio of: a) the conditional expected risk metric given that the decision-maker receives evidence that basic event j has happened; and b) the expected value of the risk metric; both the numerator and denominator fully account for epistemic uncertainty.

We now show that Definition 1 is a natural extension of the traditional RAW.

Proposition 1 *If*
1) $q_{i_k} = q_{i_k}(\lambda_{i_k})$
2) $f_{\mathbf{\Lambda}}(\boldsymbol{\lambda}) = \prod_k f_{i_k}(\lambda_{i_k})$
3) $f_{\mathbf{\Lambda}|\varphi_j=1}(\boldsymbol{\lambda}|\varphi_j = 1) = \prod_k f_{i_k|\varphi_j=1}(\lambda_{i_k}|\varphi_j = 1)$
and
4) $\mathbf{q}^{nom} = \mathbb{E}[\mathbf{q}]$ *and* $\mathbf{q}^+ = \mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[\mathbf{q}^+|\varphi_j = 1]$
then

$$ERAW_j = RAW_j \tag{22}$$

Proof. To prove eq. (22), it is necessary to prove that

$$\frac{\mathbb{E}[U|\varphi_j=1]}{\mathbb{E}[U(\mathbf{q})]}=\frac{\sum_{k=1}^{M}\prod_{l=1}^{m_k}q^{+}_{k_l}-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q^{+}_{ir_s}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q^{+}_{irl_t}-...}{\sum_{k=1}^{M}\prod_{l=1}^{m_k}q^{nom}_{k_l}-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q^{nom}_{ir_s}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q^{nom}_{irl_t}-...} \tag{23}$$

We start with the denominator of the left hand side. By definition, it is

$$\mathcal{U}=\mathbb{E}[U(\mathbf{q})]=\mathbb{E}[\sum_{k=1}^{M}\prod_{l=1}^{m_k}q^{nom}_{k_l}-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q^{nom}_{ir_s}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q^{nom}_{irl_t}-...\ ] \tag{24}$$

Under assumption 1 and 2 and 4, Lemma 1 holds. Therefore, $\mathcal{U}=U^0$. For the numerator, by assumption 1, we have

$$U|\varphi_j=\sum_{k=1}^{M}\prod_{l=1}^{m_k}q^{+}_{k_l}(\lambda_{k_l})-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q^{+}_{ir_s}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q^{+}_{irl_t}(\lambda_{irl_t})-... \tag{25}$$

and, by assumption 3,

$$\begin{aligned}\mathcal{U}^{+}_{j}=\sum_{k=1}^{M}\prod_{l=1}^{m_k}\mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[q^{+}_{k_l}|\varphi_j=1]]-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}\mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[q^{+}_{ir_s}|\varphi_j=1]]+\\+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}\mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[q^{+}_{irl_t}|\varphi_j=1]]-...\end{aligned} \tag{26}$$

Finally, by assumption 4, the base case of the conditional probabilities is set equal to the conditional expectations, then

$$\mathcal{U}^{+}_{j}=\sum_{k=1}^{M}\prod_{l=1}^{m_k}q^{+}_{k_l}-\sum_{i,r=1,i\neq r}^{M}\prod_{s=1}^{m_j+m_r-m_{ir}}q^{+}_{ir_s}+\sum_{i,r,l=1,i\neq r\neq l}^{M}\prod_{t=1}^{m_j+m_r+m_l-m_{irl}}q^{+}_{irl_t}-... \tag{27}$$

which concludes the proof. ■

Proposition 1 establishes the conditions under which ERAW coincides with RAW for any generic system, coherent and non-coherent. In the next two propositions, we analyze ERAW in the case of two types of systems which are widely studied in the reliability literature, namely series and parallel systems.

Proposition 2 *Consider a generic system and let basic event j be in series with the Top event (single point of failure). Then,*

$$\mathcal{U}^{+}_{j}=1 \text{ and } ERAW_j=\frac{1}{\mathcal{U}} \tag{28}$$

The fact that $\mathcal{U}^{+}_{j}=1$ in Proposition 2 has the following interpretation. For single points of failure, the resolution of aleatory uncertainty on basic event j leads to certainty about the system

state (it is failed), independently of the aleatory uncertainty on the remaining components' failure probabilities and of the state-of-knowledge uncertainty on the model parameters. Note that the distribution of U collapses into a Heaviside function centered at 1 (its density, $f_U(u)$, is a Dirac-δ function centered at unity.) Eq. (28) then states that, for $ERAW_j$ to equal RAW_j it suffices that $\mathcal{U} = U^{nom}$, without requirements on the conditional failure distributions. In other words, for single points of failure, assumptions 1, 2 in Proposition 1 suffice for $ERAW_j = RAW_j$.

For parallel systems, the following holds.

Proposition 3 *Consider a system made of n components in parallel. Let*[4]

$$q_n = P(\varphi_n = 1|\varphi_{n-1} = 1, ..., \varphi_j = 1),\ q_{n-1} = P(\varphi_{n-1} = 1|\varphi_{n-2} = 1, ..., \varphi_j = 1),\ etc. \tag{29}$$

Then, in general:

1.
$$ERAW_j = \frac{\mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[q_n q_{n-1} ... q_{i+1}, q_{i-1}, ..., q_1]}{\mathbb{E}_{\mathbf{\Lambda}}[q_n q_{n-1} ... q_{i+1}, q_i, q_{i-1}, ..., q_1]} \tag{30}$$

2. *If* $f_{\mathbf{\Lambda}|\varphi_j=1}(\boldsymbol{\lambda}|\varphi_j = 1) = \prod_{s=1, s\neq j}^{n} f_s(\lambda_s)$ *then*

$$ERAW_j = \frac{1}{\mathbb{E}[q_j]} \tag{31}$$

3. *If* $\mathbb{E}[q_j] = q_j^{nom}$, *then* $ERAW_j^{Parallel} = RAW_j^{Parallel}$

Proof. Item 1. In a parallel system, it is

$$\Psi = \prod_{i=1}^{n} \varphi_i \tag{32}$$

Thus,

$$P(\Psi = 1) = P(\bigcap_{s=1}^{n} \varphi_s = 1) = P(\varphi_n = 1|\varphi_{n-1} = 1, ..., \varphi_j = 1)...P(\varphi_1 = 1|\varphi_j = 1)P(\varphi_j = 1) \tag{33}$$

$$P(\Psi = 1|\varphi_j = 1) = P(\varphi_n = 1|\varphi_{n-1} = 1, ..., \varphi_j = 1)...P(\varphi_1 = 1|\varphi_j = 1) \tag{34}$$

Recalling that $U = P(\Psi = 1)$ and $U_j^+ = P(\Psi = 1|\varphi_j = 1)$, we have

$$\mathbb{E}_{\mathbf{\Lambda}}[U] = \mathbb{E}[q_n q_{n-1} ... q_{i+1}, q_{i-1}, ..., q_1, q_i] \tag{35}$$

$$\mathbb{E}_{\mathbf{\Lambda}}[U|\varphi_j = 1] = \mathbb{E}[q_n q_{n-1} ... q_{i+1}, q_{i-1}, ..., q_1] \tag{36}$$

[4] In this case, because we have a unique MCS, the notation with the double index q_{i_k} becomes redundant. In fact, it is always $i = 1$. We then switch to a 1 subscript notation.

Thus, eq. (30) is proven.

Item 2. If $f_{\boldsymbol{\Lambda}|\varphi_j=1}(\boldsymbol{\lambda}|\varphi_j=1) = \prod_{s=1,s\neq i}^{n} f_{s|\varphi_j=1}(\lambda_s|\varphi_j=1)$, then

$$RAW_j = \frac{\mathbb{E}_{\boldsymbol{\Lambda}}[q_n q_{n-1}...q_{i+1}, q_{i-1}, ..., q_1]}{\mathbb{E}_{\boldsymbol{\Lambda}}[q_n q_{n-1}...q_{i+1}, q_{i-1}, ..., q_1]\mathbb{E}[q_i]} = \frac{1}{\mathbb{E}[q_i]} \tag{37}$$

Item 3 then follows by $\mathbb{E}[q_i] = q_i^{nom}$ and by eq. (31). ■

Item 1 in Proposition 3 reports the generic expression of ERAW for a parallel system. Item 2 specializes this expression in the case the conditional epistemic distribution is a product distribution. Item 3 adds the usual assumption that the nominal value is selected as reference value (expected value) around which the distribution of q_i is assessed.

In the next section, we illustrate the discussion and results obtained sofar by means of an illustrative example.

5 An Illustrative Example

This section describes the results and concepts introduced sofar by means of an illustrative 3-component-in-parallel system. Throughout the example, we also try and evidence the role of assumptions about stochastic and state-of-knowledge (in)dependencies.

The structure function of the system is $\Psi = \varphi_1\varphi_2\varphi_3$. Then, the system unreliability is

$$U = P(\Psi = 1) = P(\varphi_1 = 1 \cap \varphi_2 = 1 \cap \varphi_3 = 1) \tag{38}$$

Case 1: no epistemic uncertainty and stochastic dependencies. Eq. (38) becomes

$$U = P(\Psi = 1) = P(\varphi_3 = 1|\varphi_2 = 1, \varphi_1 = 1)P(\varphi_2 = 1|\varphi_1 = 1)P(\varphi_1 = 1) \tag{39}$$

If there is no epistemic uncertainty, then the decision-maker is certain about the values of the failure probabilities. For simplifying the notation a bit, we assume failure of component 3 to be stochastically independent of the failures of components 1 and 2 in the next computations. Hence, eq. (39) becomes:

$$U = P(\varphi_3 = 1)P(\varphi_2 = 1|\varphi_1 = 1)P(\varphi_1 = 1) \tag{40}$$

Setting the nominal values of the failure probabilities $q_1^{nom} = P(\varphi_1 = 1) = 1/3$, $q_2^{nom} = P(\varphi_2 = 1) = 2/7$ and $q_3^{nom} = P(\varphi_3 = 1) = 1/4$, $q_2^{+} = P(\varphi_2 = 1|\varphi_1 = 1) = 3/7$ we obtain $U^{nom} = 0.036$.

This set of probabilities also allows us to compute RAW. We need to compute the numerator in eq. (1) [the denominator is U^{nom}]. The value of the risk metric computed with $\varphi_1 = 1$ is given by: $U_1^{+} = P(\Psi = 1|\varphi_1 = 1) = P(\varphi_3 = 1)P(\varphi_2 = 1|\varphi_1 = 1)$. Hence, we have

$$RAW_1 = \frac{P(\Psi = 1|\varphi_1 = 1)}{P(\Psi = 1)} = \frac{P(\varphi_3 = 1)P(\varphi_2 = 1|\varphi_1 = 1)}{P(\varphi_3 = 1)P(\varphi_2 = 1|\varphi_1 = 1)P(\varphi_1 = 1)} = \frac{1}{q_1^{nom}} = 3 \tag{41}$$

Similarly, one obtains

$$RAW_2 = \frac{P(\Psi = 1|\varphi_2 = 1)}{P(\Psi = 1)} = \frac{P(\varphi_3 = 1)P(\varphi_1 = 1|\varphi_2 = 1)}{P(\varphi_3 = 1)P(\varphi_1 = 1|\varphi_2 = 1)P(\varphi_2 = 1)} = \frac{1}{q_2^{nom}} = \frac{7}{2} \tag{42}$$

$$RAW_3 = \frac{P(\Psi = 1|\varphi_3 = 1)}{P(\Psi = 1)} = \frac{P(\varphi_2 = 1|\varphi_1 = 1)P(\varphi_1 = 1)}{P(\varphi_3 = 1)P(\varphi_1 = 1|\varphi_2 = 1)P(\varphi_2 = 1)} = \frac{1}{q_3^{nom}} = 4 \tag{43}$$

Eqs. (41)-(43) indicate that component 3 is associated with the highest RAW, followed by components 2 and 1 [see also eq. (10)]

Case II: failure rates. In case I, the analysis is carried out at the basic event level of the reliability model. In the practice, analysts utilize additional models for assessing the failure probabilities. One of the most familiar is the exponential model in which $q_i = 1 - e^{-\lambda_i T}$, where λ_i is the failure rate corresponding to q_i. For our example, we would write $q_1 = 1 - e^{-\lambda_1 T}$, $q_2^+ = 1 - e^{-\lambda_2^+ T}$ and $q_3 = 1 - e^{-\lambda_3 T}$. Letting $T = 1$ for simplicity, we obtain

$$U = (1 - e^{-\lambda_1})(1 - e^{-\lambda_2^+})(1 - e^{-\lambda_3}) \tag{44}$$

If the analyst wishes to retain the same numerical value of U^{nom}, she needs to select the failure rates [see also Section 3] from

$$-ln(1 - q_i^{nom}) = \lambda_i^* \tag{45}$$

In our case, we have $\lambda_1^* = 0.405$, $\lambda_2^* = 0.336$, $\lambda_3^* = 0.290$ and $\lambda_2^{+*} = 0.560$. This assignment also insures invariance of the component RAW's. In fact,

$$RAW_1 = \frac{P(\Psi = 1|\varphi_1 = 1)}{P(\Psi = 1)} = \frac{(1 - e^{-\lambda_2^{+*}})(1 - e^{-\lambda_3^*})}{(1 - e^{-\lambda_1^*})(1 - e^{-\lambda_2^{+*}})(1 - e^{-\lambda_3^*})} = \frac{1}{1 - e^{-\lambda_1^*}} = 3 \tag{46}$$

One obtains similar results for components 2 and 3.

Case III: epistemic uncertainty, state-of-knowledge independence and separability. Cases I and II have discussed the computation of U and RAW in the absence of epistemic uncertainty. Suppose the decision-maker is uncertain about the values of the parameters and wishes to take such epistemic uncertainty into consideration. Let $f_{\mathbf{\Lambda}}(\lambda_1, \lambda_2, \lambda_2^+, \lambda_3)$ denote the joint density of the parameters. By eq. (19), the unconditional solution of the problem is

$$\mathcal{U} = \int\int\int\int (1 - e^{-\lambda_1})(1 - e^{-\lambda_2^+})(1 - e^{-\lambda_3}) f_{\mathbf{\Lambda}}(\lambda_1, \lambda_2, \lambda_2^+, \lambda_3) d\lambda_1 d\lambda_2 d\lambda_2^+ d\lambda_3 \tag{47}$$

Eq. (47) leads to $\mathcal{U} \neq U^{nom}$, in general. However, Lemma 1 indicates the conditions under which $\mathcal{U}$ coincides with U^{nom}. Let us explore them. First, the analyst needs to assume state-of-knowledge independence. Setting

$$f_{\mathbf{\Lambda}}(\lambda_1, \lambda_2, \lambda_2^+, \lambda_3) = f_1(\lambda_1) f_2(\lambda_2) f_{2^+}(\lambda_2^+) f_2(\lambda_3) \tag{48}$$

Because eq. (44) is separable, eq. (47) becomes

$$\mathcal{U} = \mathbb{E}_\Lambda[q_1]\mathbb{E}_\Lambda[q_2^+]\mathbb{E}_\Lambda[q_3] \tag{49}$$

As it is often the case in risk assessment of complex systems, suppose she selects lognormal distributions. For clarity, the lognormal densities are denoted by

$$f_i(\lambda_i) = \frac{1}{\lambda_i \varsigma_i \sqrt{2\pi}} e^{-\frac{(\ln(\lambda_i) - \mu_i)^2}{2\varsigma_i^2}} \tag{50}$$

where μ_i and ς_i are the parameters of the lognormal distribution. If the analyst selects the parameters of the lognormal distributions so that the expected value of the parameters is equal to their nominal value, $[\mathbb{E}[\Lambda_i] = \lambda_i^{nom}$ [5]], she obtains $\mathcal{U} = 0.012$. Thus, $\mathcal{U} \neq U^{nom}$, even if state-of-knowledge independence and separability are assumed. The difference is motivated by the fact that assumption 3 in Lemma 1 is not respected by the assignment $\mathbb{E}[\Lambda_i] = \lambda_i^{nom}$. In fact, $\mathbb{E}[\Lambda_i] = \lambda_i^{nom}$ is equivalent to imposing

$$\lambda_i^{nom} = \int \frac{1}{w \varsigma_i \sqrt{2\pi}} e^{-\frac{(\ln(w) - \mu_i)^2}{2\varsigma_i^2}} dw, \tag{53}$$

while the assignment $\mathbb{E}_\Lambda[q_i] = q_i^{nom}$ requires

$$q_i^{nom} = \int \frac{(1 - e^{-w})}{w \varsigma_i \sqrt{2\pi}} e^{-\frac{(\ln(w) - \mu_i)^2}{2\varsigma_i^2}} dw \tag{54}$$

By choosing the parameters μ_i and ς_i so eq. (54) is satisfied, all assumptions in Lemma 1 hold, and $\mathcal{U} = U^{nom}$.

Let us now come to the assessment of ERAW. We start with component 1. By eq. (21), we have $\mathcal{U}_1^+$:

$$\mathcal{U}_1^+ = \mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}[P(\varphi_3 = 1)P(\varphi_2 = 1|\varphi_1 = 1)] \tag{55}$$

Here, the notation $\mathbb{E}_{\mathbf{\Lambda}|\varphi_j=1}$ implies that we have to consider the distribution of the parameters after the decision-maker has received information that $\varphi_1 = 1$. In a Bayesian framework, given evidence $E = (\varphi_1 = 1)$, the density of any λ_s is updated according to:

$$f_{s|\varphi_j=1}(\lambda_s|\varphi_j = 1) = \frac{L(\varphi_j = 1|\lambda_s) f_s(\lambda_s)}{\int L(\varphi_s = 1|r) f_s(r) dr} \tag{56}$$

where $L(\varphi_j = 1|\lambda_s)$ is the likelihood of $\varphi_j = 1$ given λ_s. Thus, $f_{s|\varphi_j=1}(\lambda_s|\varphi_j = 1) \neq f_s(\lambda_s)$.

[5] The corresponding parameters of the lognormal distributions are found solving

$$\mu_i = \ln[(\lambda_i^{nom})^2 / \sqrt{1 + (\lambda_i^{nom})^2}] \tag{51}$$

$$\zeta_i = \sqrt{\ln[1/(\lambda_i^{nom})^2 + 1]} \tag{52}$$

where we have set the variance of Λ_i equal to unity.

We emphasize that this step is not usually performed in practice. Updates of distributions follow information and data collection at the level of the whole model and are not tied to the computation of RAW. Thus, for practical purposes $f_{s|\varphi_j=1}(\lambda_s|\varphi_j = 1) = f_s(\lambda_s)$, $\forall s \neq i$. From a degree-of belief viewpoint, setting $f_{s|\varphi_j=1}(\lambda_s|\varphi_j = 1) = f_s(\lambda_s)$ is equivalent to assuming that the event $\varphi_j = 1$ has a negligible effect on the decision-maker's state-of-knowledge of λ_s, $(s \neq i)$. Under these conditions, all assumptions in Proposition 1 are verified and we obtain $ERAW_1 = RAW_1$, $ERAW_2 = RAW_2$ and $ERAW_3 = RAW_3$.

Note that in the above discussion it has not been necessary to invoke stochastic-independence.

To now, in this section, we have carried out a search of the conditions under which RAW and ERAW produce the same ranking. The reason is as follows: if the assumptions of Proposition 1 are satisfied, then results obtained at the nominal value of the probabilities hold also in the presence of epistemic uncertainty, no matter what functional form is assigned to distributions of the parameters. Thus, a decision-maker is insured that the nominal RAW ranking is robust and does not need to propagate epistemic uncertainty.

However, the assumptions in Proposition 1 are restrictive. For instance, λ_2 and λ_2^+ are, naturally, expected to be highly correlated. Their independence is artificially used in this example for meeting all conditions under which Proposition 1 holds. The hypotheses of Proposition 1 are readily violated in realistic applications and we expect $ERAW_j \neq RAW_j$. The next two cases illustrate this point.

Case IV: complete state-of-knowledge dependence. The clearest example is the case of identical components. In this case, as noted by Apostolakis and Kaplan (1981), a unique failure rate has to be utilized. In our example, if one considers that components 1 and 3 are identical, then one needs to assign $\lambda_1 = \lambda_3$. Eq. (44) becomes:

$$U = (1 - e^{-\lambda})^2(1 - e^{-\lambda_2^+}) \tag{57}$$

whence,

$$\mathcal{U} = \mathbb{E}[(1 - e^{-\lambda})^2]\mathbb{E}[(1 - e^{-\lambda_2^+})] \tag{58}$$

Because $\mathbb{E}[(1-e^{-\lambda})^2] \neq \mathbb{E}[(1-e^{-\lambda})]^2$, $\mathcal{U} \neq U^{nom}$. Note that assumption 1 (separability) in Lemma 1 is violated in eq. (57). Furthermore, we have

$$ERAW_1 = \frac{\mathbb{E}[(1 - e^{-\lambda})]\mathbb{E}[(1 - e^{-\lambda_2^+})]}{\mathbb{E}[(1 - e^{-\lambda})^2]\mathbb{E}[(1 - e^{-\lambda_2^+})]} = \frac{\mathbb{E}[(1 - e^{-\lambda})]}{\mathbb{E}[(1 - e^{-\lambda})^2]} \tag{59}$$

Eq. (59) leads to values of $ERAW_1$ generally different from the nominal RAW_1. The reason is that both assumptions 1 and 2 in Proposition 1 are violated. Similar considerations hold for $ERAW_2$ and $ERAW_3$.

Case V: common-cause failures. A very important class of dependencies in reliability analysis is generated by the presence of common-cause failures. The literature on common-cause failure modelling is vast and a thorough discussion cannot be comprised within the scope of the present

paper. We refer to the works by Apostolakis and Moieni (1987), Mosleh (1991), Sagan (2004), Paté-Cornell et al. (2004), and the review in Hoepfer et al. (2009). We illustrate the case of the β-factor model. Assuming that our system is made of are three identical redundant components, using the β-factor model one writes:

$$U = 1 - ([1 - (1 - e^{-\lambda})^3](e^{-\frac{\beta\lambda}{1-\beta}})) \tag{60}$$

U in eq. (60) is a composite multilinear function of the parameters, but it is not separable anymore. More specifically, for the β-factor model of n identical components, note that $\mathbb{E}[(1-e^{-\lambda})^n(e^{-\frac{\beta\lambda}{1-\beta}})]$ cannot be split into $\mathbb{E}[(1-e^{-\lambda})^n] \cdot \mathbb{E}[(e^{-\frac{\beta\lambda}{1-\beta}})]$ even if λ and β are assumed independent, because of the product term $\beta\lambda$ in the second exponential of eq. (60). Hence, assumption 1 in Lemma 1 is violated and we expect $\mathcal{U} \neq U^{nom}$, in general. —

Similarly, the assumptions in Proposition 1 are violated. In general, when common-cause failures are factored into the reliability model, one expects $ERAW$ to differ from RAW. For our example, we have

$$ERAW_1 = ERAW_2 = ERAW_3 = \frac{\mathbb{E}[(1-e^{-\lambda})^2(e^{-\frac{\beta\lambda}{1-\beta}})]}{\mathbb{E}[(1-e^{-\lambda})^3(e^{-\frac{\beta\lambda}{1-\beta}})]} \tag{61}$$

Eq. (61) shows clearly that $ERAW_j \neq RAW_j$.

This section has had the purpose of describing our findings through a simple example. In the next section, we discuss ERAW, Lemma 1 and Proposition 1 for a realistic application.

6 Application to a Space PSA Model

This section describes the application of the previous findings and methodology in the context of a realistic decision-making application: providing risk-informed insights in the design of a lunar space mission. The model is presented synthetically below and we refer to Borgonovo and Smith (2011) for further details.

After the Space Shuttle accident in 2002, NASA has adopted probabilistic safety assessment (PSA) as part of its risk management procedures. The PSA model utilized in this work has been built to support decision-making in the safety assessment of the next generation of lunar space missions, in accordance with NASA's Probabilistic Risk Assessment Procedures Guide [Stamatelatos et al. (2002)]. Mission success is evaluated based on two attributes: safety and performance. The top events loss of crew (LOC) and loss of mission (LOM) are considered as proxy measures of safety and performance, respectively. LOC refers to conducting the mission safely and returning the crew safely to earth. LOM refers to successfully (or not) carrying out the lunar activities. Each key part of the mission is modelled via a fault tree, whose top-event represents either LOM or LOC. The fault trees contain the details of what has to fail in order to cause either LOM or LOC at each

phase. As observed in Borgonovo and Smith (2011), LOM and LOC are not mutually exclusive.

Figure 2 shows the main event tree utilized to model the mission.

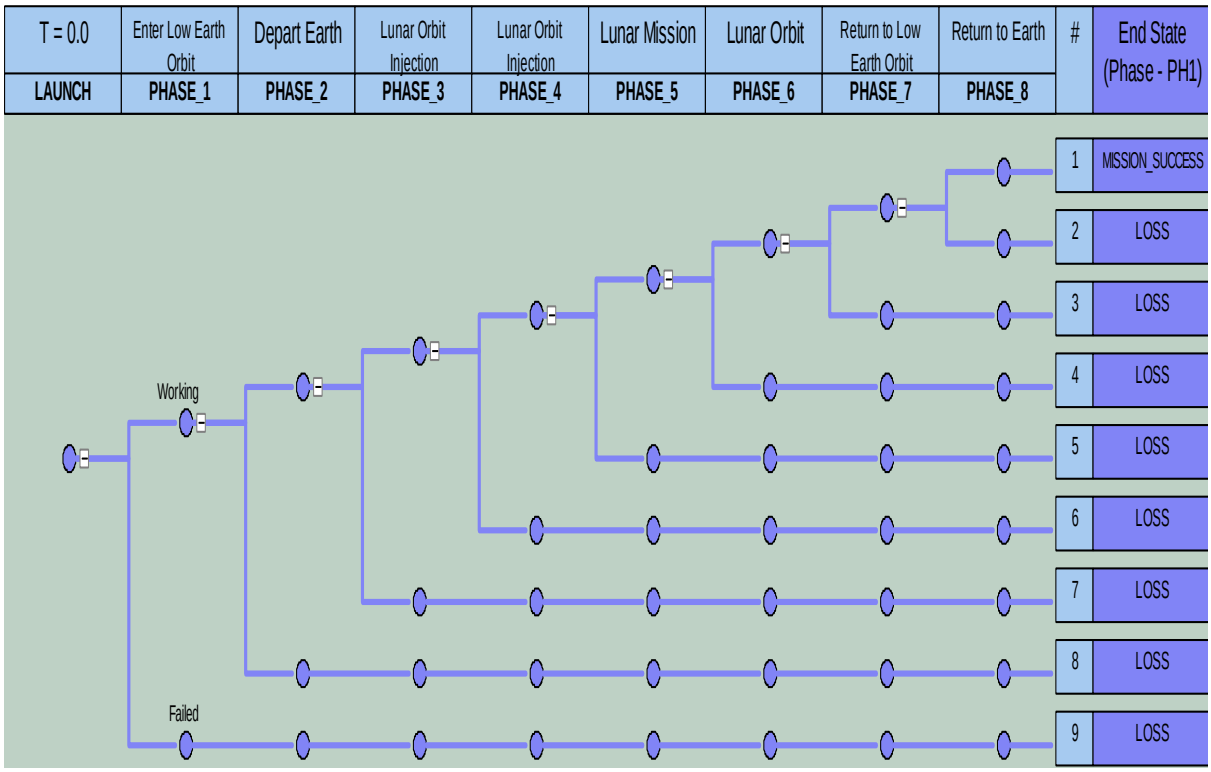

Figure 2: Mission phases for the lunar mission event tree.

The mission is divided into eight phases, from launch to earth return (Figure 2). Based on the Exploration Systems Architecture Study [NASA (2005)], NASA vehicles are composed of systems that perform 7 primary functions [see also Borgonovo and Smith (2011)]. These are: *i*) Propulsion, which comprises the main engine, the reaction control system, and the propellant tanks. *ii*) Avionics, that controls navigation, provides guidance, and performs internal state calculations receiving inputs from the crew, the sensors and external communications. *iii*) Power Supply, which includes batteries, solar arrays, electrical distribution and control subsystems. *iv*) Thermal Control, which is responsible for heat removal from various components and comprises heaters, coolers, condensate controller and mechanical equipment. *v*) Environmental Control and Life Support System, which comprises oxygen tanks, pressure regulators, sensors and mechanical equipment. *vi*) The Launch Abort System, which monitors the conditions under which the mission is aborted in the early stages. *vii*) Pyrotechnic devices, that determine component separations. In particular, the model considers 13 systems for performing these functions, which are: (1) the Active Thermal Control System, (2) The Crew Module in Orbit, (3) Crew Recovery, (4) Lunar Surface Access Module, (5) Parachutes, (6) Crew Launch Vehicle, (7) Crew Module, (8) Environmental Control and Life Support, (9) Electric Power System, (10) Launch Abort System, (11) Lunar Surface Access Module, (12) Service

Module, and (13) Structures. These systems are intrinsically interconnected and dependencies are present.

150 fault trees are developed for modelling each system at each phase of the mission. The model contains 872 basic events. The number of minimal cut sets is around 4500 at a truncation of 10^{-15}. Dependencies and common-cause failures are accounted for, these last ones using the Multiple Greek Letter method [Mosleh (1991)]. As customary in probabilistic safety assessment, lognormal state-of-knowledge distributions are assigned by analysts to the uncertain variables which, for this model, are the basic event probabilities. The parameters of the distributions are either fitted from data or elicited through expert opinion. These distributions are directly embedded in the SAPHIRE code that has been made available to the authors.

Given the model complexity, it is not possible to obtain $ERAW_j$ analytically. However, the following steps can be utilized for a numerical computation.

1) Generation of an unconditional sample of size N of the parameters, here denoted by $\widehat{\Lambda}$.

2) Evaluation of the risk metric at $\widehat{\Lambda}$, obtaining the vector of estimates $\widehat{\mathbf{U}}$ of size $1 \times N$.

3) Calculation of $\mathcal{U}$ via

$$\widehat{\mathcal{U}} = \frac{\sum_{r=1}^{N} \widehat{U}^r}{N} \tag{62}$$

where $\widehat{U}^r$ is the value of the risk metric realized in Monte Carlo run r, $r = 1, 2, ..., N$.

4) Calculation of $\mathcal{U}^+$ by setting of the Boolean variable of basic event j equal to unity as:

$$\widehat{\mathcal{U}}_j^+ = \frac{\sum_{r=1}^{N} \widehat{U}_j^{+,r}}{N} \tag{63}$$

where $\widehat{U}_j^{+,r}$ $(r = 1, 2, ..., N)$ is values of U obtained with $\varphi_j = 1$ in a generic Monte Carlo run of the repeated simulation. $\widehat{\mathbf{U}}_j$ denotes the corresponding vector of risk metric values. If dependencies are present, $\widehat{\mathbf{U}}_j$ is obtained by evaluating the model at a new sample, here denoted by $\widehat{\mathbf{\Lambda}}_j$, obtained by setting the proper conditional distributions of the parameters.

5) Estimation of RAW_j from:

$$\widehat{RAW_j} = \frac{\widehat{\mathcal{U}}_j^+}{\widehat{\mathcal{U}}} \tag{64}$$

Overall, $n \times N$ model evaluations are necessary to estimate the ERAW of n basic events following these steps.

For the present application, we utilize $N = 10000$ and apply the method to the risk metric of the NASA space PSA model described above. The overall time of the analysis is around 10 minutes on a personal PC[6].

Figure 3 displays the empirical CDFs of the risk metric obtained by applying steps 1-4. The unconditional distribution, $F_U(u)$, is plotted as a red and thick line. $F_U(u)$ in Figure 3 is the empirical CDF associated with $\widehat{\mathbf{U}}$. $F_{U|\varphi_j=1}(u)$ denotes a generic empirical CDFs associated with $\widehat{\mathbf{U}}_i$, namely the values of the risk metric obtained from the conditional sample $\widehat{\mathbf{\Lambda}}_i$ (Step 4). $\widehat{\mathbf{U}}_i$ is

[6] The PC runs on Intel Core Duo Cpu, 2.4GHz, 2.39GB RAM.

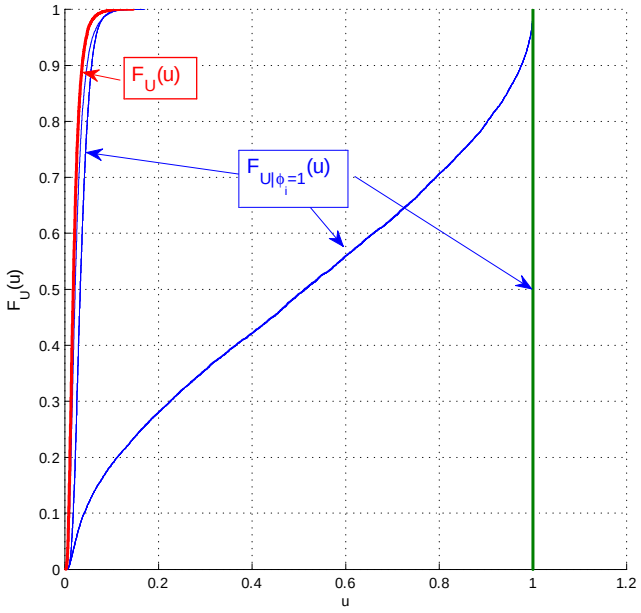

Figure 3: Unconditional distribution (thick) $[F_U(u)]$ and conditional distributions $[F_{U|\varphi_i=1}(u)]$ of the risk metric for the NASA PSA application.

then summarized in $\widehat{\mathcal{U}}_i^+$, in accordance with eq. (63), which is the numerical implementation of eq. (20).

Figure 3 confirms the discussion carried out in Sections 1-3, showing that, in the presence of epistemic uncertainty, the effect of getting to know $\varphi_j = 1$ is a modification in the distribution of U. One can also visually appreciate that some basic events leads to a slight modification of $F_U(u)$, while some basic events lead to notable discrepancies. In particular, single points of failure turn $F_U(u)$ into a Heaviside function (green vertical lines in 3; see Proposition 2 and the subsequent discussion).

Figure 4 displays the ERAW's of the 872 basic events.

We note that 393 basic events are associated with an ERAW greater than unity, with the remaining associated with ERAW equal to 1. $RAW_j = 1$ indicates that these basic events are not important as they leave the risk metric unchanged. This result is consistent with the findings in Borgonovo and Smith (2011), where 393 active basic events are registered. Among these, 51 basic events are single points of failure.

Before detailing the managerial insights of these results, let us compare the ranking induced by $ERAW$ and RAW. To do this, we perform a first numerical experiment assuming separability and independence — this is achieved by assigning $F_Q(\mathbf{q}) = \prod_{i=1}^{872} F_i(q_i)$; in other words, one samples 872 independent failure probabilities.— This experiment is performed to test Proposition 1. In fact, under the state-of-knowledge assumption, all hypothesis of Proposition 1 are verified for this

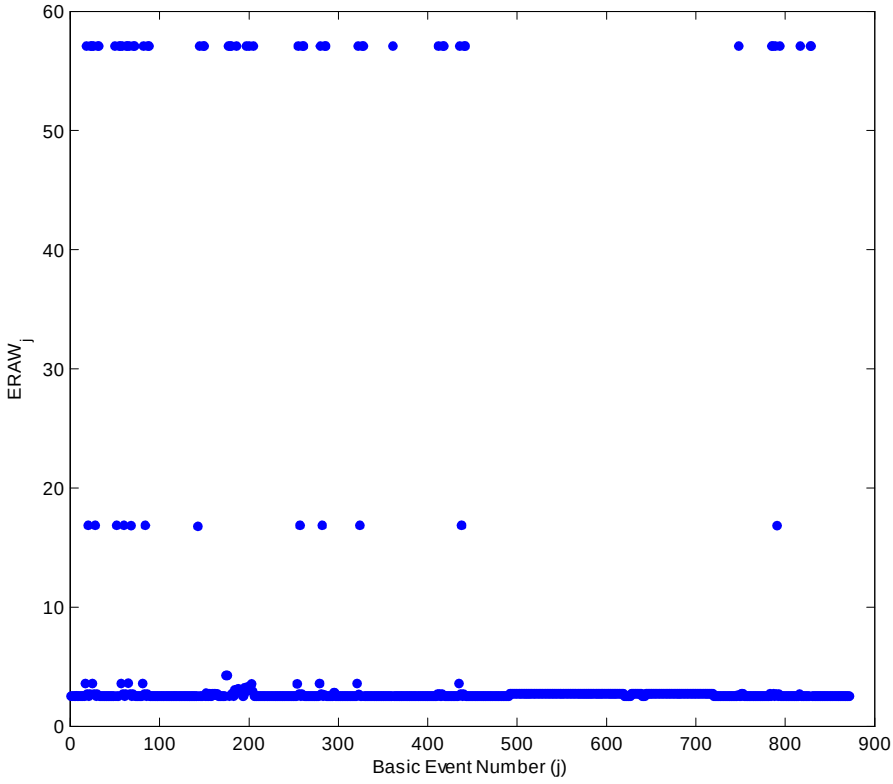

Figure 4: $ERAW_j$ estimated through the proposed algorithm for the 872 basic events. The numbers on the vertical axis are multiplied by a factor $\alpha > 0$ for privacy.

Table 1: Comparison of ERAW and RAW in the presence (abscence) of state-of-knowledge dependencies.

i	1	2	4	5	17	24
RAW	1.0003	1.0047	1.0144	1.0	1.5241	10.5953
ERAW independence	1.0003	1.0046	1.0151	1.0	1.5202	10.5176
ERAW dependence	1.0005	1.0070	1.0090	1.0	1.7878	15.9273

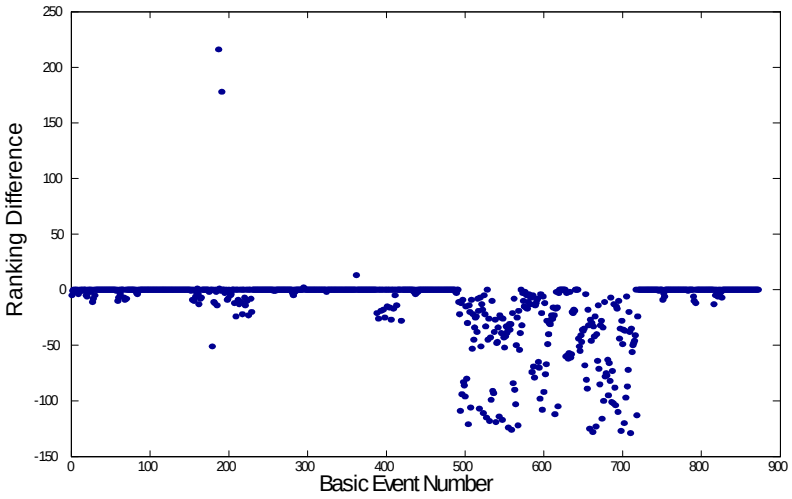

Figure 5: Differences in ranking between RAW and ERAW in the presence of state-of-knowledge dependencies.

model. Thus, we should obtain $ERAW = RAW$. The RAWs and ERAWs of representative basic events for this experiment are displayed in Table 1.

The first two lines in Table 1 confirm that, apart for numerical rounding, $ERAW = RAW$ under state-of-knowledge independence. However, for the present model, the state-of-knowledge independence is conceptually incorrect and exposes us to the pitfalls signalled by Apostolakis and Kaplan (1981). Therefore, we also performed experiments giving full credit to epistemic dependencies. The third line in Table 1 shows the results of these calculations. As expected, the values of RAW and ERAW now differ. Let us then investigate whether these differences have repercussions on the basic event ranking. The discrepancies are displayed in Figure 5.

Figure 5 shows the following:

- The majority of basic events do not undergo ranking changes. In fact, the 479 inactive basic event have both RAW and ERAW equal to unity. The 51 single points of failure rank 1^{st}, both with RAW and ERAW;

- Of the remaining 342 basic events, however, 322 undergo a ranking change. The average change is -13 positions. Nonetheless, some basic events undergo notable ranking changes.

System	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of Basic Events	142	8	6	10	18	11	216	8	48	301	9	38	57
Number of Single Points of Failure	17	3	0	0	3	1	13	3	3	0	1	4	3

Table 2: Number of single points of failure per system.

In order to assess the ranking agreement between RAW and ERAW, we utilize the Savage Score correlation coefficients. The methodology has been introduced in Iman and Conover (1987) to provide a systematic way for quantifying the agreement between ranking obtained through different criteria (see also Borgonovo et al. (2010)). Denoting the rank of basic event i by R_i, the associated Savage score is defined as:

$$SS_i = \sum_{v=R_i}^{n} \frac{1}{v} \tag{65}$$

For instance, with $n = 872$, the Savage score of the basic event ranking first is 7.35. We denote by $\mathbf{R}^{RAW}$ and $\mathbf{R}^{ERAW}$ the ranking induced by RAW and ERAW, respectively. Let $\rho_{\mathbf{R}^{RAW},\mathbf{R}^{ERAW}}$ the correlation coefficient of the raw ranks and $\rho_{\mathbf{SS}^{RAW},\mathbf{SS}^{ERAW}}$ the correlation coefficient of the corresponding Savage scores. Comparing $\rho_{\mathbf{SS}^{RAW},\mathbf{SS}^{ERAW}}$ against $\rho_{\mathbf{R}^{RAW},\mathbf{R}^{ERAW}}$ conveys information on whether agreement (or disagreement) is at the level of the problem key-drivers. In fact, SS_i emphasizes high-ranked factors. In our case, we obtain $\rho_{\mathbf{SS}^{C_1},\mathbf{SS}^{C_2}} = 0.99 > \rho_{\mathbf{R}^{RAW},\mathbf{R}^{ERAW}} = 0.97$. This result signals that ranking disagreement between RAW and ERAW is at the level of the non-relevant factors. This is reasonable, insofar the key-drivers are the 51 single points of failure, which, both with RAW and ERAW, assume the maximum RAW and ERAW values.

From an operational viewpoint, these results show risk analysts that the ranking of the most important factors obtained with RAW is robust to epistemic uncertainty. For the non-relevant factors, however, an item-by-item inspection is necessary to check whether the classification obtained with nominal values is consistent with the one obtained including epistemic uncertainty in the values of the risk measures.

ERAW results shed light on the nature of system under consideration. In fact, the model addresses the reliability of a space vehicle, namely, a rocket. Space-vehicle-design is subject to severe trade-offs for achieving the desired level of safety under tight structural constraints. For instance, weight limits pose restrictions on the number of redundant systems that can be added. This mark the main operational distinction between the system under investigation and systems such as energy production plants, and especially nuclear reactors. These can count on a much higher level of redundancy and are characterized by a lower number of single points of failure.

The question is then whether any of the 13 modelled systems is particularly vulnerable to single basic event occurrences. The answer is obtained by the examination of how single points of failure are distributed among the systems. We obtain the results reported in Table 2.

Table 2 shows that the Crew Module in Orbit (2) and the Environmental Control and Life Support systems are the most vulnerable, because 3 out of their 8 basic events individually can determine their failure. For the Active Thermal Control System (1), 17 out of the 142 basic events

are single points of failure, while for the Crew Module (7), 13 out of 216 basic events are single points of failure. Conversely, Crew Recovery (3), Lunar Surface Access Module (4) and Launch Abort System (10) show some level of redundancy, because they are not exposed to single failures.

The information in Table 2, which is obtained by giving full credit to state-of-knowledge uncertainty, points analysts to the systems that would deserve additional redundancies for improving safety, compatibly with design constraints. This information which is particularly relevant given the difficulty of adding redundancy to a system as the one analyzed in this application.

7 Conclusions

In this work, we have addressed the computation of Risk Achievement Worth (RAW) in the presence of epistemic uncertainty.

Available literature [Modarres and Agarwal (1996)] has shown that uncertainty in reliability model parameters induces variability in importance measure values, and therefore in RAW, which can prevent risk analysts from confidently entrusting the nominal value ranking. We have addressed the problem studying an extension of the definition of RAW that makes it robust with respect to epistemic uncertainty. We have called this extension ERAW.

The investigation of the conditions under which ERAW coincides with RAW has lead us to a general result concerning the assumptions under which the state-of-knowledge expected value of a reliability risk metric equals its nominal value. In details, because the unreliability (reliability) function of any system (coherent or non-coherent) is multilinear in the basic-event probabilities, then if: a) separability of the multilinear function and b) state-of-knowledge independence hold, the nominal value of the risk metric coincides with it state-of-knowledge expectations. Under these conditions, any nominal-value analysis is robust to epistemic uncertainty.

We have discussed in detail the relevance of stochastic and state-of-knowledge dependencies in the computation of RAW and ERAW by application to a three-component-in-parallel system.

The findings have then been discussed in application to a realistic risk analysis model, namely, the PSA model developed for the safety analysis of NASA next generation lunar space missions. We have illustrated the information delivered by ERAW. The analysis has registered the presence of several single points of failure, revealing a characteristic feature of the system under study. In space vehicles a limited use of redundancy is imposed by design constraints (mainly mass restrictions). We have seen that ERAW and RAW ranking differ, when state-of-knowledge dependencies are factored into the analysis. We have then discussed how analysts can benefit from the ranking comparison to obtain robust information about the risk achievement worth of basic events.

Finally, an observation on future research. The fact that component failure or basic event occurrence (E) changes the state-of-knowledge distribution of U can be used to obtain information about the effect of E on additional properties of U which are of interest to the decision-maker as, for instance, its variance, percentiles or, even, the entire state-of-knowledge distribution.

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