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Asymptotic Behavior of a Set-Statistic

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Abstract. The existence theorem of Minkowski for a polytope with given facet normals and areas is adapted to a data-analytic context. More precisely, we show that a centered, random point sample arising from an absolutely continuous distribution in $\mathbb{R}^d$ can be uniquely mapped into such a polytope almost surely. With increasing sample size, the sequence of (scaled) polytopes converges almost surely to a limiting convex body that is associated with the underlying distribution. An accompanying central limit theorem is proved using methods from the theory of empirical processes.

1. Introduction

Limit theorems of an additive type for random convex bodies were introduced in [1] and [19] where a strong law of large numbers and a central limit theorem, respectively, were shown. Since then there has been considerable activity in generalizing and extending these results; see, for example, the on-line bibliography [13] and article [20].

In this note we present a strong law and a central limit theorem of a novel type. They were motivated by recent work in exploratory data analysis that exploits the well-known existence theorem of Minkowski by mapping a multivariate point sample into what we have called the *sample Minkowski polytope* [2], [3]; to our knowledge, the approach is new to general data analysis although it has been used for particular purposes in astronomy [9] and image processing [8], [10]. The particular issue that arises is the behavior of the polytope as the size of the sample increases without bound. The answer, as we show here, is that there is convergence to a convex body that is associated with the underlying probability distribution of the sample.

To place our results in an analytical context, of the previous asymptotic results, most, if not all, have relied on identifying an arbitrary convex body with its support function, thus transferring the problem to a function analytic setting. Here, it turns out, we alternatively identify a convex body with its *surface area measure* with which it is uniquely identified up to translation. While we do not explicitly use the fact, there is an alternate operation of *Blaschke addition* of convex bodies that corresponds to addition of surface area measures; for general discussions and references, see [5], [7], and [14]–[16].

2. Background and Preliminaries

We work in $\mathbb{R}^d$ with usual inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, and unit sphere S^{d-1}; vectors are generally taken to be in column form, and transposition is denoted by $'$. For notation and background on convex bodies, see [15].

Our point of departure is the existence theorem of Minkowski [11], [12]:

Theorem 1. *Suppose that $u_1, u_2, \ldots, u_n \in S^{d-1}$ are distinct and that they linearly span $\mathbb{R}^d$. Further suppose that $f_1, f_2, \ldots, f_n$ are positive numbers such that*

$$\sum_{i=1}^{n} f_i u_i = 0.$$

Then there is a polytope $P \subset \mathbb{R}^d$ with facet normals $u_1, u_2, \ldots, u_n$ and associated facet areas ($(d-1)$–dimensional volumes) $f_1, f_2, \ldots, f_n$. P is unique up to translation.

In [2] and [3], translation-invariant questions for random samples in $\mathbb{R}^d$ are treated by adapting Theorem 1 in the following way: suppose that $X_1, X_2, \ldots, X_n$ is a random sample in $\mathbb{R}^d$, which is drawn from a probability measure μ. To avoid irrelevant complications, we assume here that μ is absolutely continuous although this can be relaxed at the expense of more complicated statements. For $n \geq d+1$, let $\overline{X}_n = (1/n) \sum_{i=1}^{n} X_i$ and define

$$U_i = \frac{X_i - \overline{X}_n}{\|X_i - \overline{X}_n\|}, \qquad F_i = \frac{1}{n} \|X_i - \overline{X}_n\|, \qquad i = 1, 2, \ldots, n. \tag{1}$$

The hypotheses of Theorem 1 hold for (1) with probability one and thus guarantee the existence of the *Minkowski polytope* P_n associated with the sample. For definiteness, we assume that P_n, and all other convex bodies to be considered, are *centered* in the sense of having the Steiner centroid at the origin [15, p. 42]. Indeed, mapping such a sample into P_n incurs no loss of information apart from location.

3. Strong Law of Large Numbers

As we shall see, the large sample behavior of P_n has convergence to a convex body that is not a polytope. To introduce the formulation, we recall that the *surface area measure*

φ of a convex body K can be described as follows: for each Borel set $A \subset S^{d-1}$, let $\partial K(A)$ be the set of points in ∂K having outward normal in A. Then the surface area measure φ is defined on the Borel σ-field of S^{d-1} by $\varphi(A) = \lambda_{d-1}(\partial K(A))$, where λ_{d-1} is the $(d-1)$-dimensional Hausdorff measure. In the case of the polytope of Theorem 1,

$$\varphi = \sum_{i=1}^{n} f_i \delta(\cdot - u_i). \tag{2}$$

The following generalization of Theorem 1 holds [15]:

Theorem 2. *Suppose that φ is a bounded measure on the Borel subsets of S^{d-1} that is not concentrated on a great circle and that satisfies*

$$\int_{S^{d-1}} u \, d\varphi = 0.$$

Then there is a unique centered convex body for which φ is the surface area measure.

We can now state the strong law of large numbers.

Theorem 3. *Suppose that μ is an absolutely continuous probability measure on $\mathbb{R}^d$ that yields a finite mean for an associated random vector X (iid copies $X_1, X_2, \ldots, X_n$). Let φ be the derived measure on S^{d-1} that satisfies, for each Borel $A \subset S^{d-1}$,*

$$\varphi(A) = E 1_A \left(\frac{X - EX}{\|X - EX\|} \right) \cdot \|X - EX\|. \tag{3}$$

Then (i) *φ is a surface area measure corresponding to a centered convex body K_φ and* (ii) *the sequence of sample Minkowski polytopes P_n based on successive iid samples $\{X_1, X_2, \ldots, X_n\}$ converges almost surely in the Hausdorff metric to K_φ.*

Proof. Without loss of generality, assume that $EX = 0$. Using (3), we have

$$\int_{S^{d-1}} u \, d\varphi = E \left(\frac{X}{\|X\|} \right) \cdot \|X\| = EX = 0,$$

so that the conditions for Theorem 2 are satisfied, thus yielding (i).

For (ii), using (1) and (2), we see that surface area measure associated with P_n is given by

$$\varphi_n = \frac{1}{n} \sum_{i=1}^{n} \delta \left(\cdot - \frac{X_i - \overline{X}_n}{\|X_i - \overline{X}_n\|} \right) \|X_i - \overline{X}_n\|.$$

It is enough to show that with probability one the sequence of measures φ_n converges weakly to φ [15, p. 393]. This will be the case if we can show that, for each g in a countable dense subset of Lipschitz continuous functions in $C(S^{d-1})$,

$$\int_{S^{d-1}} g \, d\varphi_n \xrightarrow{\text{a.s.}} \int_{S^{d-1}} g \, d\varphi. \tag{4}$$

Let $\tilde{g}$ be given by $\tilde{g}(0) = 0$ and $\tilde{g}(x) = g(x/\|x\|) \cdot \|x\|$ for $x \neq 0$. Then, as shown in Lemma 1 below, $\tilde{g}$ is also Lipschitz continuous (constant L). The existence of a mean for X then easily implies $E|\tilde{g}(X)| < \infty$ and hence

$$\frac{1}{n}\sum_{i=1}^{n} \tilde{g}(X_i) \xrightarrow{\text{a.s.}} E\tilde{g}(X).$$

Also,

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} \tilde{g}(X_i) - \frac{1}{n}\sum_{i=1}^{n} \tilde{g}(X_i - \overline{X}_n)\right| &\leq \frac{1}{n}\sum_{i=1}^{n} \left|\tilde{g}(X_i) - \tilde{g}(X_i - \overline{X}_n)\right| \\ &\leq \frac{1}{n}\sum_{i=1}^{n} L\|\overline{X}_n\| = L\|\overline{X}_n\| \xrightarrow{\text{a.s.}} 0.\end{aligned}$$

Together these imply that

$$\frac{1}{n}\sum_{i=1}^{n} \tilde{g}(X_i - \overline{X}_n) \xrightarrow{\text{a.s.}} E\tilde{g}(X),$$

which is equivalent to (4). □

Lemma 1. *Suppose that g and $\tilde{g}$ are as presented above. Then $\tilde{g}$ is Lipschitz continuous with constant $L = 2K + M$ where K is the Lipschitz constant of g and $M = \max_{\|u\|=1} |g(u)|$.*

Proof. We estimate $|\tilde{g}(y) - \tilde{g}(x)|$. If say $x = 0$, then

$$|\tilde{g}(y) - \tilde{g}(0)| = |\tilde{g}(y)| = \left|g\left(\frac{y}{\|y\|}\right) \cdot \|y\|\right| \leq M\|y\|.$$

If neither $x \neq 0$ nor $y \neq 0$, then we can write

$$\begin{aligned}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| &= \left\|\frac{x\|y\| - y\|x\|}{\|x\| \cdot \|y\|}\right\| \\ &= \left\|\frac{x\|y\| - x\|x\| + x\|x\| - y\|x\|}{\|x\| \cdot \|y\|}\right\| \\ &\leq \frac{\|\,\|y\| - \|x\|\,\|}{\|y\|} + \frac{\|y - x\|}{\|y\|} \leq 2\frac{\|y - x\|}{\|y\|}.\end{aligned}$$

It follows that

$$\begin{aligned}|\tilde{g}(y) - \tilde{g}(x)| &= \left|g\left(\frac{y}{\|y\|}\right) \cdot \|y\| - g\left(\frac{x}{\|x\|}\right) \|x\|\right| \\ &= \left|g\left(\frac{y}{\|y\|}\right) \cdot \|y\| - g\left(\frac{x}{\|x\|}\right) \cdot \|y\| + g\left(\frac{x}{\|x\|}\right) \cdot \|y\| - g\left(\frac{x}{\|x\|}\right) \cdot \|x\|\right|\end{aligned}$$

$$\leq \left| g\left(\frac{y}{\|y\|}\right) - g\left(\frac{x}{\|x\|}\right) \right| \cdot \|y\| + \left| g\left(\frac{x}{\|x\|}\right) \right| \cdot |(\|y\| - \|x\|)|$$
$$\leq K \cdot 2 \cdot \frac{\|y - x\|}{\|y\|} \cdot \|y\| + M\|y - x\|$$
$$\leq (2K + M)\|y - x\|,$$

so that $\tilde{g}$ has Lipschitz constant $L = 2K + M$. □

4. Central Limit Theorem

We turn now to the associated central limit theorem. Here, as for the strong law of large numbers, we are on new ground. Owing to the delicate asymptotics of the centering of the sample as given in (1), we have chosen to take an approach using tools from the theory of empirical processes [17], [18]. For that, some additional smoothness on g is required, specifically continuity of first partial drivatives. This is the most natural way to ensure that g has a derivative, i.e., first-order local linear approximant [21].

Theorem 4. *Let μ be the absolutely continuous probability measure on $\mathbb{R}^d$ for iid random vectors $X, X_1, X_2, \ldots, X_n \ldots$. Assume further that all second moments exist. Let g be defined and with continuous first partial derivatives in a neighborhood of S^{d-1}. Then*

$$\Delta_n = \sqrt{n}\left[\int_{S^{d-1}} g\, d\varphi_n - \int_{S^{d-1}} g\, d\varphi\right] \tag{5}$$

is asymptotically normal with mean zero and variance equal to $\mathrm{Var}[\tilde{g}(X) - (\nabla E\tilde{g}(X))'X]$.

Proof. Note that the (strengthened) conditions on g imply in a routine way that its restriction to S^{d-1} is Lipschitz continuous so that Theorem 3 holds. As in that proof, we assume without loss of generality that $EX = 0$, and alternatively write

$$\Delta_n = \sqrt{n}\left[\frac{1}{n}\sum_1^n \tilde{g}(X_i - \overline{X}_n) - E\tilde{g}(X)\right].$$

For technical reasons relating to the Donsker condition below, we need to control the behavior of $\overline{X}_n$. It will be enough simply to let $\overline{X}_n^*$ be the projection of $\overline{X}_n$ onto the unit ball of $\mathbb{R}^d$, that is, $\overline{X}_n^*$ is that point of the unit ball closest to $\overline{X}_n$; of course, with probability one, there is some (random) N such that $\overline{X}_n^* = \overline{X}_n$ for $n \geq N$.

We will show that

$$\Delta_n^* = \sqrt{n}\left[\frac{1}{n}\sum_1^n \tilde{g}(X_i - \overline{X}_n^*) - E\tilde{g}(X)\right]$$

is asymptotically normal, which will yield our result since

$$\left|\Delta_n^* - \Delta_n\right| \leq \sqrt{n}\sum_i^n \left|\tilde{g}(X_i - \overline{X}_n^*) - \tilde{g}(X_i - \overline{X}_n)\right|$$

$$\leq \sqrt{n}\sum_{i}^{n} L\|\overline{X}_n^* - \overline{X}_n\|$$
$$\leq n^{3/2}L\|\overline{X}_n^* - \overline{X}_n\| \xrightarrow{\text{a.s.}} 0.$$

Consider now the collection of functions $\{\tilde{g}_\theta\}$, where θ is a point in the unit ball of $\mathbb{R}^d$ and $\tilde{g}_\theta(x) = \tilde{g}(x-\theta)$. By the Lipschitz condition, $|\tilde{g}(x-\theta_1) - \tilde{g}(x-\theta_2)| \leq L\|\theta_1 - \theta_2\|$, we have that $\{\tilde{g}_\theta\}$ is a *Donsker class* [18, Example 19.7]. It is also the case that $\int_{\mathbb{R}^d}(\tilde{g}(x-\overline{X}_n^*) - \tilde{g}(x))^2\mu(dx) \to 0$, since the integrand is bounded above by $L^2\|\overline{X}_n^*\|^2 \to 0$. It follows that

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i - \overline{X}_n^*) \;-\; E_X\tilde{g}(X-\overline{X}_n^*)\right] = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i) \;-\; E\tilde{g}(X)\right] + o_p(1)$$

[18, Lemma 19.24]; here E_X signifies expectation with respect to X. Equivalently,

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i - \overline{X}_n^*) \;-\; E\tilde{g}(X)\right]$$
$$= \sqrt{n}\left[E_X\tilde{g}(X-\overline{X}_n^*) - E\tilde{g}(X)\right] + \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i) \;-\; E\tilde{g}(X)\right] + o_p(1). \quad (6)$$

Lemma 2 below verifies that the map $\theta \mapsto E\tilde{g}(X-\theta)$ is differentiable at $\theta = 0$. It follows that

$$E_X\tilde{g}(X-\overline{X}_n^*) = E\tilde{g}(X) - \left[\nabla E\tilde{g}(X)\right]'\overline{X}_n^* + \|\overline{X}_n^*\|\cdot\varepsilon(\|\overline{X}_n^*\|), \quad (7)$$

where ε: $\mathbb{R}^d \to \mathbb{R}^1$ vanishes at 0 and is continuous there. From (6) and (7) it follows that

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i - \overline{X}_n^*) - E\tilde{g}(X)\right]$$
$$= -\sqrt{n}\left[\nabla E\tilde{g}(X)\right]'\overline{X}_n^* + \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i) \;-\; E\tilde{g}(X)\right] + o_p(1), \quad (8)$$

since $\sqrt{n}\cdot\|\overline{X}_n^*\|\varepsilon(\|\overline{X}_n^*\|) = o_p(1)$. Since $\overline{X}_n^* \to \overline{X}_n$ almost surely, the scalar central limit theorem guarantees that the expression in (8) converges to a normal distribution with mean zero. Then the variance of the asymptotic distribution is

$$\mathrm{Var}\left[\sqrt{n}\left[-\nabla E\tilde{g}(X)\right]'\overline{X}_n + \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{g}(X_i) - E\tilde{g}(X)\right)\right]$$
$$= \mathrm{Var}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left[-\nabla E\tilde{g}(X)\right]' X_i + \tilde{g}(X_i) - E\tilde{g}(X)\right)\right]$$
$$= \mathrm{Var}\left[\tilde{g}(X) - (\nabla E\tilde{g}(X))'X\right]. \qquad \square$$

Lemma 2. *Let μ and g be as in Theorem 4. Then the map $\theta \mapsto E\tilde{g}(X-\theta)$ is differentiable at $\theta = 0$.*

Proof. It suffices to show the existence and continuity of all first partial derivatives at $\theta = 0$. Lemma 1 has $\tilde{g}$ Lipschitz continuous (constant L). Let δ_j be the vector having all elements equal to zero except for the jth element, which is set to δ. It follows that

$$\left| \frac{\tilde{g}(x-\theta-\delta_j) - \tilde{g}(x-\theta)}{\delta} \right| \le \frac{L}{|\delta|} \|x - \theta - \delta_j - x + \theta\| = L < \infty.$$

Then by the Dominated Convergence Theorem the jth partial derivative of $E\tilde{g}(X-\theta)$ is

$$\lim_{\delta \to 0} E\left[\frac{\tilde{g}(x-\theta-\delta_j) - \tilde{g}(x-\theta)}{\delta} \right] = E\left[\lim_{\delta \to 0} \frac{\tilde{g}(x-\theta-\delta_j) - \tilde{g}(x-\theta)}{\delta} \right] = E\nabla_j \tilde{g}(x-\theta).$$

We now show continuity at $\theta = 0$ by computing directly the vector of partial derivatives of $\tilde{g}(x-\theta)$ with respect to the components of θ. We note first that the partial derivative of $g[(x-\theta)/\|x-\theta\|]$ is

$$\left[\nabla g \left(\frac{x-\theta}{\|x-\theta\|} \right) \right]' \frac{\partial}{\partial \theta} \left(\frac{x-\theta}{\|x-\theta\|} \right),$$

where

$$\frac{\partial}{\partial \theta} \left(\frac{x-\theta}{\|x-\theta\|} \right) = \frac{1}{\|x-\theta\|^3} (x-\theta)(x-\theta)' - \frac{1}{\|x-\theta\|} I_d.$$

Applying the product rule to $\tilde{g}(x-\theta)$ then yields its vector of partial derivatives with respect to θ as

$$\begin{aligned} &\|x-\theta\| \frac{\partial}{\partial \theta} g \left(\frac{x-\theta}{\|x-\theta\|} \right) - g \left(\frac{x-\theta}{\|x-\theta\|} \right) \frac{(x-\theta)'}{\|x-\theta\|} \\ &\quad = \|x-\theta\| \left[\nabla g \left(\frac{x-\theta}{\|x-\theta\|} \right) \right]' \left[\frac{(x-\theta)(x-\theta)'}{\|x-\theta\|^3} - \frac{I_d}{\|x-\theta\|} \right] \\ &\quad\quad - g \left(\frac{x-\theta}{\|x-\theta\|} \right) \frac{(x-\theta)'}{\|x-\theta\|}, \end{aligned}$$

which is bounded above in norm by

$$\left| g \left(\frac{x-\theta}{\|x-\theta\|} \right) \right| + 2 \left\| \nabla g \left(\frac{x-\theta}{\|x-\theta\|} \right) \right\|.$$

Our assumptions guarantee that (for $\theta \neq x$) this is uniformly bounded above by a constant. Application of the Dominated Convergence Theorem then yields, for each $j = 1, 2, \ldots, d$,

$$\lim_{\theta \to 0} E\nabla_j \tilde{g}(X-\theta) = E \lim_{\theta \to 0} \nabla_j \tilde{g}(X-\theta) = E\nabla_j \tilde{g}(X),$$

which concludes the proof. □

Example. As an important special case of the convergence given by Theorem 4, let g be identically equal to 1. Then (5) asserts that the normalized difference of the surface areas of P_n and K_φ is asymptotically normal with mean 0 and variance equal to $\mathrm{Var}[\|X\| - (\nabla E\|X\|)'X]$.

5. Final Remarks

1. The careful reader will have noted that the surface area measure φ in Theorem 3 does not characterize the underlying probability distribution, even up to translation. This is an interesting question, and we plan to investigate the implied equivalence relation elsewhere.

2. For other recent limit theorems of innovative type, see [4] and [6], the former is similar in spirit to what we have done in the case of a planar sample although it does not impose a centering.

3. The issue of *shape* for a point sample, that is, its nature modulo scale and rigid motions, has been intensively studied in the literature. For a sample in general position, we suggest that shape may alternatively be considered to be precisely the Minkowski polytope, understood in classical geometric terms as modulo rigid motions and normalized, say, to have unit surface area.

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