Differential, Criticality and Birnbaum Importance Measures: an Application to Basic Event, Groups and SSCs in Event Trees and Binary Decision Diagrams

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Abstract

Recent works [Epstein and Rauzy (2005)] have questioned the validity of traditional fault tree/event tree (FTET) representation of probabilistic risk assessment problems. In spite of whether the risk model is solved through FTET or binary decision diagrams (BDDs), importance measures need to be calculated to provide risk managers with information on the risk/safety significance of system structures and components (SSCs). In this work, we discuss the computation of the Fussel-Vesely (FV), Criticality, Birnbaum, Risk Achievement Worth (RAW) and Differential Importance Measure (DIM) for individual basic events, basic event groups and components. For individual basic events, we show that these importance measures are linked by simple relations and that this enables to compute basic event DIMs both for FTET and BDD codes without additional model runs. We then investgate whether/how importance measures can be extended to basic event groups and components. Findings show that the estimation of a group Birnbaum or Criticality importance is not possible. On the other hand, we show that the DIM of a group or of a component is exactly equal to the sum of the DIMs of the corresponding basic events and can therefore be found with no additional model runs. The above findings hold for both the FTET and the BDD methods.

Keywords: Importance Measures, Probabilistic Risk Assessment, Event Trees, Binary Decision Diagrams.

1 Introduction

Importance measures are used by risk managers to derive information about the risk/safety significance of events, and systems structures and components (SSCs). This information is crucial in risk-informed applications both from the regulatory and the industry perspectives (Cheok *et al* (1998) [6], Vesely (1998) [14], Brewer and Canady (1999) [5], Borgonovo and Apostolakis (2001) [2], Borgonovo *et al* (2003) [3], Vinod (2003) [15].)

Importance measures utilized in the PRA realm are the Fussell-Vesely (FV), the Risk Achievement Worth (RAW), the risk reduction worth (RRW), the criticality importance factor (CIF) and the Birnbaum importance (BI) measure (Birnbaum (1969) [4], Cheok *et al* (1998) [6].) The investigation in Cheok *et al* (1998) [6] exposes several limitations in the flexibility of the above mentioned indicators in revealing the risk/safety significance of basic event groups. Borgonovo and Apostolakis (2001) [2] introduce the differential importance measure (DIM), which overcomes such limitations thanks to the additivity property. Vinod *et al* (2003) [15] present an application of DIM to risk informed inservice inspection.

Epstein and Rauzy (2005) [7] estimate the CIF, RAW and RRW importance measures of a nuclear power plant PRA. They obtain results for the same PRA model, first solved through a Fault Tree/Event Tree (FTET) method and then solved encoding the model in a Binary Decision Diagram (BDD) algorithm. The comparison shows notable discrepancies in the estimation of the importance indicators. Not only, but on a broader perspective, Epstein and Rauzy (2005) [7] question approximations made by traditional FTET codes as the rare event approximation or minimal cut set (MCS) truncation, arguing that better logical and numerical estimates can be achieved if calculations are based on BDDs. However, no matter the type of method with which a PRA is solved, to identify basic events or SSCs "*contribution to the risk* (Epstein and Rauzy (2005) [7])," importance measures need to be estimated, and it is this estimation which is the subject of the investigation of the present work.

We start by analyzing the relationships among FV, RAW, CIF, BI and DIM, illustrating that they are linked by manageable and simple expressions at the individual basic event level. These relationships allow us to show that one is capable of computing DIM with no additional model runs, from the knowledge of any of the other importance measures, independently of whether the software is utilizing the FTET or the BDD technology. Indeed, given a BDD or FTET software that estimates one or more importance measure for basic events, exploiting such relationships one can compute any of the other importance measures with no further model runs.

We next shift to the analysis of importance measures for basic event groups and components/SSCs. Findings are as follows. BI and CIF cannot be extended to multiple basic events, due to mathematical reasons. The FV importance measure and the RAW can be extended to groups, although RAW requires further model runs (see also Cheok *et al* (1998) [6]). DIM endeavours analysts with full flexibility when computing joint basic event importance and thanks to the additivity property it is possible to find the DIM of a group with no additional model runs. Since these results are linked to the definition and properties of the importance indicators, they hold both for the FTET and the BDD technologies, .

We then inquire whether/how the importance of a component can be derived from the importance of the group of basic events related to that component. The analysis shows that the DIM of a component coincides with the sum of the DIMs of the basic events related to the component, and therefore the DIM of a component can be found without additional model runs. However, it is not possible to infer a component CIF or BI from the corresponding set of basic event $CIFs$ or BIs, since they are not defined for multiple basic events.

We illustrate the above concepts by application to a numerical example, deriving the DIM of individual basic events, groups and component, both for the FTET and the BDD methods from the importance measures estimated in Epstein and Rauzy (2005).

The remainder of the work is organized as follows. In the next Section, we summarize the importance measures utilized in PRA, highlighting their interrelationships for individual basic events. In Section 3, we discuss the extension of the definition of importance measures to basic event groups and investigate the link between group and SSC importance. In Section

4, we provide numerical results and observations. Section 5 offers conclusions.

2 PRA Importance Measures: relevant relationships

From a sensitivity analysis point of view, importance measures belong to the family of local sensitivity indicators (Helton (1993) [8], Saltelli *et al* (2000) [11].)

The goal of an importance measure is to provide numerical guidance about the safety/risk significance of an SSC. With this respect, we need to recall that, as Apostolakis (2005) [1] underlines, results derived from the quantitative analysis ought not be the sole basis for a decision on the safety/risk relevance of an SSC, but they should be integrated by an expert panel peer review process.

PRA importance measures and their definitions are reported in Table 1. In Table 1, Y stands for the best estimate of the risk metric, X_j for the probability of basic event j, Y^{j+} for the value of the risk metric when basic event j is set to "True", and Y^{j-} when it is set to "False."

FV is defined as the fraction of the risk that is associated with the generic PSA element Z. Z can be a component, a system, or a basic event. For basic events, FV is defined as the ratio of probability of the union of all the minimal cutsets (MCS) containing basic event j, and the nominal value of the risk metric (Y) [Table 1.] The FV of a component is defined as the ratio of the probability of all the MCSs that contain basic events that belong to that component over the nominal risk. Analogously, a system FV is the ratio of the probability of all the MCSs that contain basic events that belong to the system of interest over the nominal risk.

RAW is defined to measure the risk that the system achieves when one of its component fails. It is defined as the ratio of Y^{j+}, the value of the risk metric when the Boolean variable associated to basic event j is set to "true". Thus, RAW_j is the ratio of the risk that is achieved when basic event j happens over the nominal risk.

Similarly, RRW is the risk (Y^{j-}) which is achieved if basic event j cannot happen, *i.e.*, its boolean variable is set to 0.

The Birnbaum importance of basic event j, BI_j, is defined as the partial derivative of the risk metric w.r.t. basic event j probability [Birnbaum (1969) [4], Cheok et al (1998) [6]]:

$$BI_j = \frac{\partial Y}{\partial X_j} \tag{1}$$

From standard reliability results, under the conditions of a coherent structure function and of independent failures [Huseby (2004)], it holds that:

$$BI_j = Y^{j+} - Y^{j-} \tag{2}$$

i.e., BI_j is the difference between the value of the risk metric achieved when basic event j happens (Y^{j+}) and when it cannot happen (Y^{j-}). Now, for the risk metric of PRA models, the partial derivative w.r.t. a probability is given by:

$$\frac{\partial Y}{\partial X_j} = Y(X_j = 1) - Y(X_j = 0) \tag{3}$$

where $Y(X_j = 1)$ is the value of the risk metric obtained susbtituting the value 1 for X_j in the model, and $Y(X_j = 0)$ the value obtained susbtituting 0 for X_j. We must note that $Y(X_j = 1)$ is not equal to Y^{j+} in the presence of dependencies (see Appendix A.) Dependencies make it necessary, after setting $X_j = 1$, to adjust the remaning probabilities conditioning over X_j [see Smith (1998) for a discussion.]

One main criticism moved towards the BI measure, consists of the fact that the importance of a component/basic event does not depend upon its own reliability/probability. The CIF importance measure, defined as:

$$CIF_j = \frac{\partial Y}{\partial X_j}\frac{X_j}{Y} = BI_J\frac{X_j}{Y} \tag{4}$$

can then be looked at as a generalization of the BI importance measure, as it normalizes BI_j through the ratio of the probability of basic event j and the nominal value of the risk metric ($\frac{X_j}{Y}$.) CIF enables to discriminate among components that have the same BI, since the less reliable component, *i.e.*, the one with higher X_j, is registered as more "critical."

We further refer to Cheok *et al* (1998) [6] and Vesely (1998) [14] for discussion of the definitions of FV, RAW, RRW, CIF and BI.

It is possible to find that manageable mathematical relationships hold between RAW, RRW, BI and CIF based on their definition for PRA models. For example, multiplying RAW_j and RRW_j times Y, and subtracting, one finds BI_j. Proceeding in a similar fashion, one finds the relationships summarized in Table 2.

Table 1: PRA Importance Measures

Importance Measure	**Symbol**	**Definition**
Fussell-Vesely	FV_j	$\frac{P[\cup MCS_j]}{Y}$
Risk Achievement Worth	RAW_j	$\frac{Y^{j+}}{Y}$
Risk Reduction Worth	RRW_j	$\frac{Y^{j-}}{Y}$
Birnbaum	BI_j	$\frac{\partial Y}{\partial X_j}$
Criticality	CIF_j	$\frac{\partial Y}{\partial X_j}\frac{X_j}{Y}$
Differential Importance Measure	DIM_j	$\frac{\frac{\partial Y}{\partial X_j}dX_j}{\sum_{s=1}^{n}\frac{\partial Y}{\partial X_s}dX_s}$

Table 2: Relationships between RAW, RRW, BI and CIF based on their definitions for PRA models

Importance Measure	RAW_j	RRW_j	BI_j	CIF_j
RAW_j	1	$\frac{BI_j}{Y} + RRW_j$ $\frac{CIF_j}{X_j} + RRW_j$	$\frac{BI_j}{Y} + RRW_j$	$\frac{CIF_j}{X_j} + RRW_j$
RRW_j	$RAW_j - \frac{CIF_j}{X_j}$	1	$RAW_j - \frac{BI_j}{Y}$	$RAW_j - \frac{CIF_j}{X_j}$
BI_j	$Y\left(RAW_j - RRW_j\right)$	$Y\left(RAW_j - RRW_j\right)$	1	$CIF_j\frac{Y}{X_j}$
CIF_j	$X_j\left(RAW_j - RRW_j\right)$	$X_j\left(RAW_j - RRW_j\right)$	$BI_j\frac{Xj}{Y}$	1

We now investigate the relationship between DIM and other PRA importance measures. Borgonovo and Apostolakis (2001) [2] define the local sensitivity measure of Y on X_j as follows:

$$DIM_j = \frac{d_j Y}{dY} = \frac{\frac{\partial Y}{\partial X_j} dX_j}{\sum_{s=1}^{n} \frac{\partial Y}{\partial X_s} dX_s} \tag{5}$$

In eq. (5), $d_j Y$ is the change in the risk metric provoked by a small change (dX_j) in the probability of basic event j, $\frac{\partial Y}{\partial X_j}$ the partial derivative of Y w.r.t. X_j, and the denominator sum is extended to all basic events ($s = 1, 2, ..., n$). Borgonovo and Apostolakis (2001) [2] show that DIM shares the following properties:

Property 1 Additivity. The joint DIM of a set of basic events is the sum of the individual basic event importances in the group. Suppose that the group is composed of basic events j, k, ..., and l. Then the importance of the group is:

$$DIM_{j,k,...,l} = DIM_j + DIM_k + ... + DIM_l \tag{6}$$

Property 2 The sum of the DIM's of all basic events equals unity, that is:

$$DIM_{1,2,...,n} = DIM_1 + DIM_2 + ... + DIM_n = 1 \tag{7}$$

Property 3 Let us consider uniform changes in the parameters (H1), *i.e.*

$$(H1) \qquad dX_j = dX_k \; \forall j, k = 1, 2, .., n \tag{8}$$

If eq. (8) holds, then

$$DIM_j^{H1} = \frac{\frac{\partial Y}{\partial X_j}}{\sum_{s=1}^{n} \frac{\partial Y}{\partial X_s}} = \frac{BI_j}{\sum_{s=1}^{n} BI_s} \tag{9}$$

Let us introduce $\alpha_{BI} = \sum_{s=1}^{n} BI_s$ and rearrange eq. (9). We have:

$$\frac{\partial Y}{\partial X_j} = BI_j = \alpha_{BI} \cdot DIM_j^{H1} \propto DIM_j^{H1} \tag{10}$$

which means that partial derivatives are proportional to DIM under an assumption of uniform changes in the parameters. This implies that ranking inputs based on BI is equivalent to state the assumption that all basic event probabilities are varied by the same (small) quantity. This result has the following practical consequence: if in a model two parameters have different dimensions, the corresponding partial derivatives cannot be compared. Thus, one cannot compare the BI an initiating event frequency to the BI of a basic event probability, since the first has units [Y/time], while the second has units [Y] (see also Borgonovo and Apostolakis (2001)[2].)

Property 4 Let us consider proportional relative changes in the parameters (H2), *i.e.*:

$$(H2) \qquad \frac{dX_j}{X_j} = \frac{dX_k}{X_k} = \omega \; \forall j, k = 1, 2, .., n \tag{11}$$

If eq. (11) holds, then:

$$DIM_j^{H2} = \frac{\frac{\partial Y}{\partial X_j}\frac{Xj}{Y}}{\sum_{s=1}^{n}\frac{\partial Y}{\partial X_s}\frac{Xs}{Y}} = \frac{CIF_j}{\sum_{s=1}^{n} CIF_s} \tag{12}$$

where CIF_j is the criticality importance factor of basic event j. Proceeding in a similar way as we did for the relationship between BI_j and DIM_j, one can rearrange the above equations expressing CIF_j as a function of DIM_j. Calling $\alpha_{CIF} = \sum_{s=1}^{n} CIF_s$, we have:

$$CIF_j = \alpha_{CIF} \cdot DIM_j^{H2} \propto DIM_j^{H2} \tag{13}$$

Eqs. (12) and (13) imply that ranking basic events with CIF is equivalent to rank them with DIM under H2, *i.e.* it is equivalent to stating the implicit assumption that all basic event probabilities are varied by the same proportion. Note that CIF_j is known also in Economics with the name of Elasticity (Samuelson (1947) [12].)

We note that Properties 3 and 4 hold for any model, independently of the type of input-output relationship.

We now report the relationship between DIM, FV and RAW for PRA models, since the link between DIM and FV also turns into a link between FV and CIF. In PRA, the risk metric (let us call it Y) assumes the following additive multiplicative form as a function of the basic event probabilities [see Borgonovo and Apostolakis (2001) [2] eq. (20)]:

$$Y = a_k X_k + OT \tag{14}$$

where X_k is the probability of basic event k, a_k the corresponding coefficient and OT the terms in the expression of Y not containing basic event k. Eq. (14) allow the following property 5 to hold, as proven in Borgonovo and Apostolakis (2001).

Property 5 The following relationships hold between DIM and FV (Borgonovo and Apostolakis (2001) [2]):

$$DIM_j^{H1} = \frac{FV_j/X_j}{\sum_{s=1}^{n} FV_s/X_s} \tag{15}$$

and

$$DIM_j^{H2} = \frac{FV_j}{\sum_{s=1}^{n} FV_s} \tag{16}$$

From eq. (16), one notes that, under H2, FV and DIM^{H2} would produce the same ranking. In fact, one can rewrite eq. (16) as:

$$FV_j = \left(\sum_{s=1}^{n} FV_s \right) \cdot DIM_j^{H2} \iff DIM_j^{H2} \propto FV_j \tag{17}$$

In their turn, eqs. (13) and (17) imply that the ranking induced by FV, CIF and DIM^{H2} is the same for models of the type of eq. (14). There is, however, one worthy remark. The relationship between FV and DIM [eq. (17)] holds only for PRA models, while the one between DIM under H2 and CIF [eqs. (12) and (13)] holds independently of the model.

We now recall the relationship between DIM and RAW. As Cheok *et al* (1998) [6] underline, computing the RAW of a basic event setting to unity the corresponding basic event probability is not entirely correct, since one ought first to resolve the PRA model with the Boolean variable set to unity, and then compute the conditional value of the risk metric. However, standard PRA software utilizes this "direct way" as a numerical approximation. If that is the case, the following relationship holds between DIM and RAW:

$$\begin{array}{cc} \text{H1} & \text{H2} \\ DIM_j^{H1} = \frac{(RAW_j-1)/(1-X_j)}{\sum_{s=1}^{n}(RAW_s-1)/(1-X_s)} & DIM_j^{H2} = \frac{(RAW_j-1)/(1/X_j-1)}{\sum_{s=1}^{n}(RAW_s-1)/(1/X_s-1)} \end{array} \tag{18}$$

From eq. (18), one notes that, in general, DIM and RAW would always produce different basic event ranking.

Note that, thanks to Table 2, eq. (16) and eq. (18), one is able to link all PRA importance measures for basic events. The presence of a relationship does not mean that they produce the same ranking; however, it has the immediate practical consequence that given a software that estimates one or more of the measures, the other ones can be gained without further model runs, independently of whether the importance measures are the output of a PRA model solved with the FTET technique or with the BDD technique. For instance, from the knowledge of FV one can compute DIM^{H2} and DIM^{H1} utilizing eqs. (15) and (16) respectively. Besides, the ranking induced by FV coincides with the ranking produced by CIF. [We would also like to refer to Table 4 in Borgonovo and Apostolakis (2001).] If, in addition, one computes RAW, then from Table 2, one can see that all the other indicators can be derived. The following pairs of measures allow the computation of all the others for equations of the form of eq. (14): (FV, RAW), (CIF, RAW)

In the next Section, we discuss how the importance of basic event groups and SSCs can be derived from the individual basic event importance.

3 Basic Event Groups and SSCs Importance

Section 2 has discussed the definition and interrelationships for importance measures of individual basic events. In this Section, we deal with how/whether the importance measure definitions can be extended from individual to basic event groups. In our analysis, we distinguish "joint importance" from SSC importance. By joint importance we mean the importance of a group (G) of basic events $j, k, ..l$, ($G = \{j, k, ..., l\}$), with the basic events that can refer to the same or different components. Concerning SSC importance we inquire whether it is necessarily true that the importance of a component can be identified with the importance of the group containing all basic events related to the component.

The relevance for applications of computing the joint importance is underlined in Cheok *et al* (1998): "*to obtain a measure of the importance of an SSC with respect to the particular application, all basic events representing the affected modes of the particular SSC should be consider as part of a group (* Cheok *et al* (1998) [6])." We then analyze the behavior of the five importance measures discussed in Section 2, beginning with DIM.

1. DIM. The computation of the DIM of a group is straightforward. In fact, it holds that $DIM_{j,k,...,l} = DIM_j + DIM_k + ... + DIM_l$, *i.e.* DIM_G is the sum of the DIMs of the basic events in the group [eq. (6)].

2. RAW. The computation of the RAW (and also RRW) for multiple basic events asks for some attention, as Cheok *et al* (1998) maintain. Extending the definition of RAW for individual basic events, Cheok *et al* (1998) propose the following RAW of a basic event group:
$$RAW_{j,k,...,l} = \frac{Y^{(j,k,...,l)+} - Y}{Y} \tag{19}$$
where $Y^{(j,k,...,l)+}$ means the value of the risk metric which is obtained by setting to "*true*" the Boolean variable of basic events $j, k, ..., l$. According to the definition it is necessary to re-run the model, as proposed in Cheok *et al* (1998). In eq. (19), $Y^{(j,k,...,l)+}$ does not need to coincide with the value of the risk metric obtained setting the basic event probabilities equal to unity (*i.e.* $X_j = X_k = ... = X_l = 1$) which is a shortcut ("direct way") taken by most of standard PRA software [Cheok *et al* (1998) and Borgonovo and Apostolakis (2001).] Appendix C in Borgonovo and Apostolakis (2001) discusses in detail the properties of $RAW_{j,k,...,l}$. In particular, it holds that $RAW_{j,k,...,l} = RAW_j = RAW_k = ... = RAW_l$ if j, k and l are related by an "or" gate.

3. FV. The FV of a basic event group can be found by extending its definition as:
$$FV_{j,k,...,l} = \frac{P\left[\cup MCS_{j,k,...,l}\right]}{Y} \tag{20}$$
where $MCS_{j,k}$ are the minimal cut sets that contain either one of basic events $j, k, ..., l$. Borgonovo and Apostolakis (2001) show that:

   $FV_{j,k,...,l} = FV_j + FV_k + ... + FV_l$ if basic events $j, k, ..., l$ appear under the same "or" gate,

   and

$FV_{j,k,...,l} = FV_j = FV_k = ... = FV_l$ if $j, k, ..., l$. appear under the same "and" gate.

4. BI. Cheok *et al* (1998) show that there is no meaningful extension of the definition of BI to group of basic events. More precisely, the BI of a set of basic events is not defined for any model. In fact, $BI_j = \frac{\partial Y}{\partial X_j}$ is the partial derivative of Y w.r.t. X_j. The natural extension of the definition would consider $BI_{j,k,...,l}$ as the partial derivative of Y w.r.t. X_j, X_k,...,X_l which is a non-defined mathematical quantity. In fact, partial derivatives are defined and account for the change of one variable at a time. The effect of joint changes is registered by the differential. With this respect, we recall that for individual basic events BI coincides with DIM under H1. Hence, to account for the joint change in basic events for uniform changes, $DIM^{H1}_{j,k,...,l}$ ought to be utilized.

5. CIF. CIF is not defined for basic event groups. As one can see from the definition in Table 1, $CIF_j = \frac{\partial Y}{\partial X_j}\frac{X_j}{Y}$, which is a normalized partial derivative. Hence, since $\frac{\partial Y}{\partial X_j}$ is not defined for multiple X_j's, it is not possible to extend the definition of CIF to multiple basic events (see Point 4.) However, we recall that CIF_j allows the ranking of basic events for proportional relative changes and this ranking coincides with the one produced by DIM^{H2}_j. Hence, to account for the importance of joint proportional changes in the parameters $DIM^{H2}_{j,k,...,l}$ is the proper importance measure.

We note that the observations in points 1-5 above hold due to the definition of the importance indicators. Thus, eventual difficulties in deriving joint importance measures hold independently of whether the indicator is estimated by a BDD or a FTET software.

We conclude this Section with some observations on whether it is possible to infer SSC importance starting from the joint importance of the corresponding basic events. We suppose that the component or SSC (C) is represented by basic events 1, 2,...,k and we denote its failure probability or unavailability with X_C. Since we want to study the link joint importance→component importance and a joint CIF or BI is not defined (see points 4, 5), we restrict our attention to FV, RAW and DIM.

Extending the formulation for basic event groups, a possible definition of the FV of a component is:

$$FV_C = \frac{P\left[\cup MCS_C\right]}{Y} \tag{21}$$

where MCS_C are the MCSs including C. Now, one only needs to specify what is meant by inclusion of a component in a MCS. This ought probably to be considered a matter of convention, and the convention is usually to consider all MCSs involving one or more of the basic events related to C.

The definition of RAW can be readily extended to a component as follows:

$$RAW_C = \frac{Y^{(C)+} - Y}{Y} \tag{22}$$

where $Y^{(C)+}$ is the new risk metric obtained with the component failed. The component RAW provides the importance of an existing condition "e.g. *a component being down.* [Vesely (1998) [14], p. 257]." Now, to reflect the dependence on the events and conditions involving the component (see Vesely (1998) [14],) a rigorous calculation of RAW_C would require one to adjust the PRA model in order to reflect the failure of C and evaluate the conditional risk metric. Smith (1998) [13] notes that to thoroughly account for the component failure one should (the next three points are direct quote from Smith (1998), p. 302):

1. Find the affected basic events in the PRA in order to change their failure probabilities to reflect the inoperable component;

2. Assess the likelihood that the affected component could be recovered;

3. Determine other impacts on PRA model such as changes to common-cause failure probabilities and initiating event frequencies.

Now, let us denote with $Y^{(C)+}$ the conditional estimate of the risk metric based on a thorough recomputation as recommended in Smith's points 1-2-3. It is clearly "$C = true$" if all the corresponding basic events $1, 2, ..., k$ are set to true. However, Smith's observations suggest that the value of the risk metric, $Y^{(1,2,...,k)+}$, obtained with basic events $1, 2, ..k$ set to true, could differ from $Y^{(C)+}$, if no readjustments are applied. In virtue of eqs. (19) and (22), $Y^{(C)+} \neq Y^{(1,2,...,k)+}$, implies $RAW_C \neq RAW_{1,2,...k}$, *i.e.* the RAW of a component is not necessarily equal to the $RAW_{1,2,...k}$ of the corresponding basic event group.

Let us now turn to DIM and ask the question of whether the DIM of a component is related to the DIMs of the corresponding basic events. The definition of DIM for component C is:

$$DIM_C = \frac{d_C Y}{dY} = \frac{\frac{\partial Y}{\partial X_C} dX_C}{\sum_{s=1}^{n} \frac{\partial Y}{\partial X_s} dX_s} \tag{23}$$

where $d_C Y$ denotes the change in Y provoked by a change in the failure probability or unavailability of C, $\frac{\partial Y}{\partial X_C}$ is the partial derivative of Y w.r.t. X_C, dX_C is the (small) change in $C's$ failure probability. As far as the interpretation of eq. (23) is concerned, we note that eq. (23) addresses the problem of measuring the relative "*risk change*" provoked by a (small) change in an SSC unavailability, stated in Vesely (1998), p. 258. We now investigate whether DIM_C has some relationship to the DIM of the basic events corresponding to component C. The component failure probability or unavailability, X_C, depends in a complicated way on the basic event probabilities:

$$X_C = X_C(X_1, X_2, ..., X_k) \tag{24}$$

Eq. (24) can be differentiated to produce:

$$dX_C = \sum_{i=1}^{k} \frac{\partial X_C}{\partial X_i} dX_i \tag{25}$$

Substituting eq. (25) in eq. (23), DIM_C can be rewritten as:

$$DIM_C = \frac{d_C Y}{dY} = \frac{\frac{\partial Y}{\partial C}\frac{\partial C}{\partial X_1} dX_1 + \frac{\partial Y}{\partial C}\frac{\partial C}{\partial X_2} dX_2 + ... + \frac{\partial Y}{\partial C}\frac{\partial C}{\partial X_k} dX_k}{\sum_{s=1}^{n} \frac{\partial Y}{\partial X_s} dX_s} \tag{26}$$

Now, let us look at the partial derivatives:

$$\frac{\partial Y}{\partial X_C}\frac{\partial X_C}{\partial X_i} \tag{27}$$

We recall that, thanks to the well known differentiation result called the "Chain Rule" it holds for any function that:

$$\frac{\partial Y}{\partial X_i} = \frac{\partial Y}{\partial X_C}\frac{\partial X_C}{\partial X_i} \tag{28}$$

Eq. (28) allows one to write:

$$DIM_C = \frac{d_C Y}{dY} = \frac{\frac{\partial Y}{\partial X_1} dX_1 + \frac{\partial Y}{\partial X_2} dX_2 + ... + \frac{\partial Y}{\partial X_k} dX_k}{\sum_{s=1}^{n} \frac{\partial Y}{\partial X_s} dX_s} \tag{29}$$

Eq. (29) states that the DIM of a component [eq. (23)] is the sum of the DIMs of basic events $1, 2, ..., k$ that refer to the component.

Finally, we note that the above results or observations (*i.e.* impossibility of defining CIF and BI for multiple basic events, procedures for the derivation of RAW for basic events and components, interpretation of FV for components and properties of DIM) depend on the definition of the importance measures themselves and hence hold independently of whether one is using FTETs or BDDs.

4 DIM for basic events, groups and components for BDDs and FTETs

In this Section, we present numerical examples that illustrate the calculation of DIM for individual basic events, groups and components based on the results of Sections 2 and 3.

We first deal with how individual basic event DIMs can be computed both for FTET and BDD codes, through the relationships presented in Section 2. Numerical results are gained with reference to Epstein and Rauzy (2005) [7] in which basic event importance measures are estimated for a PRA model making use both of a BDD and an FTET code. To compute DIM, one can choose either one of the relationships between DIM and the FV, BI, CIF of RAW indicators. Figure 1 reports the values of the first 20 basic events CIF shown in Table 9 of Epstein and Rauzy (2005) [7].[1]

We start with utilizing the CIF_j estimated via a BDD code reported in the second column of Table 9 in Epstein and Rauzy (2005) (Figure 1.) From the knowledge of CIF, one can immediately compute DIM_j^{H2}: employing eqs. (12) and (13), one can determine

[1] The discussion has the purpose of highlighting the method, and therefore it suffices to consider the 20 basic events in Table 9 of Rauzy. However, for the sake of precision, Epstein and Rauzy (2005)'s original sequence contains more than 1000 basic events. Hence, our results should be used only for illustration purposes. As the reader can easily see, our method can be immediately extended to a Table listing all the 1000 basic events CIFs.

Rank	CIF(*S*,*e*)/BDD	CIF(*S*,*e*)/cutsets
1	0.605467	0.81045
2	0.441697	0.615231
3	0.288971	0.172044
4	0.141314	0.135865
5	0.137265	0.223926
6	0.137265	0.223926
7	0.137265	0.223926
8	0.0622317	0.041397
9	0.0622317	0.041397
10	0.0622317	0.041397
11	0.0622284	0.041397
12	0.0622284	0.041397
13	0.0622284	0.041397
14	0.0573555	0.0511157
15	0.0573555	0.0511157
16	0.0522112	0.0355426
17	0.052187	0.0355426
18	0.0429735	0.0302301
19	0.0429735	0.0302301
20	0.0429735	0.0302301

Figure 1: Table 9 from Epstein and Rauzy (2005) shows the CIF for 20 basic events, estimated with BDDs (second column) and FTETs (third column).

$\alpha_{CIF} = \sum_{j=1}^{n} CIF_j$. In our case, we have: $\alpha_{CIF} = 2.610$. DIM_j^{H2} is then found by dividing each CIF_j in the second column of Figure 1 by α_{CIF}. The results are reported in the second column of Figure 2.

Adopting the same procedure, the calculation of DIM_j^{H2} estimated by FTET codes is readily performed utilizing as input the basic event CIF_j offered in column 3 of Figure 1. The results are reported in the third column of Figure 2.

We note that in the above discussion, DIM^{H2} has been found with no additional model runs, independently of whether a BDD or an FTET code output were under consideration.

One can also find DIM_j^{H1} and BI_j from the numbers of Figure 1. In fact, one can re-write the definition of CIF (Table 1) as:

$$\frac{\partial Y}{\partial X_j} = CIF_j \cdot \frac{Y}{X_j} = BI_j \tag{30}$$

Eq. (30) restates the relationship between CIF_j and BI_j (see also Table 2) and enables one to calculate $\frac{\partial Y}{\partial X_j}$ by a simple manipulation. For our numerical exercise, given the CIF_j of

j	DIM H2 BDD	DIMH2 Event Trees
1	0.23192	0.27776
2	0.16919	0.21086
3	0.11069	0.05896
4	0.05413	0.04656
5	0.05258	0.07675
6	0.05258	0.07675
7	0.05258	0.07675
8	0.02384	0.01419
9	0.02384	0.01419
10	0.02384	0.01419
11	0.02384	0.01419
12	0.02384	0.01419
13	0.02384	0.01419
14	0.02197	0.01752
15	0.02197	0.01752
16	0.02000	0.01218
17	0.01999	0.01218
18	0.01646	0.01036
19	0.01646	0.01036
20	0.01646	0.01036

Figure 2: Basic events DIMs for a BDD or a FTET code, are derived from the relationship between DIM and CIF with no further model runs.

j	Xj	BI (BDD)	DIM H1 (BDD)	BI (Cutsets)	DIM H1 (Cutsets)
1	2E-11	60.5	0.066	81.0	0.088
2	2.2E-11	40.2	0.043	55.9	0.061
3	2.6E-11	22.2	0.024	13.2	0.014
4	2.4E-11	11.8	0.013	11.3	0.012
5	3.4E-11	8.1	0.009	13.2	0.014
6	2.8E-11	9.8	0.011	16.0	0.017
7	4.2E-11	6.5	0.007	10.7	0.012
8	3.8E-11	3.3	0.004	2.2	0.002
9	3.6E-11	3.5	0.004	2.3	0.002
10	2.4E-11	5.2	0.006	3.4	0.004
11	4.2E-12	29.6	0.032	19.7	0.021
12	1.4E-12	88.9	0.096	59.1	0.064
13	1.8E-12	69.1	0.075	46.0	0.050
14	1E-12	114.7	0.124	102.2	0.111
15	8E-13	143.4	0.155	127.8	0.138
16	1E-12	104.4	0.113	71.1	0.077
17	1.6E-12	65.2	0.071	44.4	0.048
18	1.8E-12	47.7	0.052	33.6	0.036
19	1.9E-12	45.2	0.049	31.8	0.034
20	1.94E-12	44.3	0.048	31.2	0.034

Figure 3: DIM^{H1} and BI for basic events computed from the CIF output of the BDD and FTET models in Epstein and Rauzy (2005).

the BDD code (Column 2 of Figure 1), one finds the BI_j reported in Column 3 of Figure 3. Utilizing eq. (9), one can then obtain DIM_j^{H1} following the same procedure utilized for obtaining DIM^{H2}. In this case, $\alpha_{BI} = 924$ and one finds the values of DIM_j^{H1} reported in Column 4 of Figure 3. Figure 3 also shows the result for BI and DIM^{H1} obtained from the knowledge of CIF_j estimated via the FTET code of Figure 1. For completeness, in Figure 3, we have also reported the X_j values we adopted in our calculations[2].

Figure 4 reports the ranking obtained with CIF_j/DIM_j^{H2} and BI_j/DIM_j^{H1} for BDDs. One notes that there is some discrepancy. Numerically, the explanation is straightforward.

[2]The X_j were not directly available in Epstein and Rauzy (2005). We have utilized fictitious values respecting the indications available in the paper, which specified a value of Y around 10^{-9}, and and truncation at $10^{-4}Y$. Again, our calculations have only a demonstrative purpose, *i.e.* to show the methodology.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
CIF / DIM^{H2}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
BI / DIM^{H1}	7	11	13	14	16	15	17	20	19	18	12	4	5	2	1	3	6	8	9	10

Figure 4: Comparison of Basic Event Ranking with CIF/DIM^{H2} vs BI/DIM^{H1}.

Table 3: Coponent Importance with DIM

	H1 (BDD)	**H1 (FTET)**	**H2 (BDD)**	**H2 (FTET)**
DIM_E	0.51	0.42	0.42	0.43
DIM_O	0.49	0.58	0.58	0.57

Since $BI_j = CIF_j \cdot \frac{Y}{X_j}$, if X_j is very small, a parameter low ranked with CIF can jump up in the ranks with BI. In Appendix A, we discuss the use of Savage Scores (Iman and Conover (1987) [10]) as a tool for comparing the ranking agreement of different importance indicators.

The last task of our numerical analysis is the determination of DIM for basic event groups and components. Let us divide the basic events into two groups, the "evens" $E = (X_2, X_4, ..., X_{20})$ and the "odds" $O = (X_1, X_3, ..., X_{19})$. Thanks to the additivity property, the importance of the two groups can be found by adding of the importances of the basic events in the groups, and hence requires no further model runs. That is $DIM_E = DIM_2 + DIM_4 + .. + DIM_{20}$, and similarly for DIM_O. We have the results in Table 3.

The second and third columns of Table 3 report DIM_E^{H1} and DIM_O^{H1} estimated by the BDD and the FTET codes respectively. The sum is performed over the DIMs in Columns 4 and 6 of Figure 3. The fourth and fifth columns of Table 3 display DIM_E^{H2} and DIM_O^{H2} which are sum of the DIMs in Figure 2. Under H2, O results as the most important group, both with the BDD and the FTET estimates. Under H1, O is the most important group if one utilizes an FTET code, while the order is reversed if one were to utilize a BDD code. This result shows that the ranking can depend on the technology adopted. In this case, BDD estimates are more accurate according to Epstein and Rauzy (2005). However, the importance of groups is found without further model runs independently of whether one is using a BDD or a FTET method, thanks to the properties of DIM.

Finally, one can note that if E and O were components, Table 3 would also produce the exact differential importance of the components.

5 Conclusions

In this work, we have dealt with importance measures for basic events, groups and SCCs of PRA models solved utilizing the Fault Tree/Event Tree method or the Binary Decision Diagrams technology.

Our investigation has concerned the following indicators: Fussel-Vesely (FV), Criticality Importance Factor (CIF), Birnbaum (BI), Risk Achievement Worth (RAW) and differential importance measure (DIM.) We have seen that simple mathematical relationships hold among these importance measures at the individual basic event level of PRA models. These relationships have allowed us to show that one is capable of computing DIM with no additional model runs, given a software that estimates any of the other importance measures, independently of whether the software is utilizing the FTET or the BDD technology.

We have then turned our attention to how the five importance measures address the importance of basic event groups. We have seen that CIF and BI cannot be extended to multiple basic events, independently of whether one is using an FTET or BDD based code. However, the FV, DIM and RAW importance measures can be defined for groups. In particular, the DIM of groups is straightforward thanks to the additivity property.

We have then investigated whether, once the importance of a group has been defined, the importance of a component can be found from the importance of the group of basic events referring to that component. We have seen that the answer depends on the measure. It is not possible to use the CIF or BI of a basic event group to find the CIF or BI of a component, since CIF and BI are not defined for groups. For RAW, we have argued that if one applies a rigorous conditional calculation of the risk metric as suggested in Smith (1998), the RAW of a component might not coincide with the RAW of the corresponding basic event group.

As far as DIM is concerned, it has been possible to show that the DIM of a component is the sum of the DIM's of the basic events referring to that component.

The analysis has revealed that difficulties in the computation of CIF and BI for groups hold due to the definition of the indicators, independently of whether one is using an FTET or a BDD software. However, the DIM of groups or components can be inferred directly from the DIMs of the corresponding basic events with no additional model runs (both for FTET and BBDs).

Numerical results have demonstrated the computation of DIM for individual basic events, basic event groups and components by application to both the FTET and BDD code results presented in Epstein and Rauzy (2005).

A note on further research. Subject of further investigation could be to explore whether BDDs allow greater flexibility than FTETs in performing the readjustment procedure presented in Smith (1998) [13], thus allowing better estimates of component and group RAW while maintaining a comparable computational time.

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6 Appendix A: Computation of the Birnbaum Measure and the Rare Event Approximation

In this Appendix, we would like to place some further considerations on the compuation and additivity of the Birnbaum measure. Consider a fault tree where a component is represented by 3 failure modes. System failure is achieved when the component fails. Then we have:

$$Y = p(1) + p(2) + p(3) - p(12) - p(13) - p(23) + p(123) \tag{31}$$

If one assumes the rare event approximation (we'll relax such an approximation shortly), then

$$Y = X_1 + X_2 + X_3 \tag{32}$$

Now, it is clear that

$$Y_1^+ = 1 \tag{33}$$

while if one simply sets X_1 to unity, one would find:

$$Y_1(1) = 1 + X_2 + X_3 \tag{34}$$

Note that $Y_1(1)$ is a probability greater than unity. The error is due to the fact that one is using the rare event approximation and, at the same time, violating it by setting one of the probabilities equal to 1.

Thus, the Birnbaum importance of failure mode 1 (denoted as BI_1) is, using eq. (33):

$$BI_1 = Y_1^+ - Y_1^- = 1 - X_2 - X_3 \tag{35}$$

(which is correct) or, using eq. (34):

$$\widetilde{BI_1} = Y_1(1) - Y_1(0) = 1 \tag{36}$$

We have used the notation $\tilde{\cdot}$ to signal that $\widetilde{BI_1}$ is an approximatrion of BI_1. Clearly $\widetilde{BI_1} \cong BI_1$ if X_2 and X_3 are small. Similarly, for basic event 2, one gets:

$$BI_2 = Y_2^+ - Y_2^- = 1 - X_1 - X_3 \tag{37}$$

or:

$$\widetilde{BI_2} = Y_2(1) - Y_2(0) = 1 \tag{38}$$

and again $\widetilde{BI_2} \cong BI_2$ if if X_1 and X_3 are small. Suppose now that one wants the joint Birnbaum importance of basic events 1 and 2, $BI_{1,2}$. Neglecting the reservations provoked by the fact that partial derivatives are not defined for joint variables, one has to compute the risk obtained when both basic events 1 and 2 happen:

$$Y^+_{1,2} = 1 \tag{39}$$

and subtract the value of the risk metric obtained when both 1 and 2 do not happen:

$$Y^-_{1,2} = X_3 \tag{40}$$

Note that, the direct numerical substitution of $X_1 = X_2 = 1$ would lead to:

$$Y_{1,2}(1) = 2 + X_3 \tag{41}$$

which, in this case, is a probability greater than 2 (things are getting worse.) Thus, using eq. (39) one gets:

$$BI_{1,2} = Y^+_{1,2} - Y^-_{1,2} = 1 - X_3 \tag{42}$$

(which is correct) or, using eq. (41):

$$\widetilde{BI_{1,2}} = Y_{1,2}(1) - Y_{1,2}(0) = 2 \tag{43}$$

Note that in this case $\widetilde{BI_{1,2}} \neq BI_{1,2}$ even when X_3 is small. One needs some further observations:

1 One can note that:

$$\widetilde{BI_1} + \widetilde{BI_2} = \widetilde{BI_{1,2}} \tag{44}$$

which is a consequence of the approximations made in the calculation. Sometimes this result leads to consider that the Birnbaum importance is additive for disjoint basic events.

2 We would like to point out that the system we have considered possesses a coherent structure function and independent failures. Under these assumptions, general reliability theory shows that BI is the probability that the system works given that the basic event(s) of interet has(have) not happened and is failed otherwise. Thus, BI (of one or more basic events) cannot be greater than unity. There follows that $\widetilde{BI_{1,2}}$ is not only numerically wrong, but also conceptually and therefore the above mentioned additivity is only the numerical result of an erroneous approximation.

Finally, let us relax the rare event approximation and see what happens. Still assuming independence, eq. (31) becomes:

$$Y = X_1 + X_2 + X_3 - X_1X_2 - X_1X_3 - X_2X_3 + X_1X_2X_3 \tag{45}$$

Now, one has:

$$BI_1 = Y_1^+ - Y_1^- = \widetilde{BI_1} = Y_1(1) - Y_1(0) = 1 - X_2 - X_3 + X_2X_3 \tag{46}$$

and, similarly:

$$BI_2 = Y_2^+ - Y_2^- = \widetilde{BI_2} = Y_2(1) - Y_2(0) = 1 - X_1 - X_3 + X_1X_3 \tag{47}$$

The joint importance, is, still:

$$BI_{1,2} = Y_{1,2}^+ - Y_{1,2}^- = 1 - X_3 = Y_2(1) - Y_2(0) = \widetilde{BI_2} \tag{48}$$

and note that still $BI_{1,2} \neq BI_1 + BI_2$, *i.e.*, the Birnbaum measure is not additive.

7 Appendix B: Savage Scores

The following lines have the purpose of illustrating how Savage Score correlation coefficients (Iman and Conover (1987) [10]) can be utilized to compare the ranking agreement of different importance measures.

The Savage Score of each basic event (SS_i) is defined as:

$$SS_i = \sum_{t=r(i)}^{n} 1/t \quad i = 1, 2, ..., n \tag{49}$$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
CIF / DIM^{H2}	3.6	2.6	2.1	1.8	1.5	1.3	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.3	0.2	0.2	0.1	0.1
BI / DIM^{H1}	1.1	0.7	0.5	0.4	0.3	0.3	0.2	0.1	0.1	0.2	0.6	1.8	1.5	2.6	3.6	2.1	1.3	1.0	0.9	0.8

Figure 5: Savage Scores corresponding to the Ranking in Figure 4.

where $r(X_i)$ is the rank of basic event i, and n is the number of basic events. Figure 5 reports the results for the basic event SS_i corresponding to the ranking in Figure 4.After transforming the ranks of CIF/DIM^{H2} and BI/DIM^{H1} in Figure 4 in the corresponding SS_i through eq. (49), the correlation coefficient on the two series can be computed (Iman and Conover (1987) [10].) and results in a value equal to -0.32. The presence of a minus sign indicates a trend for reversal in ranks.