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# Recursive methods for incentive problems ☆

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ABSTRACT

Many dynamic incentive problems have a primal recursive formulation in which utility promises serve as state variables. We associate dual recursive formulations with these problems. We make transparent the connections between recursive primal and dual approaches, relate value iteration under each and give conditions for such value iteration to be convergent to the true value function.



## 1. Introduction

Dynamic incentive models have received widespread application in finance and macroeconomics. They have been used to provide micro-foundations for market incompleteness, firm capital structure and bankruptcy law. In macroeconomics, first Ramsey and later more general Mirrlees models have informed thinking on tax policy and social insurance. In each of these varied cases the associated dynamic incentive problem recovers optimal equilibrium payoffs and outcomes from a game played by a population of privately informed or uncommitted agents and, often, a committed mechanism designer or principal. Equilibrium restrictions from the game provide the problem's constraints and given additive separability of payoffs over histories, tractable recursive primal and dual formulations are available. In contrast to many other problems in economics, however, this recursivity is often implicit and these formulations must be recovered from the payoff/constraint structure via the addition of constraints that define state variables or through the manipulation of a Lagrangian. Recursive formulations of dynamic incentive problems have been developed in different contexts by Kydland and Prescott (1980), Abreu et al. (1990), Green (1987), Spear and Srivastava (1987), Fernandes and Phelan (2000), Judd et al. (2003) and Marcet and Marimon (2011). Each of these papers uses or develops a particular method and several consider a particular application. Our goal is to provide a unified treatment of recursive primal and dual approaches for dynamic incentive problems.

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We use basic results from the theory of dynamic programming and duality, especially conjugate function duality, to do so. We emphasize practical issues associated with the application of these methods and identify when particular methods are valid. We relate value iteration under each method and give conditions for both primal and dual value iteration to converge to the true value function.

Many dynamic economic problems include amongst their primitives a state space and a non-empty valued constraint correspondence. Given the current state, the correspondence restricts current choices inclusive of the subsequent state. In contrast, dynamic incentive problems are usually formulated in terms of incentive constraints that are non-recursive and forward-looking. State spaces and constraint correspondences must be recovered from these. A well-known approach re-expresses incentive constraints in terms of utility promises and supplements them with promise-keeping conditions that enforce past promises. Thus, utility promises serve as states. We develop the promise approach in a framework general enough to handle a variety of frictions including limited commitment and hidden information, as well as combinations of the two. Our framework accommodates problems with and problems without a committed principal. In the former, the promise approach yields an optimal payoff for the principal as a function of an initial promise to agents; in the latter it gives the set of equilibrium payoffs available to all players. In this second case, indicator functions are used to represent equilibrium payoff sets, permitting a recasting of the set-theoretic treatment of equilibrium payoffs in Abreu et al. (1990) in terms of value functions.

A difficulty with the promise approach is that some promises may not be attainable with any incentive-feasible allocation or plan. Several contributions have followed a two step procedure: in a first step, an “endogenous state space” of feasible promises is approximated; in a second, the optimal value function restricted to this state space is calculated.[1] We pursue an alternative approach. By assigning minus infinite payoffs to infeasible states, we jointly handle the value function and the endogenous state space, encoding the latter as the “effective domain” of the former.[2] An elaboration of standard arguments implies that the optimal value function solves a Bellman equation. By way of a partial converse, we give conditions for the optimal value function to be the limit of a value iteration. In economics, it is standard to consider problems with norm-bounded value functions, contractive Bellman operators and uniformly convergent (with respect to the norm) value iteration. Dynamic incentive problems with possibly non-finite value functions do not fit into this class. For these problems, hypo- and epi-convergence provide useful alternatives to uniform convergence. They relate well to optimization and to the dual approaches discussed below. We give conditions for the optimal value function to be the limit of a hypo-convergent value iteration and for the limits of policies computed during this iteration to be optimal.

Non-finite value functions remain problematic from the point of view of practical computation since they introduce arbitrarily large discontinuities or arbitrarily large steepness at the boundaries of their effective domains. In addition, in many dynamic incentive problems constraints run across current shock realizations. In these cases, the promise-keeping formulation does not permit a decomposition across time and shocks. Instead, choices contingent on all current shock realizations must be solved for simultaneously raising the dimension of the optimizations that define the Bellman operator. These issues motivate the development of dual approaches.

Constrained optimizations can be re-expressed using Lagrangians that incorporate some or all of the constraints. In these re-expressed problems, a sup–inf operation over choices and Lagrange multipliers replaces a sup operation over choices alone. In this form the optimization is referred to as a primal problem. The sequencing of sup and inf is important. By interchanging them a dual problem is obtained. We express the dynamic incentive Bellman operator in sup–inf terms and then interchange these operations to obtain its dual. From the dual Bellman operator, we extract a component dual operator $\mathscr{D}$. This operator relocates calculations to an alternative space of value functions defined on a domain of agent Pareto weights. $\mathscr{D}$ combines a Pareto weight-domain value function with a law of motion for such weights and a current payoff function over choices and multipliers; it then performs a dual inf–sup operation with respect to current choices and incentive multipliers on the resulting objective. The updated Pareto weights encode continuation rewards and penalties much as updated promises do. Under certain regularity conditions, a promise-domain value function can be uniquely paired with a weight-domain one via a mathematical operation called conjugacy. Under related conditions, $\mathscr{D}$ and the original (promise-domain) Bellman operator are tightly connected. In particular, our earlier results on (hypo-)convergent promise-domain value iteration imply (epi-)convergent Pareto weight-domain value iteration. In the latter case, the limiting value function is the optimal one from a Pareto problem (and is the conjugate of the optimal promise-domain value function). Working with dual operator $\mathscr{D}$ has certain computational advantages. It is often possible to restrict attention to real-valued weight-domain value functions and avoid the difficulties associated with infinite-valued functions and endogenous effective domains that emerge in the promise-domain setting. In addition, $\mathscr{D}$ decomposes the supremum operation into a family of shock-contingent supremum operations tied together by incentive multipliers. Thus, the high(er) dimensional supremum operations encountered in the primal promise-keeping formulation are replaced with groups of simpler supremum operations, further assisting calculations.

The paper proceeds as follows. After a brief literature review, Section 2 gives examples. Section 3 describes a framework that can accommodate a large class of dynamic incentive problems. In Section 4, a recursive promise-keeping formulation is outlined and a primal Bellman operator introduced. In Section 5, sufficient conditions for convergent primal value iteration

[1] For example, see Abraham and Pavoni (2008).

[2] If $f : X \to \mathbb{R} \cup \{-\infty\}$, then the effective domain of $f$ is the set $\{x \in X\colon\ f(x) > -\infty\}$.

are given. Section 6 defines the dual Bellman operator and extracts the component $\mathscr{D}$ from it. Sufficient conditions for convergent dual value iteration are provided and the merits of the dual approach discussed. Section 7 gives extensions.

*Literature* Papers by Green (1987), Spear and Srivastava (1987) and Abreu et al. (1990) provide early applications of the promise approach to dynamic incentive problems with and without committed principals. Further applications are provided by, inter alia, Fernandes and Phelan (2000), Kocherlakota (1996) and Rustichini (1998). Of these, the latter paper is most related to ours; it also uses monotone iterative methods and hypo-convergence. However, in other dimensions our respective approaches are different. Rustichini (1998) does not consider dual approaches and confines himself to limited commitment problems, we generalize these aspects. On the other hand, we confine ourselves to problems with a time additively separable structure, Rustichini does not.

We build on Messner et al. (2011). In that earlier contribution, we paired a sequential primal problem with a sequential dual problem and then obtained a recursive formulation of the latter. Thus, we "dualized" and then "recursivized". In the current paper we reverse these roles, first deriving a recursive primal problem and then "dualizing" this. The approach in Messner et al. (2011) has the advantage that it permits a significant weakening of the conditions for equality of optimal primal and dual values and solutions. The approach in the current paper is useful for directly relating primal and dual Bellman operators. Messner et al. (2011) showed how the recursive dual approach can be applied to non-additively separable problems and provide an application to hidden action moral hazard. In the current paper, we focus on time additively separable, infinite horizon incentive problems. We discuss in detail the properties of primal and dual Bellman operators and their fixed points in these settings.

In an important and influential contribution, Marcet and Marimon (1999) (revised: 2011) develop a recursive saddle point method for a class of dynamic contracting problems. Let $\mathscr{L}: A \times \Lambda \to \mathbb{R}$ denote a Lagrangian defined on a space of actions and multipliers. The saddle operation is given by: $\text{saddle}_{(a|\lambda)}\mathscr{L}(a,\lambda) = \mathscr{L}(a^*,\lambda^*)$, where $a^* \in \arg\max_A \mathscr{L}(a,\lambda^*)$ and $\lambda^* \in \arg\min_\Lambda \mathscr{L}(a^*,\lambda)$. Marcet and Marimon (1999) seek a recursive decomposition of this operation. Here the maximization and minimization are done in parallel, each taking the solution of the other as given. In contrast, we interchange sup–inf with inf–sup operations (in each case the operations are performed in sequence), justifying the interchange with appeals to standard duality results. As described above, we tie a dual (inf–sup) Bellman operator acting on weight-domain value functions to a primal (sup–inf) Bellman operator acting on promise-domain value functions. In particular, we relate monotone value iteration under each approach. Marcet and Marimon (1999)/(2011) consider their recursive saddle approach in isolation. Apart from establishing the connection between approaches, this also allows us to considerably relax the boundedness conditions necessary for convergent value iteration. In contrast to us, Marcet and Marimon (2011) explicitly incorporate physical state variables such as capital, we do not. On the other hand, they exclude problems with private information and focus on ones with a committed principal or government. We extend both of these elements. Marcet and Marimon (2011) construct Lagrangians that include incentive constraints (and, implicitly, the law of motion for promises), but exclude the law of motion for physical state variables. As a result, their recursive formulation features a mixture of primal and dual state variables. We keep all laws of motion in either primal or dual form. Thus, we use either primal (promise) or dual (multiplier) state variables, but never a mixture.

Judd et al. (2003) develop methods for computing the value sets of repeated games (in which no player is committed). Their outer approximation method can be interpreted as a variation on our dual approach in which only the promise-keeping constraints are dualized (i.e. brought up into the Lagrangian). They couple this with an approximation procedure for the resulting value functions.

Passing from a primal to a dual problem introduces the possibility of a difference between the optimal dual and primal values (a "duality gap"). If a saddle point exists at each primal state variable, then there are no duality gaps and all primal solutions are maximizers for the corresponding dual problem. However, the dual problem may introduce extraneous solutions, see Messner and Pavoni (2004). This issue is resolved in strictly concave settings (in which the set of dual maximizers is a singleton). Recently, Cole and Kubler (2010) provide a clever method for recovering optimal primal solutions from an augmented problem formulated on a dual state space.

Our value iteration approach relies on quite general monotone arguments. Messner et al. (2012) give conditions for the dual Bellman operator to be contractive.

## 2. Examples

To motivate the subsequent analysis and set notation, we begin with three examples. The first example illustrates our approach in a simple setting. The remaining examples show how the approach can be applied to more complex economic problems. We refer back to these examples throughout the paper.

**Example 1** *(One-sided limited commitment).* A committed principal shares risk with an uncommitted agent. Both live for an infinite number of periods $t \in \mathbb{N}$. Let $\{s_t\}_{t=1}^{\infty}$ be a shock process taking values in a finite set $\mathbb{S} = \{1,\dots,S\}$ and evolving according to a Markov transition $Q$ from $s_0 \in \mathbb{S}$. Let the endowment of goods in each current state be given by $\omega: \mathbb{S} \to \mathbb{R}_{++}$.

A consumption process $\{c_t\}_{t=1}^{\infty}$, $c_t : \mathbb{S}^t \to \mathbb{R}_+$ is resource-feasible if for all $t \in \mathbb{N}$, $s^t \in \mathbb{S}^t$, $c_t(s^t) \in [0, \omega(s_t)]$. Conditional on the first period shock, the principal and the agent value resource-feasible consumption processes $c^\infty$ according to:

$$U^c(s', c^\infty) = \sum_{t=1}^{\infty} \beta^{t-1} E\big[v\big(c_t(s^t)\big) \bigm| s_1 = s'\big], \tag{1}$$

where $\beta \in (0,1)$ and $v : \mathbb{R}_+ \to \mathbb{R}$ satisfies the following assumption.

**Assumption 1.** $v : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing and strictly concave.

Date 0 evaluations of $c^\infty$ are given by $\sum_{\mathbb{S}} U^c(s', c^\infty) Q(s, s') = \sum_{t=1}^{\infty} \beta^{t-1} E[v(c_t(s^t)) \mid s_0 = s]$. Let $c^\infty$ denote the agent's consumption process, with the principal receiving the residual $\omega(s_t) - c_t(s^t)$ at each $s^t$. $c^\infty$ is incentive-feasible if it is resource-feasible and it provides the agent with utility in excess of her outside option $B$:

$$\forall t, s^t, \quad v\big(c_t(s^t)\big) + \beta \sum_{s' \in \mathbb{S}} U^c\big(s', c_{t+1}^\infty(s^t)\big) Q(s_t, s') \geqslant B(s_t),$$

where $c_{t+1}^\infty(s^t) = \{c_{t+j}(s^t, \cdot)\}_{j=1}^{\infty}$ and $B : \mathbb{S} \to \mathbb{R}$ satisfies:

$$B(s) = b(s) + \beta \sum_{s' \in \mathbb{S}} B(s') Q(s, s'), \quad b : \mathbb{S} \to \mathbb{R}.$$

**Assumption 2.** $\forall s$, $b(s) \in [v(0), v(\omega(s))]$.

It is convenient to re-express the analysis in terms of utility net of the outside option. Given $c^\infty$ the agent's net utility plan $y^\infty = \{y_t\}_{t=1}^{\infty}$ is for all $t$, $s^t$, $y_t(s^t) = v(c_t(s^t)) - b(s_t)$. Let $h = v^{-1}$ and $C(s) = \{y\colon \omega(s) - h(y + b(s)) \geqslant 0\}$. Assumption 1 ensures that each $C(s)$ is compact, Assumption 2 that each $C(s)$ is non-empty. Let $\Omega_0 = \{y^\infty \mid \forall t, y_t(\cdot, s_t) \in C(s_t)\}$ denote the set of resource-feasible net utility plans. Conditional on $s_1 = s'$, the net value of a plan $y^\infty \in \Omega_0$ to the agent is:

$$U(s', y^\infty) = \sum_{t=1}^{\infty} \beta^{t-1} E\big[y_t(s^t) \bigm| s_1 = s'\big];$$

conditional on $s_0 = s$, its value to the principal is:

$$F(s, y^\infty) = \sum_{t=1}^{\infty} \beta^{t-1} E\big[f\big(s_t, y_t(s^t)\big) \bigm| s_0 = s\big],$$

where $f(s', y) = v(\omega(s') - h(y + b(s')))$. A plan in $\Omega_0$ is incentive-feasible if it delivers a non-negative net utility to the agent for all $t$, $s^{t-1}$, $s$,

$$y_t(s^{t-1}, s) + \beta \sum_{s' \in \mathbb{S}} U\big(s', y_{t+1}^\infty(s^{t-1}, s)\big) Q(s, s') \geqslant 0. \tag{2}$$

Let $\Omega_1$ denote the set of incentive-feasible plans. Absent further constraints, the principal's problem at $s_0 = s$ is:

$$\sup_{\Omega_1} F(s, y^\infty). \tag{3}$$

The first step in our analysis is to show how the non-recursive problem (3) can be embedded into a family of "perturbed" problems that are recursive. For each $(s, x) \in \mathbb{S} \times \mathbb{R}$ augment $\Omega_1$ with the "promise-keeping" condition:

$$x = \sum_{s' \in \mathbb{S}} U(s', y^\infty) Q(s, s'),$$

to obtain:

$$\Omega_1(s, x) = \left\{ y^\infty \in \Omega_1 \;\middle|\; x = \sum_{s' \in \mathbb{S}} U(s', y^\infty) Q(s, s') \right\}.$$

Plans in $\Omega_1(s, x)$ are incentive-feasible and deliver a lifetime utility of $x$ to the agent conditional on prior shock $s$. We will say that a shock-promise pair $(s, x)$ is *incentive-feasible* if $\Omega_1(s, x) \neq \emptyset$. For the current problem it is easy to see that $(s, x)$ is incentive-feasible if and only if $x \in [0, \overline{U}(s)]$, where $\overline{U}(s) = \sum_{t=1}^{\infty} \beta^{t-1} E[v(\omega_t(s^t)) \mid s_0 = s] - B(s)$. However, in more complicated settings the set of incentive-feasible $(s, x)$ pairs is harder to characterize. Since one of our objectives is to show how to derive and solve a recursive formulation of the principal's problem without first determining the incentive-feasible set, we proceed as if this set is unknown.

The associated family of *promise-constrained* problems is, for each $(s, x) \in \mathbb{S} \times \mathbb{R}$,

$$W^*(s, x) = \sup_{\Omega_1(s,x)} F\big(s, y^\infty\big), \tag{4}$$

where $W^*(s, x) = -\infty$ if $\Omega_1(s, x) = \emptyset$ and $(s, x)$ is not incentive-feasible. Problem (3) may be solved by recovering solutions to (4) at $x^* \in \arg\max_{x \in \mathbb{R}} W^*(s, x)$.

A recursive formulation of the promise-constrained problems (4) is obtained by defining the choice set $A = \prod_{s' \in \mathbb{S}} C(s') \times \mathbb{R}^S$ and the constraint correspondence $\Gamma : \mathbb{S} \times \mathbb{R} \to 2^A$,

$$\Gamma(s, x) = \left\{ \big(y, x'\big) \in A \;\middle|\; \begin{array}{l} x = \sum_{s' \in \mathbb{S}} \{y_{s'} + \delta x'_{s'}\} Q\big(s, s'\big) \\ y_{s'} + \delta x'_{s'} \geqslant 0,\ s' \in \mathbb{S} \end{array} \right\}.$$

$\Gamma(s, x)$ gives current utilities and continuation utility promises consistent with keeping the utility promise $x$ given shock $s$ and the current incentive constraints. $\Gamma$ is used to define the *Bellman operator* $\mathscr{B}$ on $\mathscr{W} = \{W : \mathbb{S} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}\}$ according to:

$$W \in \mathscr{W},\ \forall (s, x) \in \mathbb{S} \times \mathbb{R}, \quad \mathscr{B}(W)(s, x) = \sup_{\Gamma(s,x)} \sum_{s' \in \mathbb{S}} \big\{ f\big(s', y_{s'}\big) + \beta W\big(s', x'_{s'}\big) \big\} Q\big(s, s'\big). \tag{5}$$

By fairly standard arguments, $W^*$ satisfies the Bellman equation $W^* = \mathscr{B}(W^*)$. However, when defined on all of $\mathbb{S} \times \mathbb{R}$, $W^*$ is *extended real-valued*, i.e. can take the value $-\infty$. In Section 5, the monotonicity of $\mathscr{B}$ and convergence concepts for extended real-valued functions are used to show that $W^*$ may be obtained as the limit of a convergent value iteration. Alternatively, dual methods may be used. We describe these next.

The current problem has one promise-keeping constraint and $S$ incentive constraints. The first of these is an equality constraint, the remainder are inequality constraints. Let $\Phi = \mathbb{R} \times \mathbb{R}^S_+$ denote the corresponding multiplier set with element $(\mu, \{Q(s, s')\eta^{s'}\})$. Given $W \in \mathscr{W}$, define the Lagrangian $\mathscr{L}_s(W) : A \times \Phi \to \mathbb{R}$ according to:

$$\begin{aligned} \mathscr{L}_s(W)\big(y, x'; \mu, \eta\big) = {} & \sum_{\mathbb{S}} \big\{ f\big(s', y_{s'}\big) + \beta W\big(s', x'_{s'}\big) \big\} Q\big(s, s'\big) + \mu \sum_{\mathbb{S}} \{ y_{s'} + \delta x'_{s'} \} Q\big(s, s'\big) \\ & + \sum_{\mathbb{S}} Q\big(s, s'\big) \eta^{s'} \big[ y_{s'} + \delta x'_{s'} \big]. \end{aligned}$$

Since the infimum operation $\inf_\Phi \mathscr{L}_s(W)(y, x'; \mu, \eta) - \mu x$ attaches $-\infty$ values to choices $(y, x')$ outside of $\Gamma(s, x)$ and $\sum_{\mathbb{S}} \{f(s', y_{s'}) + \beta W(s', x'_{s'})\} Q(s, s')$ values to those inside, the Bellman operator $\mathscr{B}$ may be re-expressed in "sup–inf" form as:

$$\mathscr{B}(W)(s, x) = \sup_A \inf_\Phi \mathscr{L}_s(W)\big(y, x'; \mu, \eta\big) - \mu x. \tag{6}$$

By interchanging supremum and infimum operations a dual Bellman operator is derived:

$$\mathscr{B}^D(W)(s, x) = \inf_\Phi \sup_A \mathscr{L}_s(W)\big(y, x'; \mu, \eta\big) - \mu x.$$

Given Assumptions 1 and 2, $\mathscr{B}^D(W) = \mathscr{B}(W)$ (see Section 6). Application of $\mathscr{B}^D$ simplifies calculations since it decomposes the supremum operation in (5) or (6) allowing each $(y_{s'}, x'_{s'})$ to be solved for separately conditional on the multipliers $\mu$ and $\eta$, which are solved for in the outer inf. However, its application still involves potentially extended real-valued value functions.

We extract from $\mathscr{B}^D$ an operator $\mathscr{D}$ that relocates calculations to an alternative space of value functions defined on a "Pareto weight" domain. The key advantage of doing so is that for many problems, including this one, these functions can be restricted to be real-valued. Grouping terms involving $y_{s'}$ and $x'_{s'}$ together, the Lagrangian $\mathscr{L}_s(W)$ may be reorganized to give:

$$\mathscr{L}_s(W)\big(y, x'; \mu, \eta\big) = \sum_{\mathbb{S}} \big\{ f\big(s', y_{s'}\big) + \big(\mu + \eta^{s'}\big) y_{s'} + \beta \big( W\big(x'_{s'}\big) + \big(\mu + \eta^{s'}\big) x'_{s'} \big) \big\} Q\big(s, s'\big).$$

Consequently, $\mathscr{B}^D$ may be rewritten as:

$$\begin{aligned} \mathscr{B}^D(W)(s, x) = \inf_{\mu \in \mathbb{R}} \inf_{\eta \in \mathbb{R}^S_+} \sum_{\mathbb{S}} \Big\{ & \Big( \sup_{y_{s'} \in C(s')} f\big(s', y_{s'}\big) + \big(\mu + \eta^{s'}\big) y_{s'} \Big) \\ & + \beta \Big( \sup_{x'_{s'} \in \mathbb{R}} W\big(x'_{s'}\big) + \big(\mu + \eta^{s'}\big) x'_{s'} \Big) \Big\} Q\big(s, s'\big) - \mu x. \end{aligned}$$

It follows that application of $\mathscr{B}^D$ can be broken into three steps. In the first step, the continuation value function is relocated to the space of Pareto weights by solving:

$$\forall s', \mu' \in \mathbb{S} \times \mathbb{R}, \quad V(s', \mu') = \sup_{x' \in \mathbb{R}} W(s', x') + \mu' x'.$$

In the second step, the value function $V$ is combined with the (dual) current objectives:

$$\nu_{s'}(\mu, \eta) = \sup_{y \in C(s')} f(s', y) + (\mu + \eta^{s'}) y,$$

and the updating functions, $\mu'_{s'}(\mu, \eta) = \mu + \eta^{s'}$, to give a new objective over $\eta$. The infimum of this objective over $\eta$ is then calculated:

$$\mathscr{D}(V)(s, \mu) = \inf_{\eta \in \mathbb{R}^S_+} \sum_{\mathbb{S}} \{\nu_{s'}(\mu, \eta) + \beta V(s', \mu'_{s'}(\mu, \eta))\} Q(s, s'). \tag{7}$$

In the third step the value function $\mathscr{D}(V)$ is returned to the promise-domain with an infimum operation over $\mu$:

$$\mathscr{B}^D(W)(s, x) = \inf_{\mu \in \mathbb{R}} \mathscr{D}(V)(s, \mu) - \mu x.$$

Now consider extracting the operator $\mathscr{D}$ (defined in (7)) from $\mathscr{B}^D$. Under Assumptions 1 and 2,

$$W^*(s, x) = \inf_{\mu \in \mathbb{R}} V^*(s, \mu) - \mu x, \tag{8}$$

where $V^*$ is a fixed point of $\mathscr{D}$: $V^* = \mathscr{D}(V^*)$. Moreover, $V^*$ is the value function from the Pareto problem:

$$V^*(s, \mu) = \sup_{\Omega_1} F(s, y^\infty) + \mu \sum_{s' \in \mathbb{S}} U(s', y^\infty) Q(s, s').$$

Significantly, since $\Omega_1$ is non-empty and $F$ and $U$ are bounded on $\Omega_1$, $V^*$ is real-valued. Thus, (3) or (4) may be solved by iterating with $\mathscr{D}$ on a space of *real-valued functions* to obtain $V^*$. $W^*$ may then be recovered as in (8) or, alternatively, the optimal value in (3) may be obtained directly as $V^*(s, 0)$.

**Example 2** *(Hidden information: IID shocks).* Our next example is a hidden information problem of the sort considered by Atkeson and Lucas (1992) and Farhi and Werning (2007). An agent experiences privately observed taste shocks $\{s_t\}_{t=1}^\infty$ over her life. These shocks take values in the finite set $\mathbb{S}$ and are assumed to be i.i.d. with per period distribution $Q \in \mathbb{R}^S$. The agent values consumption processes $c^\infty = \{c_t\}_{t=1}^\infty$, $c_t : \mathbb{S}^t \to \mathbb{R}_+$, according to:

$$\sum_{t=1}^{\infty} \delta^{t-1} E[s_t v(c_t(s^t))],$$

where $\delta \in (0, 1)$ and $v : \mathbb{R}_+ \to Y = [\underline{y}, \bar{y})$, $-\infty < \underline{y} < \bar{y} \leqslant \infty$.

We re-express the analysis in terms of utility-from-consumption. Given a consumption process $c^\infty$, the corresponding utility plan is $y^\infty = \{y_t\}_{t=1}^\infty$, where $y_t(s^t) = v(c_t(s^t))$. Let $U(s', y^\infty) = \sum_{t=1}^\infty \delta^{t-1} E[s_t y_t(s^t) \mid s_1 = s']$ denote the agent's lifetime payoff from utility plan $y^\infty$ conditional on first period shock $s'$ and let $\Omega_0 = \{y^\infty \mid \forall t, y_t : \mathbb{S}^t \to Y\}$ denote the set of plans. Without loss of generality attention may be restricted to utility plans that induce the agent to truthfully report the shocks she receives. Such plans satisfy the incentive-compatibility conditions, for all $t$, $s^{t-1}$, $s' \neq \widehat{s}$,

$$s' y_t(s^{t-1}, s') + \delta \sum_{s'' \in \mathbb{S}} U(s'', y_{t+1}^\infty(s^{t-1}, s')) Q(s'') \geqslant s' y_t(s^{t-1}, \widehat{s}) + \delta \sum_{s'' \in \mathbb{S}} U(s'', y_{t+1}^\infty(s^{t-1}, \widehat{s})) Q(s''), \tag{9}$$

where $s'$ denotes a true shock, $\widehat{s}$ a lie and $y_{t+1}^\infty(s^t)$ is the continuation of $y^\infty$ after $s^t$. Let $\Omega_1$ denote the set of plans in $\Omega_0$ satisfying these constraints.

A principal seeks to maximize a discounted stream of utility to the agent net of a resource cost. The principal's per period objective is given by $f(s', y) = s' y - \psi h(y)$, where $\psi \in \mathbb{R}_{++}$ is the shadow price of resources and $h : Y \to \mathbb{R}_+$ is the inverse of $v$. The following assumption is imposed directly on $h$.

**Assumption 3.** $h$ is strictly convex, increasing, continuous and differentiable on the interior of $Y$. In addition, $h(\underline{y}) = 0$, $\lim_{y \uparrow \bar{y}} h(y) = \infty$, $\lim_{y \downarrow \underline{y}} h'(y) = 0$ and $\lim_{y \uparrow \bar{y}} h'(y) = \infty$.

The principal uses a discount factor $\beta \in [0, 1)$; her lifetime payoff is given by: $F(y^\infty) = \sum_{t=1}^\infty \beta^{t-1} E[f(s_t, y_t(s^t))]$. Given Assumption 3, $f$ is bounded above and $F$ is well defined, though possibly $-\infty$-valued. Atkeson and Lucas (1992) assume

$\delta = \beta$; Farhi and Werning (2007) assume $\delta < \beta$. The planner's problem is $\sup_{\Omega_1} F(y^\infty)$. The promise-perturbed version of this problem is obtained, as previously, by augmenting $\Omega_1$ with a "promise-keeping" condition:

$$W^*(x) = \sup_{\Omega_1(x)} F\left(y^\infty\right), \tag{10}$$

with:

$$\Omega_1(x) = \left\{ y^\infty \in \Omega_1 \;\middle|\; x = \sum_{s' \in \mathbb{S}} U\left(s', y^\infty\right) Q\left(s'\right) \right\}$$

and $W^*(x) = -\infty$ if $\Omega_1(x) = \emptyset$. $W^*$ satisfies a Bellman equation $W^* = \mathscr{B}(W^*)$,

$$\mathscr{B}(W)(x) = \sup_{\Gamma(x)} \sum_{s' \in \mathbb{S}} \left\{ f\left(s', y_{s'}\right) + \beta W\left(x'_{s'}\right) \right\} Q\left(s'\right),$$

with choice set $A = Y^S \times \mathbb{R}^S$ and constraint correspondence $\Gamma : \mathbb{R} \to 2^A$:

$$\Gamma(x) = \left\{ \left(y, x'\right) \in A \;\middle|\; \begin{array}{l} x = \sum_{s' \in \mathbb{S}} \left[s' y_{s'} + \delta x'_{s'}\right] Q\left(s'\right) \\ s' y_{s'} + \delta x'_{s'} \geqslant s' y_{\widehat{s}} + \delta x'_{\widehat{s}}, \; \left(s', \widehat{s}\right) \in \mathbb{M} \end{array} \right\}.$$

$\mathbb{M} = \{(s', \widehat{s}) \in \mathbb{S}^2, s \neq \widehat{s}\}$ gives the set of $M = S(S-1)$ constraint indices.

Proceeding through the same steps as before and assigning $\eta^{s',\widehat{s}}$ to the $(s', \widehat{s})$-incentive constraint, we obtain the dual operator:

$$\mathscr{D}(V)(\mu) = \inf_{\eta \in \mathbb{R}^M_+} \sum_{\mathbb{S}} \left\{ \nu_{s'}(\mu, \eta) + \beta V\left(\mu'_{s'}(\mu, \eta)\right) \right\} Q\left(s'\right), \tag{11}$$

where now

$$\nu_{s'}(\mu, \eta) = \sup_{y \in Y} f\left(s', y\right) + \left( \mu s' + \sum_{\widehat{s} \neq s'} \frac{\eta^{s',\widehat{s}}}{Q\left(s'\right)} s' - \sum_{\widehat{s} \neq s'} \frac{\eta^{\widehat{s},s'}}{Q\left(s'\right)} \widehat{s} \right) y,$$

and the updating function is

$$\mu'_{s'}(\mu, \eta) = \frac{\delta}{\beta} \left( \mu + \sum_{\widehat{s} \neq s'} \frac{\eta^{s',\widehat{s}}}{Q\left(s'\right)} - \sum_{\widehat{s} \neq s'} \frac{\eta^{\widehat{s},s'}}{Q\left(s'\right)} \right).$$

Once more $\mathscr{D}$ relocates calculations to an alternative space of value functions defined on a "Pareto weight" domain. Under Assumption 3 and additional regularity conditions that we describe below $V^* = \mathscr{D}(V^*)$ where $V^*$ is the value function from the Pareto problem:

$$V^*(\mu) = \sup_{\Omega_1} F\left(y^\infty\right) + \mu \sum_{s' \in \mathbb{S}} U\left(s', y^\infty\right) Q\left(s'\right).$$

**Example 3** *(Hidden information: Markov shocks).* Suppose that everything is as in Example 2 except that now shocks evolve according to a Markov process with transition $Q$ from $s_0$. The planner's problem is: $\sup_{\Omega_1} F(s, y^\infty)$, with $\Omega_1$ modified to include Markov shocks and $F(s, y^\infty) = \sum_{t=1}^\infty \beta^{t-1} E[f(s_t, y_t^\infty(s^t)) \mid s_0 = s]$. As in Fernandes and Phelan (2000), to obtain a recursive problem, it is necessary to augment $\Omega_1$ with "ex post" promise-keeping constraints that compel promises conditional on each period 1 shock, i.e. for each $x \in \mathbb{R}^S$,

$$\Omega_1(x) = \left\{ y^\infty \in \Omega_1 \;\middle|\; \forall s' \in \mathbb{S}, x_{s'} = U\left(s', y^\infty\right) \right\}.$$

The associated family of *promise-constrained* problems is then, for each $(s, x) \in \mathbb{S} \times \mathbb{R}^S$,

$$W^*(s, x) = \sup_{\Omega_1(x)} F\left(s, y^\infty\right), \tag{12}$$

where again $W^*(s, x) = -\infty$ if $\Omega_1(x) = \emptyset$ (and $x$ is not incentive-feasible). This time direct characterization of the set of incentive-feasible promises is more complicated.

To derive the recursive form of the promise-constrained problems (12), let $A = Y^S \times \mathbb{R}^{S \times S}$ and modify the constraint correspondence $\Gamma : \mathbb{R}^S \to 2^A$ to include the ex post promise-keeping constraints:

$$\Gamma(x) = \left\{ \left(y, x'\right) \in A \;\middle|\; \begin{array}{l} x_{s'} = s' y_{s'} + \delta \sum_{s''} x'_{s',s''} Q\left(s', s''\right) \\ s' y_{s'} + \delta \sum_{s''} x'_{s',s''} Q\left(s', s''\right) \geqslant s' y_{\widehat{s}} + \delta \sum_{s''} x'_{\widehat{s},s''} Q\left(s', s''\right), \; \left(s', \widehat{s}\right) \in \mathbb{M} \end{array} \right\}.$$

Define the Bellman operator $\mathscr{B}$ on $\mathscr{W} = \{W : \mathbb{S} \times \mathbb{R}^S \to \mathbb{R} \cup \{-\infty\}\}$ according to:

$$\mathscr{B}(W)(s, x) = \sup_{\Gamma(x)} \sum_{s' \in \mathbb{S}} \left\{ f\left(s', y_{s'}\right) + \beta W\left(s', x'_s\right) \right\} Q\left(s, s'\right). \tag{13}$$

Again standard arguments give $W^* = \mathscr{B}(W^*)$ and once more $W^*$ is *extended* real-valued as a function on $\mathbb{S} \times \mathbb{R}^S$ (and as noted the set upon which it is finite is more complicated).

Following the same steps as before, we obtain the dual operator:

$$\mathscr{D}(V)(s, \mu) = \inf_{\eta \in \mathbb{R}^M_+} \sum_{\mathbb{S}} \left\{ \nu_{s,s'}(\mu, \eta) + \beta V\left(s', \mu'_{s,s'}(\mu, \eta)\right) \right\} Q\left(s, s'\right), \tag{14}$$

where:

$$\nu_{s,s'}(\mu, \eta) = \sup_{y \in Y} f\left(s', y\right) + \left( \mu_{s'} s' + \sum_{\widehat{s} \neq s'} \frac{\eta^{s',\widehat{s}}}{Q\left(s, s'\right)} s' - \sum_{s \neq s'} \frac{\eta^{s,s'}}{Q\left(s, s'\right)} \widehat{s} \right) y$$

and the weight updating function is $\mu_{s,s'} = \{\mu_{s,s',s''}\}_{s'' \in \mathbb{S}}$ with:

$$\mu'_{s,s',s''}(\mu, \eta) = \frac{\delta}{\beta} \left( \mu_{s'} + \sum_{\widehat{s} \neq s'} \frac{\eta^{s',\widehat{s}}}{Q\left(s, s'\right)} - \sum_{\widehat{s} \neq s'} \frac{\eta^{\widehat{s},s'}}{Q\left(s, s'\right)} \frac{Q\left(\widehat{s}, s''\right)}{Q\left(s', s''\right)} \right).$$

Again the operator $\mathscr{D}$ relocates calculations to an alternative space of value functions defined on a "Pareto weight" domain and $V^* = \mathscr{D}(V^*)$, where $V^*$ is the value function from the Pareto problem.

## 3. A framework for dynamic incentive problems

The previous section sketched in the context of examples how recursive and dual recursive problems may be associated with dynamic incentive problems. In the remainder of the paper we formalize and generalize these ideas. We give sufficient conditions for optimal value and dual value functions to be fixed points of relevant Bellman operators and show how these operators may be used to find the associated value functions and (restrictions on) the associated policy functions. We begin by presenting a dynamic incentive framework general enough to encompass many economic applications including the three from the previous section. The framework incorporates a principal who can commit and has no private information and a single incentive-constrained agent. In Section 7 we show how this framework may be extended to encompass problems with multiple agents and/or without a committed principal.

Let $Y \subset \mathbb{R}^m$ denote a set of actions and $\mathbb{S}$ a finite set of shock values as before. Identify time with the natural numbers: $t \in \mathbb{N}$. Let $s_t$ be an $\mathbb{S}$-valued random variable describing the shock in period $t$ and $s^t$ an $\mathbb{S}^t$-valued random variable describing shock histories up to period $t$. Let $C : \mathbb{S} \to 2^Y \setminus \{\emptyset\}$ denote a correspondence giving the set of actions potentially available to the agent in each shock state.

A plan is a sequence $y^\infty = \{y_t\}_{t=1}^\infty$ with $y_t : \mathbb{S}^t \to Y$ and $y_t(s^t) \in C(s_t)$. Let $Q$ denote a Markov transition for shocks, which, given our finiteness assumption, we identify with an $S \times S$-matrix of positive transition probabilities $\{Q(s, s')\}$. Let $u : \operatorname{Graph} C \to \mathbb{R}$ give the agent's per period payoff as a function of the current shock and action and let $\delta \in [0, 1)$ denote the agent's discount factor. Given $s_1 = s$, the agent's lifetime payoff function from an action plan $y^\infty$ is:

$$U\left(s, y^\infty\right) := \liminf_{T \to \infty} \sum_{t=1}^{T} \delta^{t-1} E\left[u\left(s_t, y_t\left(s^t\right)\right) \mid s_1 = s\right].$$

Define:

$$\Omega_0 = \left\{ y^\infty \;\middle|\; \forall t,\, y_t\left(s^t\right) \in C(s_t),\ \forall s \in \mathbb{S},\, U\left(s, y^\infty\right) = \lim_{T \to \infty} \sum_{t=1}^{T} \delta^{t-1} E\left[u\left(s_t, y_t\left(s^t\right)\right) \mid s\right] \in \mathbb{R} \right\}.$$

We restrict attention to plans in $\Omega_0$. Let $y_t^\infty(s^{t-1})$ denote the continuation of plan $y^\infty$ following the shock history $s^{t-1}$.

The restrictions in $\Omega_0$ are supplemented with forward-looking incentive constraints. These constraints guarantee that a given plan is optimal for an agent relative to some set of feasible deviations. The key piece of constraint structure in the examples is the linearity of incentive constraints in continuation payoffs, see, for example, (2) or (9). However, in other dimensions the constraints vary: in the hidden information problems they run across states, while in limited commitment problems they are state specific. This leads us to write down a constraint structure that is additive across time and states and linear in agent continuation payoffs, but is otherwise general. Constraints are constructed from three elements: an index set $\mathbb{M}$, a family of functions $\{u^m\}$ and a family of continuation payoff weights $\{q^m_{s,s'}\}$. The index $m \in \mathbb{M} := \{1, \ldots, M\}$ identifies the constraint. The function $u^m : \operatorname{Graph} C \to \mathbb{R}$ describes how current actions enter the $m$-th constraint. Continuation payoffs enter the incentive constraints linearly; $q = \{q^m_{s,s'}\}$ weights them. Specifically, $q^m_{s,s'}$ weights the continuation payoff following the $s$-th current action and $s'$-th future shock in the $m$-th constraint. Collecting these elements together, we have the following definition.

**Definition 1.** A plan $y^\infty = \{y_t\}_{t=1}^\infty \in \Omega_0$ is *incentive-feasible* if for each $t \in \mathbb{N}$, $s^{t-1}$ and $m \in \mathbb{M}$, $G^m(y_t^\infty(s^{t-1})) \geqslant 0$, where

$$G^m(y^\infty) = \sum_{s\in\mathbb{S}}\left[u^m(s, y_1(s)) + \delta \sum_{s'\in\mathbb{S}} U(s', y_2^\infty(s)) q^m_{s,s'}\right]. \tag{15}$$

Let $\Omega_1$ denote the set of incentive-feasible plans (belonging to $\Omega_0$).

The function $G^m$ allows any current action $y_1(s)$ and continuation payoff $U(s', y_2^\infty(s))$ to enter any constraint $m$. By restricting $u^m$ and the weights $q^m$ appropriately, the constraints in the examples are recovered. For example, the constraints in the limited commitment problem (2) may be obtained from (15) by setting, for each current shock $m \in \mathbb{S}$,

$$u^m(s, y) = \begin{cases} y & \text{if } s = m, \\ 0 & \text{if } s \neq m \end{cases} \quad \text{and} \quad q^m_{s,s'} = \begin{cases} Q(s, s') & \text{if } s = m, \\ 0 & \text{if } s \neq m. \end{cases}$$

Thus, actions and payoffs not associated with the $m$-th shock are zeroed out of the $m$-th constraint. Note that here the weights $q^m_{s,s'}$ may be factored as $q^m_s Q(s, s')$, where $q^m_s = 1$ if $s = m$ and 0 otherwise.

Similarly, the constraints (9) in the hidden information problem are obtained from (15) by setting $\mathbb{M} = \{(m, m') \in \mathbb{S}, m \neq m'\}$ and for the $(m, m')$-th constraint (i.e. the constraint in which the truthful report $m$ is compared to the lie $m'$),

$$u^{m,m'}(s, y) = \begin{cases} my & \text{if } s = m, \\ -my & \text{if } s = m', \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad q^{m,m'}_{s,s'} = \begin{cases} Q(s') & \text{if } s = m, \\ -Q(s') & \text{if } s = m', \\ 0 & \text{otherwise.} \end{cases}$$

Again the continuation payoff weights $q^{m,m'}_{s,s'}$ have a simple structure: they may be factored as $q^{m,m'}_s Q(s')$, where $q^{m,m'}_s = 1$ if $s = m$ ("truth"), $-1$ if $s = m'$ ("lie"), and 0 otherwise. For hidden information problems with Markov shocks the functions $u^{m,m'}$ remain the same, but the payoff weights become:

$$q^{m,m'}_{s,s'} = \begin{cases} Q(m, s') & \text{if } s = m, \\ -Q(m, s') & \text{if } s = m', \\ 0 & \text{otherwise.} \end{cases}$$

These weights can no longer be factored into a part that depends on the constraint $(m, m')$ and a part that depends on the future shock $s'$ as in the i.i.d. case.

Let $f : \text{Graph}\, C \to \mathbb{R}$ denote the principal's per period payoff as a function of the current shock and action. Together $f$, a discount factor $\beta \in [\delta, 1)$ and $Q$ give the principal's lifetime payoff, $F : \mathbb{S} \times \Omega_1 \to \overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, where:

$$F(s, y^\infty) := \liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} E[f(s_t, y_t) \mid s_0 = s],$$

and $F$ depends on the date 0 not the date 1 shock.

**Assumption 4.** $f$, $\beta$, $Q$, $\Omega_1$ are such that

(i) $\Omega_1 \neq \emptyset$,
(ii) for all $s \in \mathbb{S}$ and $y^\infty \in \Omega_1$, $F(s, y^\infty) = \lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} E[f(s_t, y_t) \mid s_0 = s] \in \mathbb{R} \cup \{-\infty\}$ and
(iii) $\sup_{\mathbb{S}\times\Omega_1} F(s, y^\infty) \in \mathbb{R}$.

Assumptions 1, 2 and 3 guarantee Assumption 4 in the examples of Section 2. The principal's problem is:

$$\sup_{\Omega_1} F(s, y^\infty). \tag{16}$$

## 4. A recursive primal formulation

We now derive a recursive formulation for the class of incentive problems presented in Section 3. This derivation is not an immediate application of standard results as in, for example, Stokey et al. (1989). In standard recursive settings a state space and a recursive constraint correspondence are given as part of the primitives of the problem. From these the set of feasible plans is constructed. In contrast, in many incentive problems this derivation is reversed: the set of (incentive) feasible plans is given and from it the state space and the recursive constraint correspondence are recovered. The recursive structure in many dynamic incentive settings is, thus, implicit.

In specific settings, Green (1987), Kocherlakota (1996) and Fernandes and Phelan (2000) have shown how a state space and a recursive constraint correspondence may be associated with a contracting problem and, hence, a recursive formulation obtained. In their formulations states are identified with utility promises to agents. We show that this approach is applicable to the general dynamic incentive problems of Section 3.

We identify two cases. These cases determine whether ex ante promises (conditional on past shocks) or ex post promises (conditional on present shocks) are needed in the recursive formulation and, hence, determine the dimension of the state space. The cases are distinguished by the structure of the payoff weights $q$ in their constraints. In Case 1, for all $m \in \mathbb{M}$ and $s, s' \in \mathbb{S}$, the constraint weights may be factored as $q^m_{s,s'} = q^m_s Q(s, s')$ and we say that they are *decomposable*. In this case, the incentive constraints (15) can be rewritten as:

$$\sum_{s \in \mathbb{S}} \left[ u^m\left(s, y_t\left(s^{t-1}, s\right)\right) + \delta q^m_s \sum_{s' \in \mathbb{S}} U\left(s', y^\infty_{t+1}\left(s^{t-1}, s\right)\right) Q\left(s, s'\right) \right] \geqslant 0, \tag{17}$$

with the $q^m_s$ term factored through the summation over next period shocks $s'$. Thus, the continuation plan $y^\infty_{t+1}(s^{t-1}, s)$ affects the constraint functions at $s^{t-1}$ only insofar as it affects $\sum_{s' \in \mathbb{S}} U(s', y^\infty_{t+1}(s^{t-1}, s)) Q(s, s')$. In Case 2, some (or all) of the weights $q^m_{s,s'}$ cannot be factored as $q^m_s Q(s, s')$ and we say that the weights $q$ are *non-decomposable*. In this case, the impact of $y^\infty(s^{t-1}, s)$ on the constraint functions is not adequately summarized by $\sum_{s' \in \mathbb{S}} U(s', y^\infty_{t+1}(s^{t-1}, s))\ Q(s, s')$. Instead, the entire vector $\{U(s', y^\infty_{t+1}(s^{t-1}, s)\}_{s' \in \mathbb{S}}$ must be used. Our one-sided commitment problem belongs to Case 1, our hidden information problems with i.i.d. and Markov shocks belong to Cases 1 and 2 respectively.

For Case 1 problems, set $N = 1$ and for Case 2 problems, set $N = S$. In what follows $N$ will give the dimension of the state space. Define the family of *promise-perturbed problems* by, for all $(s, x) \in \mathbb{S} \times \mathbb{R}^N$,

$$W^*(s, x) = \sup_{\Omega_1(s,x)} F\left(s, y^\infty\right) \tag{18}$$

where in Case 1:

$$\Omega_1(s, x) = \left\{ y^\infty \in \Omega_1 \;\middle|\; x = \sum_{s' \in \mathbb{S}} U\left(s', y^\infty\right) Q\left(s, s'\right) \right\}, \tag{19a}$$

and in Case 2:

$$\Omega_1(s, x) = \left\{ y^\infty \in \Omega_1 \;\middle|\; \text{for each } s' \in \mathbb{S},\ x_{s'} = U\left(s', y^\infty\right) \right\}. \tag{19b}$$

The variable $x \in \mathbb{R}^N$ is referred to as a promise. Thus, (18) augments the original problem (16) with additional promise-keeping constraints (contained in (19)). We retain our convention that $W^*(s, x) = -\infty$ if $\Omega_1(s, x) = \emptyset$. This is natural since we do not at this stage know which $(s, x)$-states are consistent with non-empty constraint sets and is consistent with our later treatment of dual problems. It does, however, mean that $W^*$ is an extended real-valued function on all of $\mathbb{S} \times \mathbb{R}^N$. Let $\operatorname{Dom} W^* = \{(s, x) \in \mathbb{S} \times \mathbb{R}^N \mid \Omega_1(s, x) \neq \emptyset\}$ denote the *effective domain* of $W^*$ on which $W^*$ is not equal to $-\infty$. It coincides with the set of incentive-feasible shock-promise pairs. Definition 2 describes a mild regularity condition that excludes certain pathological functions.

**Definition 2.** A function $W : \mathbb{S} \times \mathbb{R}^N \to \overline{\mathbb{R}}$ is *sup-proper* if for each $s \in \mathbb{S}$, $W(s, \cdot)$ is nowhere equal to $\infty$ and not everywhere equal to $-\infty$. Let $\mathscr{W}$ denote the set of sup-proper functions with domain $\mathbb{S} \times \mathbb{R}^N$.

A consequence of Assumption 4 is that $W^*$ is sup-proper (see Appendix A). Let $A := \prod_{\mathbb{S}} C(s) \times \mathbb{R}^{NS}$. Define the case 1 constraint correspondence $\Gamma : \mathbb{S} \times \mathbb{R}^N \to 2^A$:

$$\Gamma(s, x) = \left\{ \left(y, x'\right) \in A \;\middle|\; \begin{array}{l} x = \sum_{s' \in \mathbb{S}} \left[u\left(s', y_{s'}\right) + \delta x'_{s'}\right] Q\left(s, s'\right) \\ \sum_{s' \in \mathbb{S}} \left[u^m\left(s', y\right) + \delta q^m_{s'} x'_{s'}\right] \geqslant 0,\ m \in \mathbb{M} \end{array} \right\} \tag{20}$$

and the case 2 correspondence:

$$\Gamma(s, x) = \left\{ \left(y, x'\right) \in A \;\middle|\; \begin{array}{l} x_{s'} = u\left(s', y_{s'}\right) + \delta \sum_{s'' \in \mathbb{S}} x'_{s',s''} Q\left(s', s''\right),\ s' \in \mathbb{S} \\ \sum_{s' \in \mathbb{S}} \left[ u^m\left(s', y\right) + \delta \sum_{s'' \in \mathbb{S}} q^m_{s',s''} x'_{s',s''} \right] \geqslant 0,\ m \in \mathbb{M} \end{array} \right\}. \tag{21}$$

**Definition 3.** The *Bellman operator* $\mathscr{B}$ is defined on $\mathscr{W}$ according to, for $W \in \mathscr{W}$ and each $(s, x) \in \mathbb{S} \times \mathbb{R}^N$,

$$\mathscr{B}(W)(s, x) = \sup_{\Gamma(s,x)} \sum_{s' \in \mathbb{S}} \left\{ f\left(s', y_{s'}\right) + \beta W\left(s', x'_{s'}\right) \right\} Q\left(s, s'\right).$$

The corresponding *Bellman equation* is given by: $W = \mathscr{B}(W)$, $W \in \mathscr{W}$.

Note that since $W \in \mathscr{W}$, $\mathbb{S}$ is finite and $f$ is real-valued on Graph $C$, $\mathscr{B}(W)$ is well defined, though possibly $-\infty$-valued. Proposition 1 below establishes that $W^*$ satisfies the Bellman equation and that solutions to (18) are consistent with the associated optimal policy correspondence. A partial converse to Proposition 1 is supplied in Section 5.

**Proposition 1.** *If Assumption 4 holds, then $W^* = \mathscr{B}(W^*)$. Also, if $y^\infty$ solves (18) at $(s_0, x_1)$, then there is a corresponding promise plan $x^\infty = \{x_t\}_{t=1}^\infty$ such that $(y_1, x_2) \in G(s_0, x_1)$ and for each $t > 1$ and $s^{t-1}$, $(y_t(s^{t-1}), x_{t+1}(s^{t-1})) \in G(s_{t-1}, x_t(s^{t-1}))$ where:*

$$G(s,x) = \operatorname*{arg\,max}_{\Gamma(s,x)} \sum_{s' \in \mathbb{S}} \bigl\{ f\bigl(s', y_{s'}\bigr) + \beta W^*\bigl(s', x'_{s'}\bigr) \bigr\} Q\bigl(s, s'\bigr).$$

**Proof.** See Appendix A. □

*A difficulty with $\mathscr{B}$* In contrast to standard problems, in (18) there is no *exogenously* given state space on which the constraint correspondence $\Omega_1$ is non-empty and $W^*$ real-valued. The effective domain of $W^*$, $\mathrm{Dom}\, W^* = \{(s,x) \in \mathbb{S} \times \mathbb{R}^N \mid \Omega_1(s,x) \neq \emptyset\}$, constitutes such a state space, but it is endogenous and must be solved for along with $W^*$.[3]

One approach is to solve for $W^*$ in two steps. In the first an approximation to the domain of $W^*$ is obtained, in the second the finite-valued restriction of $W^*$ to the approximated domain is calculated.[4] Alternatively, a value iteration using $\mathscr{B}$ may be performed directly on (a subset of) $\mathscr{W}$. This effectively combines the steps of the first approach and allows for the joint determination of $W^*$ and Dom $W^*$. It is the approach we take. Analytical results are available once an appropriate convergence concept for extended real-valued functions has been settled upon. In what follows, we use the concept of *hypo-convergence* and give sufficient conditions for value iteration under $\mathscr{B}$ to be hypo-convergent to the true value function. We remark that our approach is sufficiently flexible to allow for non-compact valued constraint correspondences $\Gamma$. This is useful for problems in which the incentive constraints do not restrict future utility promises to a bounded set and later in dual formulations when multipliers are similarly unrestricted.

## 5. Dynamic programming

We begin by introducing a useful convergence concept for extended real-valued functions.

**Definition 4.** A sequence of functions $\{W_n\}_{n=1}^\infty$, $W_n : \mathbb{S} \times \mathbb{R}^N \to \overline{\mathbb{R}}$, *hypo-converges* to $W : \mathbb{S} \times \mathbb{R}^N \to \overline{\mathbb{R}}$ written $W_n \xrightarrow{h} W$ if for each $s \in \mathbb{S}$ and any $x \in \mathbb{R}^N$,

(1) $\forall x_n \to x$, $\limsup_n W_n(s, x_n) \leqslant W(s,x)$ and
(2) $\exists x_n \to x$, $\liminf_n W_n(s, x_n) \geqslant W(s,x)$. $W$ is called the *hypo-limit* of $\{W_n\}_{n=1}^\infty$.

Hypo-convergent sequences of functions have several desirable properties. In particular, if $W_n \xrightarrow{h} W$ and $x_n \in \operatorname{arg\,max}_{x \in \mathbb{R}^N} W_n(s,x)$, then all cluster points of the sequence $\{x_n\}$ maximize $W(s, \cdot)$. Thus, hypo-convergent value iteration allows policies as well as values to be approximated. We will also use the following definition.

**Definition 5.** Let $Z \subset \mathbb{R}^n$ and $g : Z \to \overline{\mathbb{R}}$. $g$ is *upper-level bounded* if for each $v \in \mathbb{R}$ the *upper level set* $\text{u-lev}_v g = \{z \in Z \mid g(z) \geqslant v\}$ is bounded (and possibly empty).

A real-valued function defined on a bounded domain is immediately upper-level bounded. We impose the following condition.[5]

**Assumption 5.** $C$ is closed and non-empty valued. $f : \mathrm{Graph}\, C \to \mathbb{R}$ is upper semicontinuous and upper-level bounded. Each $u^m : \mathrm{Graph}\, C \to \mathbb{R}$ is upper semicontinuous and $u : \mathrm{Graph}\, C \to \mathbb{R}$ is continuous.

Theorem 1 below combines Assumptions 4 and 5 with the requirement that there exists an upper bounding function $W_0$ for $W^*$ satisfying certain properties. Under these conditions, it establishes that a value iteration from $W_0$ (hypo-)converges to a fixed point of $\mathscr{B}$.

[3] In intertemporal consumption problems, No Ponzi game conditions render some initial debt levels infeasible and, hence, define an endogenous state space. For these problems the endogenous state space is a collection of intervals indexed by the shock and unbounded at the top. The lower bounds are called "natural debt limits".
[4] See, for example, Abraham and Pavoni (2008).
[5] *Language convention*: if a function $g : \mathbb{S} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is such that each $g(s, \cdot)$ satisfies a property (e.g. upper semicontinuity, concavity, etc.), we will simply say that $g$ possesses that property.

**Theorem 1.** *Let Assumptions 4 to 5 hold. Let $W_0$ be sup-proper, upper semicontinuous and upper-level bounded with* (1) $W_0 \geqslant \mathscr{B}(W_0)$ *and* (2) $W_0 \geqslant W^*$. *For* $n = 0, 1, 2, \ldots,$ *let* $W_{n+1} = \mathscr{B}(W_n)$. *Then* $W_n \xrightarrow{h} W_\infty$ *with* $W_\infty \geqslant W^*$ *and* $W_\infty = \mathscr{B}(W_\infty)$.

**Proof.** See Appendix A. □

The assumptions made in the examples of Section 2 ensure Assumption 4. They also ensure Assumption 5 up to the closed valuedness of $C$ in the hidden information problems.[6] In each of the examples it may be verified that the value function from the incentive unconstrained problems $\overline{W}(s,x) = \sup_{\Omega_0(s,x)} F(s, y^\infty)$, where $\Omega_0(s,x)$ omits the incentive constraints from $\Omega_1(s,x)$, satisfies the requirements on $W_0$.

Although Theorem 1 gives conditions for convergent value iteration, it does not ensure that the limit of the iteration is the true value function $W^*$. To guarantee this Theorem 2 below imposes a further restriction on $W_0$. Let $\|\cdot\| : \mathbb{R}^N \to \mathbb{R}_+$ denote the Euclidean norm on $\mathbb{R}^N$.

**Definition 6.** $W \in \mathscr{W}$ is *sup-coercive in expectation* if for all $s_0 \in \mathbb{S}$ and promise plans $x^\infty = \{x_{t+1} : S^t \to \mathbb{R}^N\}$ with $\limsup_{t\to\infty} \delta^t E[\|x_{t+1}\| \mid s_0] \neq 0$, $\limsup_{t\to\infty} \beta^t E[W(s_t, x_t) \mid s_0] = -\infty$.

$W$ is sup-coercive in expectation if an "exploding" promise process implies divergence of the discounted expected value of $W$ to $-\infty$.

**Theorem 2.** *Let the assumptions of Theorem 1 hold. Assume additionally that $W_0$ is sup-coercive in expectation. For* $n = 0, 1, \ldots,$ *let* $W_{n+1} = \mathscr{B}(W_n)$. *Then* $W_n \xrightarrow{h} W^*$.

**Proof.** See Appendix A. □

Sup-coercivity in expectation is satisfied if $W$ is finite on a bounded set and is otherwise $-\infty$. Evidently, the incentive unconstrained value function $\overline{W}$ satisfies this condition if the set $\{(s,x)\colon\ \Omega_0(s,x) \neq \emptyset\}$ is bounded, as is the case in the one-sided limited commitment problem or the hidden information problems with bounded agent utility. When agent utility is unbounded above in the hidden information problems, the Inada condition and convexity of $h$ ensure that $f$ and, hence, $\overline{W}$ satisfy sup-coercivity properties.[7]

The following corollary is a straightforward consequence of the previous result's proof. It asserts that any plan generated by the optimal policy correspondence $G$ from the recursive problem solves the original sequence problem (18).

**Corollary 1.** *Let the assumptions of Theorem 2 hold. If $y^\infty$ satisfies $(y_t(s^{t-1}), x_{t+1}(s^{t-1})) \in G(s_{t-1}, x_t(s^{t-1}))$ for each shock history $s^{t-1}$ and a promise plan $x^\infty$, then $y^\infty$ solves (18) at $(s_0, x_1)$. In particular, if $W^*(s_0, x_1) > -\infty$, then there is an action plan satisfying the preceding condition and, hence, an optimal solution to (18) at $(s_0, x_1)$.*

**Proof.** See Appendix A. □

We now show that the value iteration considered above not only generates sequences of values that converge to the optimal value, but also sequences of approximate *policies* that converge to optimal policies. This result exploits implications of hypo-convergence.

**Proposition 2.** *Let the assumptions of Theorem 2 hold and let $W_n$ be defined as in that proposition. Assume that $W^*(s,x) > -\infty$ and for each $n \in \mathbb{N}$ and $\varepsilon > 0$, let*:

$$G^n_\varepsilon(s,x) = \left\{ (y, x') \in \Gamma(s,x) \,\middle|\, \sum_{s' \in \mathbb{S}} \left\{ f\left(s', y_{s'}\right) + \beta W_n\left(s', x'_{s'}\right) \right\} Q\left(s, s'\right) + \varepsilon > W_{n+1}(s,x) \right\}.$$

*For any pair of sequences $\{\varepsilon_n\}$ and $\{y_n, x'_n\}$ with $\varepsilon_n \downarrow 0$ and $(y_n, x'_n) \in G^n_{\varepsilon_n}(s,x)$, all cluster points of $\{y_n, x'_n\}$ belong to $G(s,x)$.*

**Proof.** See Appendix A. □

[6] In our hidden information problems, $C(s)$ is identified with the range of the agent's utility function: $[\underline{y}, \overline{y})$. If $\overline{y} \in \mathbb{R}$ (i.e. does not equal $\infty$), then $C(s)$ is not closed-valued. A small modification of our arguments deals with this case, see Remark 2 in Appendix B.

[7] For example, if $\beta = \delta$ and shocks are i.i.d., $\overline{W}(x) = \frac{1}{1-\beta} E[s\overline{y}_s(x) - \psi h(\overline{y}_s(x))]$ and $\lim_{x\to\infty} \overline{y}_s(x) = \infty$, where $\overline{y}_s$ is the (static) optimal policy function from the incentive-unconstrained problem. If $\lim_{t\to\infty} \beta^t x_t > 0$, then $\lim_{t\to\infty} \beta^t \overline{W}(x_t) \leqslant \lim_{t\to\infty} \beta^t x_t (1 - \frac{\psi \bar{s}}{1-\beta} \frac{h(E[\overline{y}(x_t)])}{E[\overline{y}(x_t)]}) = -\infty$. The concavity of $\overline{W}$ then gives the desired result.

## 6. Dual recursive formulations

In this section, we use a Lagrangian to pair the Bellman operator $\mathscr{B}$ with its dual counterpart $\mathscr{B}^D$. We show that when restricted to a particular domain of functions, $\mathscr{B}$ and $\mathscr{B}^D$ coincide. Then we decompose the dual Bellman operator $\mathscr{B}^D$. We extract one of its components, which we label $\mathscr{D}$, and show that, under certain conditions, value iteration can be conducted with $\mathscr{D}$ rather than $\mathscr{B}$. Our focus on $\mathscr{D}$ is motivated by practical computational considerations. In addition, our tying of $\mathscr{D}$ to $\mathscr{B}$ allows us to derive convergent value iteration results for $\mathscr{D}$ as easy consequences of the results in the previous section.

### 6.1. A first dual formulation

We focus on Case 1 problems with decomposable constraint weights, deferring the extension to Case 2 settings until Section 7. Recall the definition of $\mathscr{B}$:

$$\mathscr{B}(W)(s,x) = \sup_{\Gamma(s,x)} \sum_{\mathbb{S}} \{ f(s', y_{s'}) + \beta W(s', x'_{s'}) \} Q(s,s'), \tag{22}$$

with, once again, $\sup \emptyset = -\infty$ and $G(W)$ the associated policy correspondence. As described in the examples, the family of optimizations (22) may be re-expressed as sup–inf problems using the Lagrangians $\mathscr{L}_s(W): A \times \Phi \to \mathbb{R}$, where $A = \prod_{\mathbb{S}} C(s') \times \mathbb{R}^S$ is the choice set, $\Phi := \mathbb{R} \times \mathbb{R}^M_+$, with generic element $(\mu, \eta)$, is the multiplier set and:

$$\begin{aligned}\mathscr{L}_s(W)(y, x'; \mu, \eta) = {} & \sum_{\mathbb{S}} \{ f(s', y_{s'}) + \beta W(s', x'_{s'}) \} Q(s,s') + \mu \left[ \sum_{\mathbb{S}} [u(s', y_{s'}) + \delta x'_{s'}] Q(s,s') \right] \\ & + \sum_{\mathbb{M}} \eta^m \left[ \sum_{\mathbb{S}} u^m(s', y_{s'}) + \delta \sum_{\mathbb{S}} q^m_{s'} x'_{s'} \right]. \end{aligned} \tag{23}$$

As described previously, $\mathscr{B}(W)$ and $G(W)$ may be re-expressed as:

$$\mathscr{B}(W)(s,x) = \sup_A \inf_\Phi \mathscr{L}_s(W)(y, x'; \mu, \eta) - \mu x \tag{24}$$

and

$$G(W)(s,x) = \arg\max_A \inf_\Phi \mathscr{L}_s(W)(y, x'; \mu, \eta) - \mu x.$$

We refer to problems in the form (24) as primal problems.

Dual problems are obtained from primal problems by interchanging the supremum and infimum operations. In the context of (24), this leads to the "dual" operator $\mathscr{B}^D$:

$$\mathscr{B}^D(W)(s,x) = \inf_\Phi \sup_A \mathscr{L}_s(W)(y, x'; \mu, \eta) - \mu x. \tag{25}$$

We emphasize that the dual problems $\mathscr{B}^D(W)$ are obtained from the *recursive* primal problems $\mathscr{B}(W)$ and that they involve the dualization of only current constraints (i.e. the absorption of only these constraints into the Lagrangian). This "recursivize-then-dualize" approach allows us to avoid explicit treatment of the infinite dimensional dual space associated with the original sequential problem (16). In contrast, the alternative "dualize-then-recursivize" approach which pairs the sequential problem (16) with a dual and then seeks a recursive formulation of the latter necessitates consideration of this space.

The order of the infimum and supremum operations in (24) and (25) matters. Classical weak duality assures $\mathscr{B}^D(W) \geqslant \mathscr{B}(W)$, but, in general, equality of values is not guaranteed and a *duality gap*, $\mathscr{B}^D(W)(s,x) > \mathscr{B}(W)(s,x)$, is possible. Proposition 3 below rules this out. It requires similar properness, upper semicontinuity and level boundedness properties to those used in the previous section. Now, however, additional concavity assumptions are needed. To state them, we embed the family of problems (24) into a larger family that incorporates perturbations of the incentive constraints. For $W \in \mathscr{W}$ and each $(s, x, p, y, x') \in \mathbb{S} \times \mathbb{R}^{N+M} \times A$, let:

$$\widetilde{\mathscr{U}}(W)(s,x,p,y,x') := \begin{cases} \sum_{\mathbb{S}} \{ f(s', y_{s'}) + \beta W(s', x'_{s'}) \} Q(s,s') & \text{if } \widetilde{\Gamma}(s,x,p) \neq \emptyset, \\ -\infty & \text{otherwise} \end{cases}$$

where:

$$\widetilde{\Gamma}(s,x,p) = \left\{ (y, x') \in A \;\middle|\; \begin{array}{l} x = \sum_{\mathbb{S}} [u(s', y_{s'}) + \delta x'_{s'}] Q(s,s') \\ \sum_{\mathbb{S}} [u^m(s', y) + \delta q^m_{s'} x'_{s'}] \geqslant p_m,\ m \in \mathbb{M} \end{array} \right\}.$$

$\widetilde{\mathscr{U}}(W)$ is simply the principal's objective in a recursive problem in which the incentive constraints are perturbed by $p$. Infeasible choices are encoded as $-\infty$ payoffs. By setting $p=0$ our earlier problems are recovered: $\mathscr{B}(W)(s,x) = \sup_A \widetilde{\mathscr{U}}(W)(s,x,0,y,x')$. We define $\widetilde{\mathscr{U}}(W)$ to be *concave-like* if for each $s$, $(x^0, p^0, y^0, x'^0)$, $(x^1, p^1, y^1, x'^1)$ and $\lambda \in [0,1]$, there is a $(y^\lambda, x'^\lambda)$ such that $\widetilde{\mathscr{U}}(W)(s, \lambda x^0 + (1-\lambda)x^1, \lambda p^0 + (1-\lambda)p^1, y^\lambda, x'^\lambda) \geqslant \lambda \widetilde{\mathscr{U}}(W)(s, x^0, p^0, y^0, x'^0) + (1-\lambda)\widetilde{\mathscr{U}}(W)(s, x^1, p^1, y^1, x'^1)$. Note that concave-likeness does not require that $y^\lambda$ and $x'^\lambda$ be convex combinations of $y^0$ and $y^1$ and $x'^0$ and $x'^1$. Hence, it is weaker than concavity of each $\widetilde{\mathscr{U}}(W)(s,\cdot)$. It is easy to see that if $\widetilde{\mathscr{U}}(W)$ is concave-like, then $\mathscr{B}(W)$ is concave. Let $\mathscr{C}$ denote the set of concave, upper semicontinuous, sup-proper functions with domain $\mathbb{S} \times \mathbb{R}^N$.

**Assumption 6.** If $W \in \mathscr{C}$, then $\widetilde{\mathscr{U}}(W)$ is concave-like.

A sufficient condition for Assumption 6 to hold is that $f$ and $u^m$ are concave and $u$ is affine. These conditions ensure that if $W \in \mathscr{C}$, then $\widetilde{\mathscr{U}}(W)$ is not just concave-like, but is concave. They hold in the examples of Section 2. Imposing Assumption 6, the following result obtains.

**Proposition 3.** *Let Assumptions 4 to 6 hold. Assume that $W$ is sup-proper, upper semicontinuous, upper-level bounded and concave. Then $\mathscr{B}^D(W) = \mathscr{B}(W)$.*

Under the conditions of Propositions 1 and 3 (with $W^*$ substituted for $W$),

$$W^* = \mathscr{B}\big(W^*\big) = \mathscr{B}^D\big(W^*\big),$$

i.e. $W^*$ satisfies the dual Bellman equation $W^* = \mathscr{B}^D(W^*)$.

### *6.2. Decomposing $\mathscr{B}^D$ and a second dual formulation*

The dual Bellman $\mathscr{B}^D$ may be decomposed into component operators. We first give the following useful concept from convex analysis.

**Definition 7.** Given $g : \mathbb{R}^N \to \overline{\mathbb{R}}$, the function $g^\star : \mathbb{R}^N \to \overline{\mathbb{R}}$, with:

$$g^\star\big(x^\star\big) = \sup_{\mathbb{R}^N} \big\langle x, x^\star\big\rangle - g(x)$$

and $\langle\cdot,\cdot\rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ the vector dot product operation, is called the *conjugate* of $g$. The mapping of a function to its conjugate is called the *Legendre–Fenchel transform*. If $g : \mathbb{S} \times \mathbb{R}^N \to \overline{\mathbb{R}}$, then we define its conjugate to be $g^\star = \{g(s,\cdot)^\star\}_{\mathbb{S}}$ and extend the definition of the Legendre–Fenchel transform accordingly.

Conjugation and the Legendre–Fenchel transform play an important role in duality theory (see, inter alia, Rockafellar, 1970). For our purposes, it will be useful to define two related operations, for $g : \mathbb{S} \times \mathbb{R}^N \to \overline{\mathbb{R}}$, let:

$$D(g)(s,\mu) = (-g)^\star(s,\mu) = \sup_{x \in \mathbb{R}^N} \langle x, \mu\rangle + g(s,x),$$

$$L(g)(s,x) = -\big(g^\star\big)(s,x) = \inf_{\mu \in \mathbb{R}^N} -\langle x, \mu\rangle + g(s,\mu).$$

$D$ gives the "conjugate of the negative" and $L$ the "negative of the conjugate". Roughly speaking, if $W$ is a candidate value function on a domain of utility promises, then $V = D(W)$ is the corresponding value function on a domain of Pareto weights. Geometrically, it is the "upper envelope" of $W$. Conversely and under some regularity conditions, if $V$ is a candidate value function on a domain of Pareto weights, then $W = L(V)$ is the corresponding value function on a domain of utility promises. Geometrically, it is the lower envelope of $V$. Fig. 1 illustrates.

Proposition 4 shows how $\mathscr{B}^D$ can be decomposed into three steps. Loosely, the theorem asserts that applying $\mathscr{B}^D$ to a promise-domain value function $W$ is equivalent to (i) transforming $W$ into a Pareto domain value function via $D$, (ii) applying a "dual" operator $\mathscr{D}$ and (iii) converting the resulting value function back to the promise-domain via $L$. To make the statement of the theorem precise, a definition and some notation is needed. We say that $V : \mathbb{S} \times \mathbb{R}^N \to \overline{\mathbb{R}}$ is inf-proper if each $V(s,\cdot)$ is nowhere $-\infty$ and not everywhere $\infty$ (i.e. $-V$ is sup-proper). Let $\mathscr{V}$ denote the set of inf-proper functions on $\mathbb{S} \times \mathbb{R}^N$ and $\mathscr{F}$ the set of extended real-valued functions on $\mathbb{S} \times \mathbb{R}^N$. Recall also that for Case 1 problems with a single agent, $N = 1$.

**Proposition 4.** *Define $\mathscr{D} : \mathscr{V} \to \mathscr{F}$ by:*

$$\mathscr{D}(V)(s,\mu) = \inf_{\mathbb{R}^M_+} \sum_{\mathbb{S}} \big(\nu_{s,s'}(\mu,\eta) + \beta V\big(s', \mu_{s'}(\mu,\eta)\big)\big) Q\big(s,s'\big), \tag{26}$$

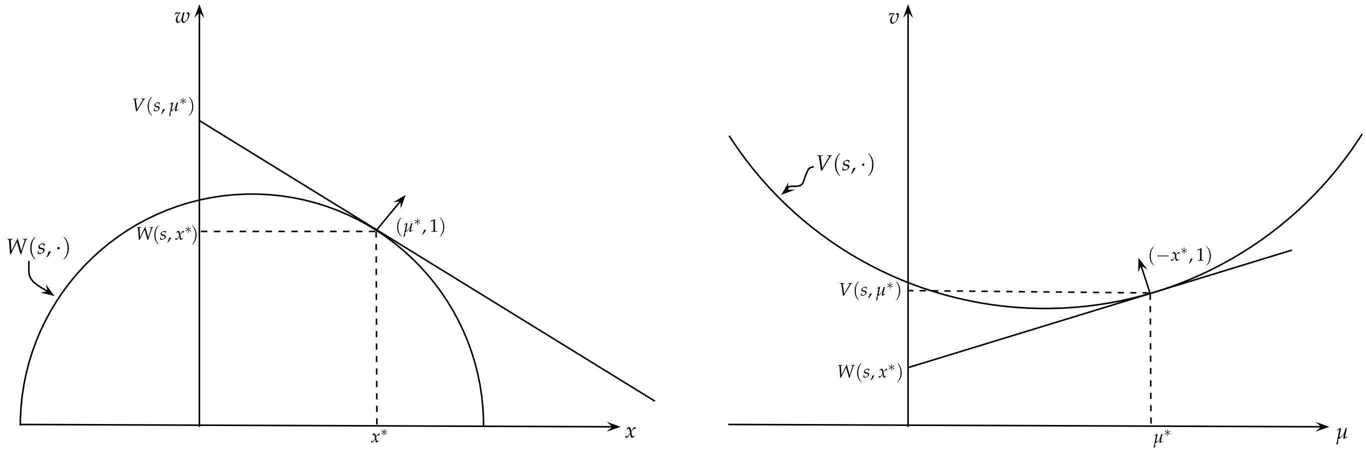


**Fig. 1.** $W(s,\cdot)$ and $V(s,\cdot)$ evaluated at $x^*$ and $\mu^*$. $V(s,\mu^*)$ is the largest vertical intercept of a line with normal $(\mu^*,1)$ that touches/intersects $W(s,\cdot)$. $W(s,x^*)$ is the smallest vertical intercept of a line with normal $(-x^*,1)$ that touches/intersects $V(s,\cdot)$.

*where*:

$$\nu_{s,s'}(\mu,\eta) = \sup_{C(s')}\left[f\left(s', y_{s'}\right) + \mu u\left(s', y_{s'}\right) + \sum_{\mathbb{M}} \frac{\eta^m u^m(s', y_{s'})}{Q(s,s')}\right]$$

*and*

$$\mu_{s'}(\mu,\eta) = \frac{\delta}{\beta}\left(\mu + \sum_{\mathbb{M}} \frac{\eta^m q^m_{s'}}{Q(s,s')}\right).$$

*Then*:

$$L\mathscr{D}D(W) = \mathscr{B}^D(W).$$

*In particular, if Assumption 4 holds and $\mathscr{B}^D(W^*) = \mathscr{B}(W^*) = W^*$, then $W^* = L\mathscr{D}D(W^*)$.*

**Proof.** See Appendix A. □

Interpretation of $\mathscr{D}$ and specific forms of it for one-sided limited commitment and hidden information problems were given in Section 2.

### 6.3. Dual value iteration

Having defined $\mathscr{D}$ and related it to $\mathscr{B}$, we now extract it and consider using it to solve dynamic incentive problems. This requires some strengthening of assumptions.

**Assumption 5′.** Assumption 5 holds. The function: $\sum_{\mathbb{S}} \nu_{s,s'}(\mu,\eta)Q(s,s')$ is bounded below.

We will also use the following definition.

**Definition 8.** $W:\mathbb{S}\times\mathbb{R}^N \to \overline{\mathbb{R}}$ is *sup-coercive* if it is bounded above and for each $s$,

$$\limsup_{\|x\|\to\infty} \frac{W(s,x)}{\|x\|} = -\infty.$$

For functions in $\mathscr{C}$, this strengthens our earlier sup-coercive in expectation condition. Previously, we introduced the concept of hypo-convergence; its dual analogue is epi-convergence. If a sequence of functions $\{W_n\}$, $W_n:\mathbb{S}\times\mathbb{R}^N \to \overline{\mathbb{R}}$, hypo-converges to $W:\mathbb{S}\times\mathbb{R}^N \to \overline{\mathbb{R}}^N$ ($W_n \xrightarrow{h} W$), then its negatives $V_n = \{-W_n\}$ are said to epi-converge to $V = -W$. We write $V_n \xrightarrow{e} V$ and call $V$ the epi-limit of the negative sequence. The main result of the paper follows.

**Theorem 3.** *Let Assumptions 4, 5′ and 6 hold. Let $W_0 \in \mathscr{C}$ be sup-coercive and suppose that for each $s \in \mathbb{S}$, $0 \in \operatorname{Dom} W_0(s,\cdot)$ and that $W_0 \geqslant \mathscr{B}(W_0) \geqslant W^*$. Define $V_0 = D(W_0)$ and $V^* = D(W^*)$. Then, $\mathscr{B}^n(W_0) \xrightarrow{h} W^*$ and $\mathscr{D}^n(V_0) \xrightarrow{e} V^*$. Also,*

$$V^*(s,\mu) = \sup_{\Omega_1} F\left(s, y^\infty\right) + \mu \sum_{\mathbb{S}} U\left(s', y^\infty\right) Q\left(s, s'\right)$$

*and* $V^* : \mathbb{S} \times \mathbb{R}^N \to \mathbb{R}$, *i.e. is real-valued.*

**Proof.** See Appendix D. □

In words, the theorem says that under certain regularity conditions on primitives and starting points, $\mathscr{B}$-value iteration (hypo-)converges to the optimal promise-domain value function, while $\mathscr{D}$-value iteration (epi-)converges to the optimal Pareto problem value function. The two iterations are tied together via a conjugacy relationship: the iterates of one are the (negatives of the) conjugates of the iterates of the other. As we have previously described, Assumptions 1, 2 and 3 ensure Assumptions 4 and 6 hold in the examples of Section 2 (if agent utility is bounded above in the hidden information problems a slightly modified version of Assumption 4 sufficient for Theorem 3 is needed). Assumption 5′ holds in these examples after possible renormalization of agent utilities. For example, in the one-sided commitment problem, for each $s'$, $0 \in C(s')$ and so $\nu_{s,s'}(\mu,\eta) \geqslant f(s',0) = v(\omega(s') - h(b(s')))$. Similarly, in the hidden information problems, utilities may be normalized so that $0 \in C(s')$ and $\nu_{s,s'}(\mu,\eta) \geqslant f(s',0) = -\psi h(0)$. $W_0$ may be identified with the value function from a problem in which the incentive constraints are not imposed (after renormalization of agent utilities to ensure $0 \in \operatorname{Dom} W_0(s,\cdot)$). In all of the examples, our assumptions ensure this function is sup-coercive.

*Why use the $\mathscr{D}$-operator?* As previously indicated, a difficulty with $\mathscr{B}$-value iteration is that it may involve extended real-valued functions. In particular, if the agent has bounded payoffs, then some agent payoffs are infeasible and at these $W^*$ is $-\infty$-valued. Thus, $W^*$ is extended real-valued. In the bounded case, we clearly also have for each $s$, $\lim_{\|x\|\to\infty} \|x\|^{-1} W^*(s,x) = -\infty$, i.e. $W^*$ is sup-coercive. By Rockafellar (1970, p. 116), for a sup-proper, concave, upper semicontinuous function, sup-coercivity is equivalent to real-valuedness of (the negative of) the conjugate. Since $V^*$ *is* the negative of the conjugate of $W^*$, we have that if Assumptions 4, 5′ and 6 hold and the agent's payoffs are bounded, then $W^*$ is extended real-valued, but $V^*$ is real-valued. Thus, by choosing the starting point of the $\mathscr{D}$-value iteration appropriately (i.e. to bound $V^*$ above, but still to be real-valued), difficulties associated with extended real-valued functions are avoided.

A further advantage of the $\mathscr{D}$ operator is that it permits the decomposition of the supremum operation across shocks. In the prototypical dynamic programming problem, e.g. Stokey et al. (1989), constraints do not run across shock realizations. Thus, it is possible to decompose problems across time and shocks, solving for current choices contingent on different current shocks separately. This is, for example, the case in standard consumer choice problems in which the budget constraint in a given shock-state does not depend on alternative unrealized shocks (and preferences are additively separable across shocks). However, it is usually not the case in incentive problems. Either the incentive constraint or, if it involves an ex ante utility promise, the promise-keeping constraint place restrictions across states and prevent choices in one shock state being solved separately from those in another. This raises the dimension of the choice variables in the sup (or sup–inf) problem associated with $\mathscr{B}$. In contrast, the inf–sup problem associated with $\mathscr{D}$ decouples shock specific supremum operations. Consider the definition of $\mathscr{D}$ given in (26). Each supremum operation:

$$\nu_{s,s'}(\mu,\eta) := \sup_{C(s')} \left[ f\left(s', y_{s'}\right) + \mu u\left(s', y_{s'}\right) + \sum_{\mathbb{M}} \frac{\eta^m u^m(s', y_{s'})}{Q\left(s, s'\right)} \right] \tag{27}$$

can be solved separately. The value functions from these component maximizations may then be embedded into the minimization over multipliers $\eta$. This inter-shock decomposition complements the inter-period decomposition implicit in $\mathscr{D}$ and further simplifies calculations. Additional simplification is possible if $u$ and each $u^m$ are affine in $y$. Notice this was the case in the examples of Section 2. In this situation, if the constraints in $C(s')$ are non-binding, then $\nu_{s,s'}$ is just a constant plus the conjugate of $-f(s',\cdot)$ evaluated at a weighted sum of multipliers. If, in addition, $f(s',\cdot)$ is an additive sum of standard functional forms, e.g. polynomial or exponential, then the conjugate of $-f(s',\cdot)$ is immediately available and no explicit maximization needs to be done. This is commonly the case in applied problems, leading to considerable simplification. For example, in the hidden information problem of Section 2, $f(s',y) = s'y - \psi h(y)$, where $h$ is the inverse of the agent's utility function. If the latter is CRRA with coefficient $\sigma$, then $h(y) = \{(1-\sigma)y\}^{\frac{1}{1-\sigma}}$. Applying standard rules for conjugation to $s'y - \psi h(y)$ (or direct calculation) gives: $\nu_{s,s'}(\mu,\eta) = d(s' + \alpha_{s'}(\mu,\eta))^{\frac{1}{\sigma}}$, where $d$ is a constant and $\alpha_{s'}(\mu,\eta)$ is the weighted sum $\alpha_{s'}(\mu,\eta) = \mu s' + \sum_{\widehat{s}} \frac{\eta^{s',\widehat{s}}}{Q(s,s')} s' - \sum_{\widehat{s}} \frac{\eta^{\widehat{s},s'}}{Q(s,s')} \widehat{s}$. No explicit optimization is needed to construct $\nu_{s,s'}$.

## 6.4. Policies

So far in this section we have focused on values rather than policies. We now give two results relating to policies. The first establishes that the optimal dual policy correspondence obtained by applying $\mathscr{D}$ to $V^* = D(W^*)$ gives necessary conditions for optimal primal policies; the second establishes that under stronger (strict concavity) assumptions it gives sufficient conditions as well. In addition, epi-convergent dual value iteration delivers a sequence of approximate policy correspondences that converge to the optimal dual policy correspondence. We will need the following definition.

**Definition 9.** Let $g : \mathbb{R}^N \to \overline{\mathbb{R}}$. The *sub-differential* of $g$ at $x$, $g(x)$ finite, is given by:

$$\partial g(x) = \{\lambda \in \mathbb{R}^N \mid \forall x' \in \mathbb{R}^N,\ g(x') \geqslant g(x) + \langle \lambda, (x' - x)\rangle\}.$$

The elements of $\partial g(x)$ are called *sub-gradients* of $g$ at $x$.

The application of $\mathscr{D}$ to the function $V^*$ induces a policy correspondence $\Psi$ where:

$$\Psi(s,\mu) = \{(\eta,\mu') \mid \eta \text{ attains the minimum in } \mathscr{D}(V^*)(s,\mu) \text{ and } \mu' = \{\mu_{s'}(\mu,\eta)\}_{s'}\}.$$

We say that $\{\eta_t, \mu_{t+1}\}$, $\eta_t : \mathbb{S}^{t-1} \to \mathbb{R}^M_+$ and $\mu_{t+1} : \mathbb{S}^t \to \mathbb{R}^N$, is generated by $\Psi$ from $(s_0, \mu_1)$ if $(\eta_1, \mu_2) \in \Psi(s_0, \mu_1)$ and for each $s^{t-1}$, $(\eta_t(s^{t-1}), \mu_{t+1}(s^{t-1})) \in \Psi(s_{t-1}, \mu_t(s^{t-1}))$. Let:

$$\Xi(s,\mu,\eta) = \left\{ y \in Y^S \;\middle|\; \forall s' \in \mathbb{S},\ y_{s'} \in \operatorname*{arg\,max}_{C(s')} f(s', \tilde{y}_{s'}) + \mu u(s', \tilde{y}_{s'}) + \sum_{\mathbb{M}} \frac{\eta^m u^m(s', \tilde{y}_{s'})}{Q(s,s')} \right\}.$$

We say that a plan $\{y_t\}$ is generated by $(\Psi, \Xi)$ from $(s_0, \mu_1)$ if there is a sequence $\{\eta_t, \mu_{t+1}\}$ generated by $\Psi$ such that each $y_t(s^{t-1}) \in \Xi(s_{t-1}, \mu_t(s^{t-1}), \eta_t(s^{t-1}))$. Proposition 5 below shows that if the family of Lagrangians $\mathscr{L}_s(W^*)(\cdot, \mu, \cdot)$ admit saddle points, then $\{y_t^*\}$ is optimal for the primal problem (18) at $(s_0, x_1)$, with $x_1$ a point of sub-differentiability of $-W^*(s_0, \cdot)$, only if it is generated by $(\Psi, \Xi)$ from $s_0$ and $\mu_1 \in \partial(-W^*)(s_0, x_1)$. In this sense, the policy correspondences $(\Psi, \Xi)$ obtained from the dual approach provide necessary conditions for an optimal primal solution.

**Proposition 5.** *Let $\{y_t^*\}$ solve (18) at $(s_0, x_1)$ and let $\{x_{t+1}^*\}$ denote a corresponding optimal promise plan. Assume that $-W^*(s_0, \cdot)$ is sub-differentiable at $x_1$ with sub-gradient $\mu_1$. Assume further that for all $(s, \mu)$, $\mathscr{L}_s(W^*)(\cdot; \mu, \cdot)$ admits a saddle point. Then, $\{y_t^*\}$ is generated by $(\Psi, \Xi)$ from $(s_0, \mu_1)$.*

**Proof.** See Appendix E. □

**Remark 1.** The Lagrangians $\mathscr{L}_s(W^*)(\cdot, \mu, \cdot)$ admit saddle points if, in addition to the assumptions of Proposition 3, a Slater condition is satisfied with respect to the incentive constraints, i.e. there is some $(y, x')$ such that for each $m \in \mathbb{M}$, $\sum_{\mathbb{S}} u^m(s', y_{s'}) + \delta \sum_{\mathbb{S}} q^m_{s'} x'_{s'} > 0$. Note that the promise-keeping constraints do not need to respect a Slater condition. The conditions in Proposition 5 seem weaker than those used by Marcet and Marimon (2011) when deriving a related result. Marcet and Marimon (2011) also require uniqueness of the optimal plan $\{y_t^*\}$.

Corollary 1 gives conditions for the existence of an optimal action and promise plan $(y^{\infty *}, x^{\infty *})$ from some $(s_0, x_1)$. Under the further conditions of Proposition 5 these plans are generated by $(\Psi, \Xi)$ from $s_0$ and some $\mu_1$. However, even under these conditions $(\Psi, \Xi)$ may generate extraneous action-plans that are not optimal for (18).[8] A simple sufficient condition that rules this out is contained in the next result.

**Proposition 6.** *Assume the following:*

(i) *$W^*(s_0, x_1) > -\infty$ and that $\mu_1 \in \partial(-W^*(s_0, x_1))$,*
(ii) *the conditions of Theorem 2 hold,*
(iii) *at each $(s, \mu)$ the Lagrangian $\mathscr{L}_s(W^*)(\cdot, \mu, \cdot)$ admits a saddle point,*
(iv) *all saddle points of $\mathscr{L}_s(W^*)(\cdot, \mu, \cdot)$ have a common primal component $(y, x')$. Then there is a unique plan $y^{\infty *}$ generated by $(\Psi, \Xi)$ from $(s_0, \mu_1)$ and $y^{\infty *}$ is optimal for (18) at $(s_0, \mu_1)$.*

**Proof.** See Appendix E. □

A sufficient condition for (iv) in Proposition 6 is that each $C(s')$ is convex, $f$ is strictly concave, $u^m$ is concave and $u$ is affine. Notice this condition holds in all of the examples of Section 2. Since, under the conditions of Theorem 3, $\mathscr{D}^n(V_0) \xrightarrow{e} V^*$, an analogue of Proposition 2 is available for the dual setting. This result establishes convergence of dual multiplier policies. Specifically, associated with the iterates $V_n = \mathscr{D}^n(V_0)$ there is a family of dual policy correspondences:

$$\Psi^n_\varepsilon(s,\mu) = \{(\eta,\mu') \mid V_n(s,\mu) + \varepsilon \geqslant J_{n-1}(s,\mu,\eta) \text{ and } \mu' = \{\mu_{s'}(\mu,\eta)\}\},$$

with: $J_n(s,\mu,\eta) = \sum_{\mathbb{S}} \{v_{s,s'}(\mu,\eta) + \beta V_{n-1}(s', \mu_{s'}(\mu,\eta))\} Q(s,s')$. Similar to Proposition 2, if $\varepsilon_n \downarrow 0$ and $\{\eta_n, \mu_n\}$ is a sequence with each $(\eta_n, \mu_n) \in \Psi^n_{\varepsilon_n}(s', \mu)$, then all cluster points of $\{\eta_n, \mu_n\}$ lie in $\Psi(s', \mu)$. In the interests of space, we omit the proof.

[8] As was pointed out by Messner and Pavoni (2004), the same issue arises in the context of Marcet and Marimon (2011)'s recursive saddle point method. Cole and Kubler (2010) provide an extension that resolves this problem in weakly concave settings.

## 7. Extensions and variations

We briefly detail some extensions.

*Case 2 extension* We have developed $\mathscr{B}^D$ and $\mathscr{D}$ in the context of Case 1 problems with decomposable weights $q^m$. The analysis may be extended to the non-decomposable setting of Case 2. In this case, following the approach of Proposition 4, we obtain:

$$\mathscr{D}(W)(s,\mu) = \inf_{\mathbb{R}^M_+} \sum_{\mathbb{S}} \left( \sup_{C(s')} \left[ f\left(s', y_{s'}\right) + \mu_{s'} u\left(s', y_{s'}\right) + \sum_{\mathbb{M}} \frac{\eta^m u^m(s', y_{s'})}{Q(s,s')} \right] + \beta W\left(s', \mu_{s,s'}(\mu,\eta)\right) \right) Q(s,s'),$$

where:

$$\mu_{s,s'}(\mu,\eta) = \left\{ \frac{\delta}{\beta} \left[ \mu_{s'} + \sum_{\mathbb{M}} \frac{\eta^m q_{s',s''}}{Q(s,s')Q(s',s'')} \right] \right\}_{s'' \in \mathbb{S}} \in \mathbb{R}^S.$$

$\mu = \{\mu_{s'}\} \in \mathbb{R}^S$ is an (ex post) Pareto weight vector. The components $\mu_{s'}$ are contingent on the shock and are updated to give new Pareto weight vectors $\mu_{s'}(\mu,\eta) \in \mathbb{R}^S$. Our results relating $\mathscr{D}$ to $\mathscr{B}^D$ and $\mathscr{B}$ go through for this case. In particular under the conditions of an appropriately modified version of Theorem 3, epi-convergent $\mathscr{D}$-iteration to $V^*$, the value function from an (ex post) Pareto problem, is obtained. The $\mathscr{D}$-operator can then be used to compute value functions from problems with persistent private information, such as in Section 2.

*Multiple agents* We have focused on single agent problems. Multiple agent problems may be accommodated at the cost of a corresponding increase in the dimension of the state space. Consider a problem with $I$ agents indexed by $i \in \mathbb{I} := \{1, \ldots, I\}$ each of whom has a per period payoff function $u^i : \text{Graph}\, C \to \mathbb{R}$, discount factor $\delta^i$, constraint set $\mathbb{M}^i$ with $M_i$ constraints, constraint functions $u^{i,m} : \text{Graph}\, C \to \mathbb{R}$ and decomposable constraint weights $q^{i,m}_{s'}$. Let $\mu = \{\mu^i\} \in \mathbb{R}^I$ denote Pareto weights for these agents and $\eta = \{\eta^{i,m}\} \in \mathbb{R}^M_+$, $M = \sum_{\mathbb{I}} M_i$, multipliers on their incentive constraints. The problem formulation is as before except there are now $I + \sum_{\mathbb{I}} M_i$ constraints. The corresponding $\mathscr{D}$ operator is:

$$\mathscr{D}(W)(s,\mu) = \inf_{\mathbb{R}^M_+} \sum_{\mathbb{S}} \left( \sup_{C(s')} \left[ f\left(s', y_{s'}\right) + \sum_{\mathbb{I}} \mu^i u^i\left(s', y_{s'}\right) + \sum_{\mathbb{I}} \sum_{\mathbb{M}^i} \frac{\eta^{i,m} u^{i,m}_{s'}(y_{s'})}{Q(s,s')} \right] + \beta W\left(s', \mu_{s,s'}(\mu,\eta)\right) \right) Q(s,s'),$$

with:

$$\mu_{s,s'}(\mu,\eta) = \left\{ \frac{\delta^i}{\beta} \left[ \mu^i + \sum_{\mathbb{M}^i} \frac{\eta^{i,m} q^{i,m}_{s'}}{Q(s,s')} \right] \right\}_{i \in \mathbb{I}}.$$

All earlier results go through for this case with appropriate modification of notation. The extension to multiple agents with Case 2 constraints is similar.

*Equilibrium problems without a principal* To motivate the analysis in this section consider the following example.

**Example 4** *(Risk sharing with limited commitment).* A population $\mathbb{I} = \{1, \ldots, I\}$ of agents receive shocks to its endowment of goods and the outside utility options of its members. As before $\{s_t\}$ is an underlying $\mathbb{S}$-valued shock process with Markov transition $Q$. A consumption process $c^\infty = \{c^i_t\}$ now gives the consumption of every agent $i$ after all histories $s^t$. The $i$-th agent evaluates her personal consumption process according to (1) and has an outside utility option $B^i$ that evolves according to: $B^i(s) = b^i(s) + \beta \sum_{s' \in \mathbb{S}} B^i(s') Q(s,s')$, $b^i : \mathbb{S} \to Y$. Given the endowment map $\omega : \mathbb{S} \to \mathbb{R}_{++}$, incentive-feasibility requires: $\forall t, s^t$, $\sum_{\mathbb{I}} c^i_t(s^t) \leqslant \omega(s_t)$ and for all $i$, $t$, $s^t$, $v(c^i_t(s^t)) + \beta U^c(s_t, c^{i,\infty}_{t+1}(s^t)) \geqslant B^i(s_t)$. We seek to characterize the set of incentive-feasible plans and/or the set of Pareto optimal incentive-feasible plans.

Re-expressing the problem in terms of net utility, a plan $y^\infty = \{y^i_t\}$ is incentive-feasible if for all $t$, $s^t$, $y_t(s^t) \in C(s_t)$, where:

$$C(s) = \left\{ y \;\middle|\; \omega(s) - \sum_{i \in \mathbb{I}} h\left(y^i + b^i(s)\right) \geqslant 0 \right\},$$

and for all $i$, $t$, $s^{t-1}$, $s$, $y^i_t(s^{t-1}, s) + \beta \sum_{s' \in \mathbb{S}} U^i(s', y^{i,\infty}_{t+1}(s^{t-1}, s)) Q(s,s') \geqslant 0$, with $U^i(s, y^\infty) = \sum_{t=1}^\infty \beta^{t-1} E[y^i_t(s^t) \mid s_1 = s]$. Let $\Omega_1$ denote the set of incentive-feasible plans.

This environment lacks a committed principal. We set $f = 0$ and, hence, $F = 0$. Thus, the immediate analogue of (16) is the problem $\sup_{\Omega_1} F(y^\infty)$ which takes the value 0 if $\Omega_1 \neq \emptyset$ and, given our convention, $-\infty$ otherwise. The optimal value of this problem simply expresses whether the set of incentive-feasible plans is empty or not. The promise perturbed version of this problem augments $\Omega_1$ with promise-keeping constraints for each of the $I$ players:

$$W^*(s,x) = \sup_{\Omega_1(s,x)} F\left(y^\infty\right),$$

where: $\Omega_1(s,x) = \{y^\infty \in \Omega_1 \mid \forall i \in \mathbb{I}, x^i = \sum_{s'\in\mathbb{S}} U^i(s', y^\infty) Q(s,s')\}$. Now, $W^* : \mathbb{S} \times \mathbb{R}^I \to \{0, -\infty\}$ takes the value 0 if $x$ is a feasible promise profile and $-\infty$ if it is not. Formally, $W^*$ is the *indicator function* of the set of incentive-feasible shock-utility promise pairs.[9] Just as before $W^*$ satisfies a Bellman equation $W^* = \mathscr{B}(W^*)$, where[10]:

$$\mathscr{B}(W)(s,x) = \sup_{\Gamma(s,x)} \sum_{s'\in\mathbb{S}} \left\{ f\left(s', y_{s'}\right) + \beta W\left(s', x'_{s'}\right) \right\} Q\left(s,s'\right) \tag{28}$$

and:

$$\Gamma(s,x) = \left\{ \left(y, x'\right) \in \prod_{s'\in\mathbb{S}} C\left(s'\right) \times \mathbb{R}^{IS} \;\middle|\; \begin{array}{l} x^i = \sum_{s'\in\mathbb{S}} \{y_s^i + \delta x_{s'}^{i\prime}\} Q\left(s,s'\right) \\ y_{s'}^i + \delta x_{s'}^{i\prime} \geqslant 0, \; \left(s', i\right) \in \mathbb{S} \times \mathbb{I} \end{array} \right\}. \tag{29}$$

Assigning the multiplier $\eta^{s',i} Q(s,s')$ to the $(s',i)$-th incentive constraint and proceeding as before, we obtain the dual operator:

$$\mathscr{D}(V)(s,\mu) = \inf_{\eta \in \mathbb{R}_+^M} \sum_{s'\in\mathbb{S}} \left\{ \nu_{s'}(\mu,\eta) + \beta V\left(s', \mu'_{s'}(\mu,\eta)\right) \right\} Q\left(s,s'\right), \tag{30}$$

where $\nu_{s'}(\mu,\eta) = \sup_{y\in C(s')} \sum_{\mathbb{I}} (\mu^i + \eta^{s',i}) y^i$ and $\mu'_{s'} = \{\mu'_{s',i}\}$ with $\mu'_{s',i}(\mu,\eta) = \mu^i + \eta^{s',i}$. Again the operator $\mathscr{D}$ relocates calculations to the space of "Pareto weight" domain value functions and $V^* = \mathscr{D}(V^*)$, where $V^*$ is the value function from the Pareto problem:

$$V^*(s,\mu) = \sup_{\Omega_1} \sum_{\mathbb{I}} \mu^i \sum_{s'\in\mathbb{S}} U\left(s', y^\infty\right) Q\left(s,s'\right).$$

$V^*(s,\cdot)$ is also the *support function* of the feasible payoff set given shock $s$.[11] To obtain only Pareto efficient risk-sharing plans (and since the continuation of Pareto efficient plans are Pareto efficient), it is sufficient to restrict attention to $\mu \in D := \mathbb{R}_+^I \setminus \{0\}$.

More generally, to extend the analysis to problems in which all players are uncommitted, set $f = 0$ on Graph $C$. The primal problems (18) then reduce to:

$$W^*(s,x) = \begin{cases} 0 & \text{if } \Omega_1(s,x) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases} \tag{31}$$

As in Example 4, $W^*$ is an indicator function for the set of incentive-feasible shock-promise pairs $X^* = \{(s,x) \mid \Omega_1(s,x) \neq \emptyset\}$. Suppose that Graph $C$ is bounded and that $u^m$ and $u$ satisfy the conditions of Theorem 1. For each $s \in \mathbb{S}$, let $X_0(s)$ be a compact and convex set satisfying $X^*(s) \subset X_0(s)$. If $W_0$ is the indicator function for $X_0$ and $\mathscr{B}(W_0) \leqslant W_0$, then, by Theorem 2, $\mathscr{B}^n(W_0) \xrightarrow{h} W^*$.

The dual value functions $V^*(s,\cdot) = D(W^*)(s,\cdot)$ are support functions for the promise sets $X^*(s) = \{x \mid \Omega_1(s,x) \neq \emptyset\}$. If $X^*$ is bounded, then $W^*$ is extended real-valued, but, following our earlier discussion, $V^*$ is real-valued. If $X_0$ and $W_0$ are as in the previous paragraph and with a strengthening of assumptions, along the lines of Theorem 3, then iteration of $\mathscr{D}$ from $V_0 = D(W_0)$ generates a sequence of functions that epi-converge to the support functions of the incentive-feasible promise set $X^*$.[12]

Abreu et al. (1990) introduce a set-valued operator for computing the sequential equilibrium payoff sets of repeated games. Our Bellman operator $\mathscr{B}$ is quite related to their $B$-operator. There are two main differences.[13] First, we represent equilibrium payoffs with indicator functions, rather than sets. This allows us to give a more unified treatment of problems with and without a committed principal. Second, the APS $B$-operator uses the (candidate) set of equilibrium payoffs to

[9] If $P \subseteq \mathbb{R}^N$, then its indicator function is $\delta_P : \mathbb{R}^N \to \overline{\mathbb{R}}$, with $\delta_P(p) = 0$ if $p \in P$ and $-\infty$ otherwise.

[10] There is another way to obtain a recursive formulation of the problem. It involves maximizing the weighted payoff of an (incentive-constrained) agent subject to incentive-feasibility and the delivery of a utility promise to all other agents. This approach leads to a recursive formulation in which the optimal value function enters the constraint set. It is the formulation adopted in Kocherlakota (1996).

[11] If $P \subseteq \mathbb{R}^N$, then the support function is $\sigma_P : \mathbb{R}^N \to \overline{\mathbb{R}}$ with $\sigma_P(p^*) = \sup_P \langle p^*, p\rangle$. $\sigma_P$ encodes the set of supporting hyperplanes to the set $P$.

[12] Judd et al. (2003) also use support functions to represent equilibrium payoff sets. Their outer approximation formulation can be seen as combining a dualization of the promise-keeping (but not the incentive) constraints with a linear interpolation.

[13] Abreu et al. (1990) consider a class of hidden action games. These do not fall within the class of problems considered here as their incentive constraints are not additively separable in current actions and continuation payoffs. However, the APS $B$-operator is applicable to games which do fall in to this class.

determine the continuation payoff consequences of agent deviations. Our basic formulation of incentive constraints handles two cases: those in which the payoff consequences of deviation are determined by the optimal plan and those in which they are known a priori (to the modeler). The first case accommodates hidden information problems, the second limited commitment problems in which reversion to autarky characterizes the worst equilibrium outcome. An extension of our formulation allows us to compute the worst equilibrium payoff simultaneously with all payoffs that can be sustained by reversion to the worst. This extension allows us to solve problems in which public deviations are possible and worst equilibrium payoffs a priori unknown; it aligns our approach more closely with Abreu et al. (1990). In the extension, some or all of the recursive incentive constraints have the form:

$$u^i(s', y) + \delta x'^i_{s'} - \sup_{C(s'; y_{-i})} \inf_{\mathbb{R}^I} \{u^i(s', \tilde{y}) + \delta(\tilde{x}_i - W^*(s', \tilde{x}))\} \geqslant 0. \tag{32}$$

Here, $u^i(s', y) + \delta x'^i_{s'}$ gives the payoff to the $i$-th player if she adheres to the action profile $y$, while $\sup_{C(s';y_{-i})} \inf_{\mathbb{R}^I} \{u^i(s', \tilde{y}) + \delta(\tilde{x}_i - W^*(s', \tilde{x}))\}$ gives her payoff from a best defection followed by a worst incentive-feasible continuation payoff. $C(s'; y_{-i})$ is player $i$'s set of observable defections given that the other players $(-i)$ adhere to $y$; $\tilde{x} = \{\tilde{x}_i, \tilde{x}_{-i}\}$ gives continuation payoffs following a defection by $i$. $W^*$ is an indicator function for the equilibrium payoff set. Its inclusion in (32) ensures that only incentive-feasible defection penalties are chosen.[14]

Assume a set of players $\mathbb{I}$ each of whom has constraints of the form (32) (indexed by $m = (i, s') \in \mathbb{I} \times \mathbb{S}$) and that $f = 0$ on Graph $C$ and $-\infty$ otherwise. Let $W$ denote the indicator function of a closed candidate equilibrium payoff set and define:

$$\begin{aligned}\tilde{\mathscr{L}}_s(W)(y, x'; \mu, \eta) = {} & \sum_{\mathbb{S}} [f(s', y_{s'}) + \beta W(s', x'_{s'})] Q(s, s') + \sum_{\mathbb{I}} \mu^i \sum_{\mathbb{S}} [u^i(s', y_{s'}) + \delta x'^i_{s'}] Q(s, s') \\ & + \sum_{\mathbb{I}\times\mathbb{S}} \eta^{i,s'} \Big[u^i(s', y_{s'}) + \delta x'^i_{s'} - \sup_{C(s';y_{-i})} \inf_{\mathbb{R}^I} \{u^i(s', \tilde{y}) + \delta(\tilde{x}_{s'} - W(s', \tilde{x}))\}\Big].\end{aligned} \tag{33}$$

In analogy with our earlier analysis let $\tilde{\mathscr{B}}(W)(s, x) = \sup_A \inf_{\mathbb{R}^I \times \mathbb{R}^{IS}_+} \tilde{\mathscr{L}}_s(W)(y, x'; \mu, \eta) - \sum_{\mathbb{I}} \mu^i x^i$. The dual operator $\tilde{\mathscr{B}}^D$ may be obtained from this by interchanging the supremum and infimum operations. Rearranging terms and using the definition $V = D(W)$, the following analogue of the $\mathscr{D}$-operator may be extracted from $\mathscr{B}^D$:

$$\begin{aligned}\tilde{\mathscr{D}}(V)(s, \mu) = \inf_{\mathbb{R}^{IS}_+} \sum_{\mathbb{S}} \Bigg\{ & \sup_{C(s')} \sum_{\mathbb{I}} \Big[\mu^i u^i(s', y_{s'}) + \frac{\eta^{i,s'}}{Q(s, s')}\Big(u^i(s', y_{s'}) - \sup_{C(s';y_{-i})} u^i(s', \tilde{y})\Big)\Big] \\ & + \beta\Big[V(s', \mu_{s'}(\mu, \eta)) + \frac{\delta}{\beta} \sum_{\mathbb{I}} \Big(\frac{\eta^{i,s'}}{Q(s, s')} V(s', -1_i)\Big)\Big]\Bigg\} Q(s, s'),\end{aligned}$$

where $\mu_{s'}(\mu, \eta) = \{\frac{\delta}{\beta}[\mu^i + \frac{\eta^{i,s'}}{Q(s,s')}]\}_{\mathbb{I}}$ and $-1_i \in \mathbb{R}^N$ has a $-1$ in the $i$-th location and 0's elsewhere. $V(s', \cdot)$ is the support function for the $s'$-shock payoff set. If this set is compact and convex, then $-V(s', -1_i)$ gives the worst (continuation) payoff to the $i$-th agent in this state.[15] Our preceding analysis goes through for the operators $\tilde{\mathscr{B}}$, $\tilde{\mathscr{B}}^D$ and $\tilde{\mathscr{D}}$.[16] In addition, since support functions are homogenous of degree 1, $\mathscr{D}(V)(s, c\mu) = c\mathscr{D}(V)(s, \mu)$, $c > 0$, and the state space can be identified with the unit sphere in $\mathbb{R}^N$.

## 8. Conclusion

We identify a class of dynamic incentive problems that admit recursive formulations. The key property of this class is that incentive constraints are (additively) separable in the continuation utilities of incentive-constrained agents. A difficulty is that not all promised utilities are feasible. We tackle this issue by defining the optimal value to be $-\infty$ at infeasible promises and provide sufficient conditions for value iteration to generate a sequence of extended real-valued functions that hypo-converge to the true value function. Non-finite value functions are problematic from the point of view of numerical computation. This motivates us to consider dual formulations. We isolate a component dual operator that relocates the value iteration to a space of conjugate functions, these may be real-valued even when the original (promise-domain) value functions are not. In addition, the dual operator decomposes optimizations across shock realizations and, hence, simplifies them. We give sufficient conditions for the dual operator to generate a sequence of functions that epi-converge to the conjugate of the promise-domain value function. Equivalently, they converge to the value function from a corresponding recursive Pareto problem. Our results rely on a "recursize, then dualize" strategy, in contrast to other papers that reverse these roles. Consequently, we do not need to treat explicitly the infinite dimensional dual space containing multipliers on all constraints from the infinite horizon problem.

[14] $-W^*(s', \tilde{x}) = 0$ if $\tilde{x}$ is incentive feasible and $\infty$ if $\tilde{x}$ is infeasible. The latter will not be chosen in the infimum operation.

[15] Using the definition of a support function: $-V(s', -1_i) = -\sup_{X^*(s')} \langle -1_i, x\rangle = \inf\{x_i \mid x \in X^*(s')\}$.

[16] The concavity requirements in the earlier theorems now become quite demanding. Following Judd et al. (2003) lotteries may be introduced to restore concavity.

## Appendix A. Proofs from Sections 3 and 4

**Lemma 1.** *Under Assumption 4, $W^* \in \mathscr{W}$ and, for each $s$,* $\operatorname{Dom} W^*(s,\cdot) \neq \emptyset$.

**Proof.** By Assumption 4, for each $s \in \mathbb{S}$, there is some $(x, y^\infty)$ such that $y^\infty \in \Omega_1(s,x)$ and $F(s, y^\infty) > -\infty$. Hence, $W^*(s,\cdot)$ is not everywhere $-\infty$ and $\operatorname{Dom} W^*(s,\cdot) \neq \emptyset$. Also, by Assumption 4, for all $(s,x)$, $W^*(s,x) \leqslant \sup_{\mathbb{S}\times\Omega_1} F(s,y^\infty) < \infty$. Hence, each $W^*(s,\cdot)$ is sup-proper. □

**Proof of Proposition 1.** Case 1 proof is given; Case 2 proof is similar. Note first that since Assumption 4 holds, by Lemma 1, $W^*$ is sup-proper and $\mathscr{B}(W^*)$ is well defined. We first show $W^*(s,x) \leqslant \sup_{\Gamma(s,x)} \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s')$. If $W^*(s,x) = -\infty$, then the inequality is immediate. If not, let $y^\infty = (y_1, \{y_2^\infty(s')\}) \in \Omega_1(s,x)$ and for $s' \in \mathbb{S}$ let $x_{2,s'} = \sum_{\mathbb{S}} U(s'', y_2^\infty(s'))Q(s',s'')$. Then since $y^\infty \in \Omega_1(s,x)$ implies for $s' \in \mathbb{S}$, $y_2^\infty(s') \in \Omega_1$, we have $y_2^\infty(s') \in \Omega_1(s', x_{2,s'})$. So, $W^*(s', x_{2,s'}) \geqslant F(s', y_2^\infty(s'))$. Also, for $m \in \mathbb{M}$, $\sum_{\mathbb{S}}[u^m(s', y_1(s')) + \delta x_{2,s'} q^m_{s'}] = \sum_{\mathbb{S}}[u^m(s', y_1(s')) + \delta \sum_{\mathbb{S}} U(s'', y_2^\infty(s'))Q(s',s'')q^m_{s'}] \geqslant 0$ and so $(y_1, x_2) \in \Gamma(s,x)$. Thus, $\sup_{\Gamma(s,x)} \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s') \geqslant \sum_{\mathbb{S}}[f(s', y_1(s')) + \beta W^*(s', x_{2,s'})]Q(s,s') \geqslant F(s, y^\infty)$. Since $y^\infty$ was an arbitrary element of $\Omega_1(s,x)$, $\sup_{\Gamma(s,x)} \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s') \geqslant \sup_{\Omega_1(s,x)} F(s, y^\infty) = W^*(s,x)$.

Next we show $W^*(s,x) \geqslant \sup_{\Gamma(s,x)} \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s')$. If $\Gamma(s,x) = \emptyset$, then $\sup_{\Gamma(s,x)} \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s') = -\infty$ and the inequality is immediate. Let $(y, \{x'_{s'}\}) \in \Gamma(s,x)$. If $\Omega_1(s', x'_{s'}) = \emptyset$ for some $s' \in \mathbb{S}$, then again $\sum_{\mathbb{S}}[f(s', y_{s'}) + W^*(s', x'_{s'})]Q(s,s') = -\infty$ and the result follows. Otherwise, for each $s' \in \mathbb{S}$ choose $y^\infty_{s'} \in \Omega_1(s', x'_{s'})$. Then $(y, \{y^\infty_{s'}\}) \in \Omega_1(s,x)$ and $W^*(s,x) \geqslant \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta F(s', y^\infty_{s'})]Q(s,s')$. Since each $y^\infty_{s'}$ is an arbitrary element of $\Omega_1(s', x'_{s'})$, $W^*(s,x) \geqslant \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s')$. Since $(y, \{x'_{s'}\})$ is an arbitrary element of $\Gamma(s,x)$, $W^*(s,x) \geqslant \sup_{\Gamma(s,x)} \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W^*(s', x'_{s'})]Q(s,s')$.

Suppose that $y^\infty \in \Omega_1(s,x)$ solves (18) at $(s,x)$. Then, $(y_1, x_2) \in \Gamma(s,x)$, where $x_{2,s'} = \sum_{\mathbb{S}} U(s'', y_2^\infty(s'))Q(s',s'')$ and each $y_2^\infty(s') \in \Omega_1(s', x_{2,s'})$. Thus, $W^*(s,x) = \sum_{\mathbb{S}}[f(s', y_1(s')) + \beta F(s', y_2^\infty(s'))]Q(s,s') \leqslant \sum_{\mathbb{S}}[f(s', y_1(s')) + \beta W^*(s', x_{2,s'})]Q(s,s') \leqslant W^*(s,x)$. Hence, $(y_1, x_2) \in G(s,x)$ and $F(s', y_2^\infty(s')) = W^*(s', x_{2,s'})$. Applying this argument repeatedly at successive histories proves the final part of the proposition. □

## Appendix B. Proofs from Section 5

For $W \in \mathscr{W}$, define $\mathscr{U}(W)$ according to, for each $(s,x,y,x') \in \mathbb{S} \times \mathbb{R}^N \times Y^S \times \mathbb{R}^{NS}$,

$$\mathscr{U}(W)\big(s,x,y,x'\big) := \begin{cases} \sum_{\mathbb{S}}\{f(s', y_{s'}) + \beta W(s', x'_{s'})\}Q(s,s') & \text{if } (y,x') \in \Gamma(s,x), \\ -\infty & \text{otherwise.} \end{cases}$$

$\mathscr{U}(W)$ expresses the payoff of a principal with continuation objective $W$ as a function of states and choices. Infeasible state-choice combinations are assigned the value $-\infty$. We use the following definition.

**Definition 10.** Let $Z_i \subset \mathbb{R}^{N_i}$, $i = 1,2$. Given $g : Z_2 \to \overline{\mathbb{R}}$ and $v \in \overline{\mathbb{R}}$, $\text{u-lev}_v g = \{z_2 \in Z_2 \mid g(z_2) \geqslant v\}$ is called the $v$-upper level set of $g$. $g$ is said to be *u-level* (*upper-level*) *bounded* if for every $v \in \mathbb{R}$ the set $\text{u-lev}_v g$ is bounded (and possibly empty). Let $g : Z_1 \times Z_2 \to \overline{\mathbb{R}}$. $g$ is said to be *u-level bounded locally in* $z_1$ if for each $z_1 \in Z_1$ and $v \in \mathbb{R}$ there is a neighborhood $\mathscr{N}$ of $z_1$ and a bounded set $B$ such that $\text{u-lev}_v g(z'_1, \cdot) \subset B$ for each $z'_1 \in \mathscr{N}$.

Local u-level boundedness in $z_1$ of $g : Z_1 \times Z_2 \to \overline{\mathbb{R}}$ ensures that the upper level sets of $g(z_1,\cdot)$ do not expand arbitrarily rapidly in the parameter $z_1$. This property is assured if $g$ is finite and continuous on the graph of a continuous, compact-valued correspondence, but is much weaker.

**Lemma 2.** *Let Assumption 5 hold. If $W$ is upper semicontinuous, upper-level bounded and sup-proper, then $\mathscr{U}(W)$ is upper semicontinuous and $\mathscr{U}(W)$ is u-level bounded locally in $x$.*

**Proof.** Let $s \in \mathbb{S}$ and $v \in \mathbb{R}$. If $\text{u-lev}_v \mathscr{U}(W)(s,\cdot) = \emptyset$, then it is closed. If $\text{u-lev}_v \mathscr{U}(W)(s,\cdot) \neq \emptyset$, then let $\{x_n, y_n, x'_n\}$ be a sequence within $\text{u-lev}_v \mathscr{U}(W)(s,\cdot)$ with limit $(x,y,x')$. Since $v \in \mathbb{R}$, it follows that each $(x_n, y_n, x'_n)$ is in Graph $\Gamma(s,\cdot)$. Since $C$ is closed-valued, $u$ is continuous and $u^m$ is upper semicontinuous, $(x,y,x') \in \text{Graph}\,\Gamma(s,\cdot)$ and, since $f$ and $W$ are upper semicontinuous, $\mathscr{U}(W)(s,x,y,x') = \sum_{\mathbb{S}}[f(s', y_{s'}) + \beta W(s', x'_{s'})]Q(s,s') \geqslant \limsup_n \sum_{\mathbb{S}}[f(s', y_{s',n}) + \beta W(s', x'_{s',n})]Q(s,s') \geqslant v$. So, $\text{u-lev}_v \mathscr{U}(W)(s,\cdot)$ is closed. Since $s$ and $v$ were arbitrary, $\mathscr{U}(W)$ is upper semicontinuous. Since $f$ and $W$ are upper semicontinuous, u-level bounded and sup-proper, $H(y,x') = \sum_{\mathbb{S}}\{f(s', y_{s'}) + \beta W(s', x'_{s'})\}Q(s,s')$ is u-level-bounded. So, for each $v \in \mathbb{R}$, the set $\text{u-lev}_v H$ is bounded. But for all $x \in \mathbb{R}^N$, $\text{u-lev}_v \mathscr{U}(W)(s,x,\cdot) \subset \text{u-lev}_v H$. Hence, each $\mathscr{U}(W)(s,\cdot)$ is u-level bounded locally in $x$. □

**Remark 2.** It is easy to check that Lemma 2 (and all subsequent results) continues to hold if the assumption of closed-valuedness of $C$ is replaced with the following weaker condition. For all $s$ and convergent sequences $y_n \to y$, each $y_n \in C(s)$, if $y \notin C(s)$, then $\lim_{n\to\infty} f(s, y_n) = -\infty$. This condition is implied in the hidden information examples of Section 2 by Assumption 3.

The next lemma is a direct application of Rockafellar and Wets (1998, Theorem 1.17).

**Lemma 3.** *Assume that $\mathscr{U}(W)$ is sup-proper, upper semicontinuous and u-level bounded locally in x. Then $\mathscr{B}(W)$ is sup-proper and upper semicontinuous. In addition, if $\mathscr{B}(W)(s,x) > -\infty$, then $G(W)(s,x) = \arg\max \mathscr{U}(W)(s,x,\cdot)$ is non-empty and compact. If $\mathscr{B}(W)(s,x) = -\infty$, then $G(W)(s,x) = \emptyset$. If $(y^\nu, x'^\nu) \in G(W)(s,x^\nu)$, $x^\nu \to x$ and $\mathscr{B}(W)(s,x^\nu) \to \mathscr{B}(W)(s,x) > -\infty$, then $(y^\nu, x'^\nu)$ is bounded and its cluster points lie in $G(W)(s,x)$.*

**Proof of Theorem 1.** We use p-lim to denote the pointwise limit and h-lim to denote the hypo-limit of a sequence of functions. Let $I := \{W' \mid W_0 \geqslant W' \geqslant W^*\} \subset \mathscr{W}$. By assumption, $W_0$ is sup-proper and u-level bounded; by Assumption 4 and Lemma 1, $W^*$ is sup-proper. Hence, all elements of $I$ are sup-proper and u-level bounded. It is easily shown that if $W'$ and $W''$ belong to $\mathscr{W}$ with $W' \geqslant W''$, then $\mathscr{B}(W') \geqslant \mathscr{B}(W'')$. By (1) and (2) in the theorem, Proposition 1 and the monotonicity of $\mathscr{B}$, for any $W' \in I$, $W_0 \geqslant \mathscr{B}(W_0) \geqslant \mathscr{B}(W') \geqslant \mathscr{B}(W^*) = W^*$. Thus, $\mathscr{B}: I \to I$. In particular, $\{W_n\} \subset I$ and so each $W_n$ is sup-proper and u-level bounded. By (1) in the theorem, $W_0 \geqslant \mathscr{B}(W_0) = W_1$. By the monotonicity of $\mathscr{B}$, $W_n \geqslant W_{n+1}$ implies $W_{n+1} = \mathscr{B}(W_n) \geqslant \mathscr{B}(W_{n+1}) = W_{n+2}$. Thus, by induction, $\{W_n\}$ is a non-increasing sequence. Hence, since each $W_n \geqslant W^*$, $\{W_n\}$ is pointwise convergent with $\operatorname{p-lim} W_n = \inf_n W_n =: W_\infty \geqslant W^*$. Moreover, $W_n \geqslant W_\infty$ implies that for all $n$, $W_{n+1} = \mathscr{B}(W_n) \geqslant \mathscr{B}(W_\infty)$. Consequently, $W_\infty = \operatorname{p-lim} W_{n+1} \geqslant \mathscr{B}(W_\infty)$. In addition, for all $n$ and all $(s,x)$, $\infty > W_0(s,x) \geqslant \mathscr{B}(W_0)(s,x) = \sup_A \mathscr{U}(W_0)(s,x,y,x') \geqslant \sup_A \mathscr{U}(W_n)(s,x,y,x') = \mathscr{B}(W_n)(s,x) = W_{n+1}(s,x) \geqslant W^*(s,x) \geqslant -\infty$, with the last inequality strict for some $(s,x)$. Hence, for each $n$, $\mathscr{U}(W_n)$ is sup-proper.

$W_n$ is sup-proper and u-level bounded. If, in addition, $W_n$ is upper semicontinuous, then Assumption 5 holds and, by Lemma 2, $\mathscr{U}(W_n)$ is upper semicontinuous and u-level bounded locally in $x$ (as well as sup-proper). Hence, by Lemma 3, $W_{n+1}$ is upper semicontinuous. Since $W_0$ is upper semicontinuous, by induction each $W_n$ is upper semicontinuous and each $\mathscr{U}(W_n)$ is upper semicontinuous and u-level bounded locally in $x$.

If $W_\infty(s,x) = -\infty$, then $\mathscr{B}(W_\infty)(s,x) \geqslant W_\infty(s,x)$. If $W_\infty(s,x) > -\infty$, then each $W_{n+1}(s,x) > -\infty$ and so, by Lemma 3, $\arg\max \mathscr{U}(W_n)(s,x,\cdot)$ is non-empty and compact. Also, in this case, each $\arg\max \mathscr{U}(W_n)(s,x,\cdot)$ is contained in $\{(y,x') \mid \mathscr{U}(W_0)(s,x,y,x') \geqslant W_\infty(s,x)\}$, where the latter set is non-empty and, since $\mathscr{U}(W_0)(s,x,\cdot)$ is upper semicontinuous and u-level bounded, compact. Hence, there is a sequence $\{y_n, x'_n\}$ with each term $(y_n, x'_n) \in \arg\max \mathscr{U}(W_n)(s,x) \subset \{(y,x') \mid \mathscr{U}(W_0)(s,x,y,x') \geqslant W_\infty(s,x)\}$. Since the latter set is compact, the sequence admits a convergent subsequence $\{y_{n_\nu}, x'_{n_\nu}\}$ with limit $(y_\infty, x'_\infty) \in \{(y,x') \mid \mathscr{U}(W_0)(s,x,y,x') \geqslant W_\infty(s,x)\}$. Now,

$$\begin{aligned}\mathscr{B}(W_\infty)(s,x) &= \sup_{(y,x')} \mathscr{U}(W_\infty)\big(s,x,y,x'\big) \geqslant \mathscr{U}(W_\infty)\big(s,x,y_\infty,x'_\infty\big) = \lim_{\nu\to\infty} \mathscr{U}(W_{n_\nu})\big(s,x,y_\infty,x'_\infty\big) \\ &\geqslant \lim_{\nu\to\infty} \limsup_{\tilde\nu \geqslant \nu} \mathscr{U}(W_{n_\nu})\big(s,x,y_{n_{\tilde\nu}},x'_{n_{\tilde\nu}}\big) \geqslant \lim_{\nu\to\infty} \limsup_{\tilde\nu \geqslant \nu} \mathscr{U}(W_{n_{\tilde\nu}})\big(s,x,y_{n_{\tilde\nu}},x'_{n_{\tilde\nu}}\big) \\ &= \limsup_{\tilde\nu} \mathscr{U}(W_{n_{\tilde\nu}})\big(s,x,y_{n_{\tilde\nu}},x'_{n_{\tilde\nu}}\big) = \limsup_{\tilde\nu} W_{n_{\tilde\nu}+1}(s,x) = W_\infty(s,x).\end{aligned}$$

Combining inequalities $W_\infty = \mathscr{B}(W_\infty)$. The sequence $\{W_n\}$ is a non-increasing sequence of upper semicontinuous functions with pointwise limit $W_\infty$. By Rockafellar and Wets (1998, Proposition 7.4(d)), the sequence $\operatorname{h-lim} W_n = \inf_n \operatorname{cl} W_n$, where $\operatorname{cl} W_n$ denotes the function whose hypograph is the closure of the hypograph of $W_n$. But since each $W_n$ is sup-proper and upper semicontinuous, $\operatorname{cl} W_n = W_n$ and since the sequence is non-increasing $\inf_n W_n = W_\infty$, the result follows. □

**Proof of Theorem 2.** By Theorem 1, $W_\infty \geqslant W^*$. It remains to show $W^* \geqslant W_\infty$. If $W_\infty(s,x) = -\infty$, then this is immediate. Let $W_\infty(s,x) > -\infty$. Then, by Theorem 1, $\mathscr{B}(W_\infty)(s,x) = W_\infty(s,x) > -\infty$. Also, following the argument for $\mathscr{U}(W_n)$ in the proof of Theorem 1, $\mathscr{U}(W_\infty)$ is sup-proper. $W_\infty$ is in $I$ and so is sup-proper and u-level bounded. It is upper semicontinuous as the pointwise limit of a non-increasing sequence of upper semicontinuous functions. So, by Lemma 2, $\mathscr{U}(W_\infty)$ is upper semicontinuous and u-level bounded locally in $x$. By Lemma 3, there is a $(y_1^*, x_2^*) \in \Gamma(s,x)$ such that:

$$W_\infty(s,x) = \sum_{\mathbb{S}} \big[f\big(s', y_1^*(s')\big) + \beta W_\infty\big(s', x_2^*(s')\big)\big] Q\big(s,s'\big).$$

Since $W_\infty(s,x) > -\infty$, each $W_\infty(s', x_2^*(s')) > -\infty$. Applying this argument at successive nodes gives a sequence $\{y_t^*, x_{t+1}^*\}$ such that for each $s^{t-1}$, $(y_t^*(s^{t-1}), x_{t+1}^*(s^{t-1})) \in \Gamma(s_{t-1}, x_t^*(s^{t-1}))$, where $x_1^*(s^0) = x$, $s_0 = s$. Also for each $T$,

$$W_\infty(s,x) = \sum_{t=1}^{T} \beta^{t-1} E\big[f\big(s_t, y_t^*(s^t)\big) \mid s_0 = s\big] + \beta^T E\big[W_\infty\big(x_{T+1}^*(s^T)\big) \mid s_0 = s\big]. \tag{34}$$

Since $W_0$ is sup-proper, u-level bounded and upper semicontinuous, $W_0$ is bounded above. Thus, $W_\infty$ is bounded above and $W_\infty(s,x) \leqslant \liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} E[f(s_t, y_t^*(s^t)) \mid s_0 = s]$. In addition, since each $(y_t^*(s^{t-1}), x_{t+1}^*(s^{t-1})) \in \Gamma(s_{t-1}, x_t^*(s^{t-1}))$, in Case 1,

$$x=\sum_{t=1}^{T}\delta^{t-1}E\left[u\left(s_t,y_t^*\left(s^t\right)\right)\mid s_0=s\right]+\delta^T E\left[x_{T+1}^*\left(s^T\right)\mid s_0=s\right],$$

or in Case 2, for all $s'$, $x_{s'}=\sum_{t=1}^{T}\delta^{t-1}E[u(s_t,y_t^*(s^t))\mid s_1=s']+\delta^T E[x_{T+1}^*(s^{T+1})\mid s_1=s']$. Since $W_0$ is sup-coercive in expectation and $W_0\geqslant W_\infty$, $W_\infty$ is sup-coercive in expectation as well. Hence, if $\limsup_{T\to\infty}\delta^T E[\|x_{T+1}^*\|\mid s_0=s]\neq 0$, then $\liminf_{T\to\infty}\beta^T E[W_\infty(x_{T+1}^*(s^T))\mid s_0=s]=-\infty$. But this, (34) and the fact that $f$ is bounded above, contradict $W_\infty(s,x)>-\infty$. Thus, $\limsup_{T\to\infty}\delta^T\|E[x_{T+1}^*\mid s_0=s]\|=0$ and so, in Case 1,

$$x=\lim_{T\to\infty}\sum_{t=1}^{T}\delta^{t-1}E\left[u\left(s_t,y_t^*\left(s^t\right)\right)\mid s_0=s\right],$$

and in Case 2, for all $s'\in\mathbb{S}$, $x_{s'}=\lim_{T\to\infty}\sum_{t=1}^{T}\delta^{t-1}E[u(s_t,y_t^*(s^t))\mid s_1=s']$. By the same logic, in Case 1 for each $s^t$, $x_{t+1}^*(s^t)=\lim_{T\to\infty}\sum_{s=1}^{T}\delta^{s-1}E[u(s_{t+s},y_{t+s}^*(s^{t+s}))\mid s^t]$ and similarly for Case 2. Combining these equalities with $(y_t^*(s^{t-1},\cdot),x_{t+1}^*(s^{t-1},\cdot))\in\Gamma(s_{t-1},x_t^*(s^{t-1}))$ ensures that $\{y_t^*\}\in\Omega_1(s,x)$ and so is feasible for (18). Thus, using Assumption 4, $W^*(s,x)\geqslant\sum_{t=1}^{\infty}\beta^{t-1}E[f(s_t,y_t^*(s^t))\mid s_0=s]\geqslant W_\infty(s,x)$. □

**Proof of Corollary 1.** From the proof of Theorem 2, using the fact that $W^*=W_\infty$, if $y^\infty$ satisfies $(y_t(s^{t-1}),x_{t+1}(s^{t-1}))\in G(s_{t-1},x_{t+1}(s^{t-1}))$ for all $t$, $s^{t-1}$, then it is feasible for (18) at $(s_0,x_1)$. Hence, it has a payoff no more than $W^*(s_0,x_1)$. On the other hand, since $W^*$ is bounded above (so that $\limsup_{t\to\infty}\beta^t E[W^*(s_t,x_{t+1})\mid s_0]\leqslant 0$) and $(y_t(s^{t-1}),x_{t+1}(s^{t-1}))\in G(s_{t-1},x_{t+1}(s^{t-1}))$ for all $t$, $s^{t-1}$ and, hence, satisfies the Bellman equation with equality, the plan $y^\infty$ has a payoff no less than $W^*(s_0,x_1)$. Consequently, $y^\infty$ is optimal for (18) at $(s_0,x_1)$. That there exists such a plan $y^\infty$ is immediate from the proof of Theorem 2. □

**Proof of Proposition 2.** Let $(s,x)$ be such that $W^*(s,x)>-\infty$. Then, $\infty>W_0(s,x)\geqslant W_{n+1}(s,x)=\sup_A\mathscr{U}(W_n)(s,x,\cdot)\geqslant W^*(s,x)>-\infty$. Hence, each $\mathscr{U}(W_n)(s,x,\cdot)$ is sup-proper. By the proof of Theorem 2, it is also upper semicontinuous and u-level bounded (since $\mathscr{U}(W_n)$ is u-level bounded locally). Again by the proof of Theorem 2, h-lim $W_n=$ p-lim $W_n=W^*$. Thus, h-lim $\mathscr{U}(W_n)(s,x,\cdot)=$ p-lim $\mathscr{U}(W_n)(s,x,\cdot)=\mathscr{U}(W^*)(s,x,\cdot)$, see Maso (1993, Example 6.24). Now, $\infty>W^*(s,x)=\mathscr{B}(W^*)(s,x)=\sup_A\mathscr{U}(W^*)(s,x,\cdot)>-\infty$. Hence, $\mathscr{U}(W^*)(s,x,\cdot)$ is sup-proper. It is also upper semicontinuous (as the pointwise limit of upper semicontinuous functions). The result then follows from Rockafellar and Wets (1998, Theorem 7.33). □

## Appendix C. Proofs from Section 6

**Proof of Proposition 3.** By the conditions in the proposition, $\sum_{\mathbb{S}}\{f(s',y_{s'})+\beta W(s',x'_{s'})\}\,Q(s,s')$ is sup-proper, upper semicontinuous and u-level bounded. Hence, it has compact upper level sets and $\sup_A\sum_{\mathbb{S}}[f(s',y_{s'})+\beta W(s',x'_{s'})]Q(s,s')<\infty$. Also, $\mathscr{B}(W)(s,x)\leqslant\mathscr{B}^D(W)(s,x)=\inf_\Phi\sup_A\ \mathscr{L}_s(W)(y,x';\mu,\eta)\ -\ \mu x\leqslant\sup_A\sum_{\mathbb{S}}[f(s',y_{s'})+\beta W(s',x'_{s'})]Q(s,s')\ <\infty$, where the first inequality follows from weak duality and the second from the fact that $\mu=0$, $\eta=0$ is feasible for the inf problem. Embed the problems in (22) into the larger class: $R(s,x,p)=\sup_{\tilde{\Gamma}(s,x,p)}\sum_{\mathbb{S}}\{f(s',y_{s'})+\beta W(s',x'_{s'})\}Q(s,s')$. The constraint functions for these problems are upper semicontinuous and sup-proper. By the concavity of $W$ and Assumption 6, each $R(s,\cdot)$ is concave. Hence, the result follows from Borwein and Lewis (2006, Theorem 4.3.8). □

**Proof of Proposition 4.** The Lagrangian (23) may be arranged to give:

$$\mathscr{L}_s(W)\left(y,x';\mu,\eta\right)=\sum_{\mathbb{S}}\left(h_{s,s'}(\mu,\eta,y)+\beta\left\{W\left(s',x'_{s'}\right)+\frac{\delta}{\beta}\left(\mu+\sum_{\mathbb{M}}\eta^m\frac{q_{s'}^m}{Q\left(s,s'\right)}\right)x_{s'}\right\}\right)Q\left(s,s'\right),$$

where: $h_{s,s'}(\mu,\eta,y)=f(s',y_{s'})+\mu u(s',y_{s'})+\sum_{\mathbb{M}}\frac{\eta^m u^m(s',y_{s'})}{Q(s,s')}$. Hence, using the definitions of $\mathscr{B}^D$, $D$, $\mathscr{D}$, $L$, $\nu_{s,s'}$ and $\mu_{s,s'}$, we have:

$$\begin{aligned}L\mathscr{D}D(W)(s,x)&=\inf_{\mathbb{R}^N}\mathscr{D}D(W)(s,\mu)-\mu x\\&=\inf_{\Phi}\sum_{\mathbb{S}}\left(\nu_{s,s'}(\mu,\eta)+\beta D(W)\left(s',\mu_{s,s'}(\mu,\eta)\right)\right)Q\left(s,s'\right)-\mu x\\&=\inf_{\Phi}\sup_{A}\sum_{\mathbb{S}}\left(h_{s,s'}(\mu,\eta,y)+\beta\left\{W\left(s',x'_{s'}\right)+\frac{\delta}{\beta}\left(\mu+\sum_{\mathbb{M}}\eta^m\frac{q_{s'}^m}{Q\left(s,s'\right)}\right)x_{s'}\right\}\right)Q\left(s,s'\right)\\&=\mathscr{B}^D(W)(s,x).\end{aligned}$$

By Proposition 1, $W^*=\mathscr{B}(W^*)$. If $\mathscr{B}(W^*)=\mathscr{B}^D(W^*)$, then $W^*=\mathscr{B}(W^*)=\mathscr{B}^D(W^*)=L\mathscr{D}D(W^*)$. □

## Appendix D. Proof of Theorem 3

**Lemma 4.** *Let Assumption 5′ hold. Assume $V : \mathbb{S} \times \mathbb{R}^N \to \mathbb{R}$ is convex, then $\mathscr{D}(V)$ is convex. If $V$ is bounded below (by a real number) and $V \geqslant \mathscr{D}(V)$, then $\mathscr{D}(V)$ is real-valued and bounded below. $\mathscr{D}(V)$ (and $V$) are continuous.*

**Proof.** Recall the definitions of $h_{s,s'}$ and $\nu_{s,s'}$. For $\lambda \in [0,1]$, let $\mu^\lambda = \lambda\mu^1 + (1-\lambda)\mu^2$ and $\eta^\lambda = \lambda\eta^1 + (1-\lambda)\eta^2$. Then:

$$\nu_{s,s'}(\mu^\lambda,\eta^\lambda) = \sup_{C(s')} h_{s,s'}(\mu^\lambda,\eta^\lambda,y) = \sup_{C(s')}\big[\lambda h_{s,s'}(\mu^1,\eta^1,y) + (1-\lambda)h_{s,s'}(\mu^2,\eta^2,y)\big]$$
$$\leqslant \lambda \sup_{C(s')} h_{s,s'}(\mu^1,\eta^1,y) + (1-\lambda)\sup_{C(s')} h_{s,s'}(\mu^2,\eta^2,y) = \lambda\nu_{s,s'}(\mu^1,\eta^1) + (1-\lambda)\nu_{s,s'}(\mu^2,\eta^2).$$

Hence, $\nu_{s,s'}$ is convex. Since $V$ is convex, it follows that $\mathscr{T}(V)(s,\mu,\eta) := \sum_{\mathbb{S}}(\nu_{s,s'}(\mu,\eta) + \beta V(s',\mu_{s,s'}(\mu,\eta)))Q(s,s')$ is convex in $(\mu,\eta)$. Hence, for any pair $(\mu^i,\eta^i)_{i=1}^2$ and $\lambda \in [0,1]$,

$$\lambda\mathscr{T}(V)(s,\mu^1,\eta^1) + (1-\lambda)\mathscr{T}(V)(s,\mu^2,\eta^2) \geqslant \mathscr{T}(V)(s,\mu^\lambda,\eta^\lambda) \geqslant \inf_{\mathbb{R}^M_+}\mathscr{T}(s,\mu^\lambda,\eta) = \mathscr{D}(V)(s,\mu^\lambda),$$

where again $\mu^\lambda = \lambda\mu^1 + (1-\lambda)\mu^2$ and similarly for $\eta^\lambda$. Since $\eta^1$ and $\eta^2$ were arbitrary, we have: $\lambda\mathscr{D}(V)(s,\mu^1) + (1-\lambda)\mathscr{D}(V)(s,\mu^2) \geqslant \mathscr{D}(V)(s,\mu^\lambda)$. Then, since $s$, $\mu^1$ and $\mu^2$ were arbitrary, $\mathscr{D}(V)$ is convex.

By assumption, both $\sum_{\mathbb{S}}\nu_{s,s'}(\cdot)Q(s,s')$ and $V$ are bounded below. Hence, $\mathscr{D}(V)$ is bounded below (by a real number). It is bounded above by $V$ and, hence, is real-valued on all of $\mathbb{R}^N$. Since real-valued, convex functions on open subsets of $\mathbb{R}^N$ are continuous, continuity of $\mathscr{D}(V)$ (and $V$) is immediate. □

We are now ready to relate $\mathscr{D}$ to $\mathscr{B}^D$ value iteration and, hence, to $\mathscr{B}$ value iteration. Let $\mathscr{B}^{D1} = \mathscr{B}^D$, $\mathscr{B}^{Dn+1} = \mathscr{B}^D \circ \mathscr{B}^{Dn}$ and similarly for $\mathscr{D}$ and $\mathscr{B}$.

**Proposition 7.** *Let Assumption 5′ hold, $W \in \mathscr{C}$ be sup-coercive and for each $s$, $0 \in \operatorname{Dom} W(s,\cdot)$ and $D(W) \geqslant \mathscr{D}D(W)$. Then for $n \in \mathbb{N}$, $\mathscr{B}^{Dn}(W) = D^{-1}\mathscr{D}^n D(W)$.*

**Proof.** Since $W \in \mathscr{C}$ is sup-coercive, $D(W)$ is convex and real-valued (see, Rockafellar and Wets, 1998, p. 479). In addition, since for each $s$, $0 \in \operatorname{Dom} W(s,\cdot)$, $D(W)$ is bounded below (by $W(\cdot,0) > -\infty$). From Lemma 4, $\mathscr{D}D(W)$ is convex and real-valued (and, hence, continuous) and bounded below. From Proposition 4, $\mathscr{B}^D(W) = L\mathscr{D}D(W)$.

We now proceed by induction. Suppose that for some $n$, $\mathscr{D}^n D(W)$ is convex, real-valued (and, hence, continuous) and bounded below and $\mathscr{B}^{Dn}(W) = L\mathscr{D}^n D(W)$. Then: $\mathscr{B}^{Dn+1}(W) = \mathscr{B}^D L\mathscr{D}^n D(W) = L\mathscr{D}DL\mathscr{D}^n D(W) = L\mathscr{D}^{n+1}D(W)$, where the first equality is by assumption, the second is from Proposition 4 and the third is by the assumption on $\mathscr{D}^n D(W)$ and the fact that $g = (g^\star)^\star$ if $g$ is convex and real-valued (in fact, this equality holds under much weaker conditions). By the monotonicity of $\mathscr{D}$, $D(W) \geqslant \mathscr{D}D(W) \geqslant \mathscr{D}^n D(W) \geqslant \mathscr{D}^{n+1}D(W)$. Then, by the assumption on $\mathscr{D}^n D(W)$ and Lemma 4, $\mathscr{D}^{n+1}D(W)$ is convex, real-valued and bounded below. Hence, by induction $\mathscr{B}^{Dn}(W) = L\mathscr{D}^n D(W)$ for all $n$. Moreover, $L = D^{-1}$ on the space of convex and real-valued functions giving the result. □

**Proof of Theorem 3.** Since $W_0$ is in $\mathscr{C}$ and is sup-coercive, it is also upper-level bounded and sup-coercive in expectation. The assumptions in the theorem and Theorem 2 imply that $\mathscr{B}^n(W_0) \xrightarrow{h} W^*$. Proposition 3 then implies that $\mathscr{B}^{Dn}(W_0) \xrightarrow{h} W^*$. $W_0 \in \mathscr{C}$ is sup-coercive and for each $s$, $0 \in \operatorname{Dom} W_0(s,\cdot)$. Hence, as described in the proof of Proposition 7, $D(W_0)$ is convex, real-valued, continuous and bounded below. From Lemma 4, $\mathscr{D}D(W_0)$ is convex and bounded below. In addition, it is finite at 0 and, hence, inf-proper. From Propositions 3 and 4 and the assumptions, $W_0 \geqslant \mathscr{B}(W_0) = \mathscr{B}^D(W_0) = L\mathscr{D}D(W_0)$ and so $D(W_0) \geqslant DL\mathscr{D}D(W_0)$. Now, since $\mathscr{D}D(W_0)$ is convex and inf-proper, $DL\mathscr{D}D(W_0)$ coincides with the closure of $\mathscr{D}D(W_0)$ (see, for example, Bertsekas, 2009, pp. 84–85). But since the bounding function $D(W_0)$ is real-valued, $\mathscr{D}D(W_0)$ and its closure coincide (see Bertsekas, 2009, Proposition 1.3.15, p. 40). Hence, $D(W_0) \geqslant \mathscr{D}D(W_0)$ and so $\mathscr{D}D(W_0)$ is real-valued and continuous. By Proposition 7, for each $n$, $\mathscr{B}^{Dn}(W_0) = L\mathscr{D}^n D(W_0) = L\mathscr{D}^n(V_0)$. Consequently, $-L\mathscr{D}^n(V_0) \xrightarrow{e} -W^*$. Also, $\mathscr{D}^n(V_0) = DL\mathscr{D}^n(V_0) = (-L\mathscr{D}^n(V_0))^\star$. The continuity of the Legendre–Fenchel transform with respect to epi-convergence (Rockafellar and Wets, 1998, Theorem 11.34) then implies $\mathscr{D}^n(V_0) \xrightarrow{e} (-W^*)^\star = V^*$. Finally, $V^*(s,\mu) = (-W^*)^\star(s,\mu) = \sup_{x\in\mathbb{R}} W^*(s,x) + \langle x,\mu\rangle = \sup_{x\in\mathbb{R}}\sup_{\Omega_1(x)} F(s,y^\infty) + \langle x,\mu\rangle = \sup_{\Omega_1} F(s,y^\infty) + \mu\sum_{\mathbb{S}} U(s',y^\infty)Q(s,s')$. $W^*$ is sup-coercive since it lies below $W_0$ and so $V^*$ is real-valued. □

## Appendix E. Proofs of Propositions 5 and 6

**Proof of Proposition 5.** Fix $(s_0,x_1)$ and let $\{y_t^*\}$ be as in the proposition. By Proposition 1, there is a corresponding promise plan $\{x_{t+1}^*\}$ with $(y_1^*,x_2^*) \in G(s_0,x_1)$ and for each $t$ and $s^{t-1}$, $(y_t^*(s^{t-1}),x_{t+1}^*(s^{t-1})) \in G(s_{t-1},x_t^*(s^{t-1}))$. By assumption, $\mu_1 \in \partial(-W^*)(s_0,x_1)$. Hence, from Rockafellar (1974, Theorem 16, p. 40), we have that: $W^*(s_0,x_1) = \sup_A \inf_{\mathbb{R}^M_+} \mathscr{L}_{s_0}(W^*)(y,x';\mu_1,\eta) - \mu_1 x_1$. Now:

$$
\begin{aligned}
W^*(s_0, x_1) &= \sup_A \inf_{\mathbb{R}_+^M} \mathscr{L}_{s_0}(W^*)(y, x'; \mu_1, \eta) - \mu_1 x_1 \\
&\geqslant \inf_{\mathbb{R}_+^M} \mathscr{L}_{s_0}(W^*)(y_1^*, x_2^*; \mu_1, \eta) - \mu_1 x_1 \\
&= \inf_{\mathbb{R}_+^M} \sum_{\mathbb{S}} \{ f(s', y_1^*(s')) + \beta W^*(s', x_2^*(s')) \} Q(s_0, s') + \sum_{\mathbb{M}} \eta^m \Big[ \sum_{\mathbb{S}} u^m(s', y_1^*(s')) + \delta \sum_{\mathbb{S}} q_{s'}^m x_2^*(s') \Big] \\
&\quad + \mu_1 \Big[ \sum_{\mathbb{S}} [u(s', y_1^*(s')) + \delta x_2^*(s')] Q(s_0, s') - x_1 \Big] = W^*(s_0, x_1),
\end{aligned}
$$

where the last equality uses the fact that $(y_1^*, x_2^*) \in G(s_0, x_1)$ and so the promise-keeping constraint holds with equality and the incentive constraint with weak inequality. Hence,

$$
(y_1^*, x_2^*(\cdot)) = \arg\max_A \inf_{\mathbb{R}_+^M} \mathscr{L}_{s_0}(W^*)(y, x'; \mu_1, \eta).
$$

By assumption, $\mathscr{L}_{s_0}(W^*)(\cdot; \mu_1, \cdot)$ has a saddle point and so, as a corollary of Rockafellar and Wets (1998, Theorem 11.50), there is an $\eta_1^* \in \arg\min_{\mathbb{R}_+^M} \sup_A \mathscr{L}_{s_0}(W^*)(y, x'; \mu_1, \eta)$ with $(y_1^*, x_2^*) \in \arg\max_A \mathscr{L}_{s_0}(W^*)(y, x'; \mu_1, \eta_1^*)$. Let $\mu_2^* = \{\mu_{s_0, s'}(\mu_1, \eta_1^*)\}$. Using the definition of $\mathscr{D}(V^*)$, $(\eta_1^*, \mu_2^*) \in \Psi(s_0, \mu_1)$ and $y_1^* \in \Xi(s_0, \mu_1^*, \eta_1^*)$. In addition, $x_2^*(s') \in \arg\max_{\mathbb{R}} \mu_2^*(s') x'_{s'} + W^*(s', x'_{s'})$ and so $\mu_2^*(s') \in \partial(-W^*)(s', x_2^*(s'))$. Iterating on this argument gives the desired result. □

**Proof of Proposition 6.** By (i) and (ii) and Corollary 1, there is an optimal action plan $y^{\infty *}$ at $(s_0, x_1)$ with a corresponding optimal promise plan $x^{\infty}$ at $(s_0, x_1)$. By (i) $\mu_1 \in \partial(-W^*(s_0, x_1))$. It remains to show that $y^{\infty *}$ is the unique plan generated by $(\Psi, \Xi)$ from $(s_0, \mu_1)$. Suppose $\mu_t^*(s^{t-1}) \in \partial(-W^*(s_{t-1}, x_t^*(s^{t-1})))$. By (iii) and the proof of Proposition 5, $(y_t^*(s^{t-1}), x_{t+1}^*(s^{t-1}))$ is the primal part of a saddle point of $\mathscr{L}_{s_{t-1}}(W^*)(\cdot, \mu_t^*(s^{t-1}), \cdot)$. Then, by (iv), $(y_t^*(s^{t-1}), x_{t+1}^*(s^{t-1}))$ is the primal component of all saddle points of $\mathscr{L}_{s_{t-1}}(W^*)(\cdot, \mu_t^*(s^{t-1}), \cdot)$. Thus, while there may be multiple minimizers in the problem defined by $\mathscr{D}(V^*)(s_{t-1}, \mu_t^*(s^{t-1}))$ there is a unique maximizer and this must be $(y_t^*(s^{t-1}), x_{t+1}^*(s^{t-1}))$. In particular, for all minimizing multipliers $\eta_t^*(s^{t-1})$, the updated weights $\mu_{t+1}^*(s^t) = \mu'_{s_t}(\mu_t^*(s^{t-1}), \eta_t^*(s^{t-1}))$ are sub-differentials of $-W^*(s_t, x_{t+1}^*(s^t))$. So, by induction, for all $t$ and $s^{t-1}$, and all multiplier plans $\{\eta_t^*, \mu_{t+1}^*\}$ generated by $\Psi$ from $(s_0, \mu_1)$, $y_t^*(s^{t-1})$ is the unique element of $\Xi(s_{t-1}, \mu_t^*(s^{t-1}), \eta_t^*(s^{t-1}))$. Hence, $y^{\infty *}$ is the unique plan generated by $(\Psi, \Xi)$ from $(s_0, \mu_1)$ as desired. □

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