# The Dual Approach to Recursive Optimization: Theory and Examples*

Nicola Pavoni[†] Christopher Sleet[‡] Matthias Messner[§]

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**Abstract**

We bring together the theories of duality and dynamic programming. We show that the dual of a separable dynamic optimization problem can be recursively decomposed. We provide a dual version of the principle of optimality and give conditions under which the dual Bellman operator is a contraction with the optimal dual value function its unique fixed point. We relate primal and dual problems, address computational issues and give examples.

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## 1 Introduction

Many dynamic economic relationships can be formulated as planning problems subject to limited commitment or private information frictions. These problems often admit recursive formulations in which utility promises to agents serve as state variables. Under these formulations, the planner's objective is a weighted sum of initial promise values, while future promise values enter into the problems' incentive constraints.[1] A direct (primal) approach to calculating optimal values and policies in

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[†]Department of Economics, Bocconi University and IGIER, Via Roentgen 1, I-20136, Milan, Italy; IFS and CEPR, London; pavoni.nicola@gmail.com.

[‡]Tepper School of Business, Carnegie Mellon University, Pittsburgh PA 15217; csleet@andrew.cmu.edu.

[§]Bocconi University and IGIER, 20136 Milano, Italy; matthias.messner@unibocconi.it.

[1]Examples are given in the paper. For further examples see Ljungqvist and Sargent (2012).

such settings requires prior recovery of the endogenous set (or correspondence) of feasible state variables. To avoid this difficulty, we propose an alternative "recursive dual" approach. The approach replaces a dynamic (economic) optimization with its dual and reformulates the latter as a dynamic programming problem on a dual co-state space. We show that recursive economic problems always have recursive duals and that the associated (dual) Bellman operator is a contraction on an appropriate domain of candidate value functions. The recursive dual solves the dual rather than the original "primal" problem. We give sufficient conditions for the optimal payoffs and solutions from the dual and the primal to coincide. For problems in which these conditions are not satisfied, we provide a numerical check for primal optimality.

To introduce the dual recursive approach, we use a risk sharing problem with limited commitment and recursive preferences. In this problem, a planner seeks to maximize a Pareto weighted sum of agent utilities subject to resource and no default constraints. The dual of the problem is defined using a Lagrangian that incorporates these constraints and laws of motion for utility promises to the agents. It involves an outer minimization over multipliers (on all resource and no default constraints) and costates (on all laws of motion) and an inner maximization over consumption allocations. The recursive structure of the original problem is inherited by the dual with the costates serving as dual state variables. In particular, the dual value function (on a domain of costates) is a fixed point of a dual Bellman operator. This operator updates candidate value functions via an outer minimization over current constraint multipliers and future costates and an inner maximization over current consumption and future utility promises. Additive separability of the Lagrangian can be exploited to break the inner maximization into a family of simpler maximizations that are coordinated by the multipliers and costates. These simpler maximizations can be solved in parallel and sometimes have analytical solutions avoiding the need for numerical maximization in the inner step completely. The optimal dual value function is convex and positively homogenous in costates and finite on $\mathcal{S} \times \mathbb{R}_{+}^{n_I}$, where $\mathcal{S}$ is the set of shocks and $\mathbb{R}_{+}^{n_I}$ the set of costates for the $n_I$ agents. The positive homogeneity property implies that the dual value function is fully defined on the compact set $\mathcal{S} \times \mathcal{C}_{+}$, where $\mathcal{C}_{+}$ is the intersection of the unit sphere with $\mathbb{R}_{+}^{n_I}$. Thus, the dual state space is simple and is immediately given. This is in contrast to the usual (primal) recursive approach to this problem in which the state space is the endogenous set of incentive-feasible utility promises. The dual Bellman operator is a contraction (on an appropriately defined interval of functions and with respect to an alternative Thompson metric). Thus, it is natural to apply value iteration to calculate the optimal value function. We do so, calculating solutions to two and three agent problems. The strict concavity of the objective and constraint functions combined with the structure of the constraint set ensures that the allocation computed via the recursive dual approach is (up to numerical error) a solution to the original problem.

The remainder of the paper extends and rigorously proves in a general environment results discussed in the limited commitment setting. Specifically, it considers a family of dynamic recursive optimizations that encompasses many economic applications. The objective and constraints of these optimizations are functions of current actions and recursively evolving state variables. In the context of particular applications, the state variables have interpretations as capital, utility promises or inflation. They may be backward-looking (i.e. functions of past actions and shocks and an initial condition such as physical capital) or forward-looking (i.e. functions of current and future actions and shocks as the utility promises were in the limited commitment case). As before, a dual problem is associated with these optimizations via a Lagrangian. In general, to accommodate laws of motion for state variables that are non-linear in states, the Lagrangian must explicitly include these laws of motion along with other constraints.[2] In this general setting we formally develop the computational merits of the recursive dual approach. We give conditions for the dual Bellman operator to be a contraction on a space of convex, positively homogenous and readily approximable functions. As in the limited commitment case, the (co)state space may be restricted to the corresponding unit sphere or its intersection with the non-negative orthant. We describe how in the general case the dual Bellman again involves minimizations over multipliers that coordinate low dimensional and sometimes analytically solvable maximizations. Finally, we provide theoretical conditions for the equality of dual and primal values and stronger conditions for equality of solutions. For situations in which these conditions fail, we provide a numerical ex post check of dual-primal equivalence.

Derivation of the contraction result presents special challenges. The recursive dual features an unbounded value function and an unbounded constraint correspondence. For problems with unbounded value functions, a common procedure following Wessels (1977), is to show that there is a set of functions closed and bounded with respect to a more permissive weighted sup norm that contains the optimal value function and on which the Bellman is a contractive self-map. However, this approach requires that the continuation state variables and, hence, the continuation value function cannot vary "too much" on the graph of the constraint correspondence. Since the dual Bellman operator permits the choice of multipliers

[2] If laws of motion are linear in states, then state variables can be substituted from the problem via iteration on their laws of motion. Both states and laws may then be excluded from the Lagrangian.

from an unbounded set, this condition is only guaranteed in the dual setting if additional non-binding constraints on multipliers are found. Instead, we show that the Bellman is contractive with respect to an alternative metric on a space of functions sandwiched between two (unbounded) functions. We show through examples that such bounding functions are often available.

The paper proceeds as follows. After a brief literature review, Section 2 uses a limited commitment model with recursive preferences to introduce the recursive dual approach. Section 3 introduces a general class of stochastic, infinite-horizon decision problems. In Section 4, a decision maker's problem from this class is paired with its dual and a recursive formulation of the latter obtained. A Bellman-type principle of optimality for the dual problem is established. Section 5 gives a contraction result for recursive dual problems. Section 6 relates values and solutions to primal and dual problems. Section 7 concludes. The published appendix contains proofs. Details of numerical implementation, additional calculations from the limited commitment case and further examples are contained in online appendices. The important special case of problems with laws of motion and constraints that are quasi-linear in primal states is also considered there.

**Literature** Our method is related to, but distinct from, that of Marcet and Marimon (2011). These authors propose solving dynamic optimizations by recursively decomposing a saddle point operation. They restrict attention to concave problems with constraints (including laws of motion) that are linear in forward-looking state variables. They substitute forward-looking states out of the problem using their laws of motion and absorb a subset of constraints into a Lagrangian. Laws of motion for backward-looking primal states (e.g. capital) are left as explicit restrictions. They then recursively decompose a saddle point of this Lagrangian (on the constraint set defined by the backward-looking laws of motion). This approach requires that a saddle point exists in all sub-problems after all histories.

Our approach cleanly separates dualization of the primal from recursive decomposition of the dual. We show that the latter is available under weak separability conditions, much weaker than those assumed by Marcet and Marimon (2011): the recursive dual is available for all recursive problems and can be used to obtain an upper bound on payoffs. Theoretical sufficient conditions for equality of optimal dual and primal values and solutions are stronger than those guaranteeing recursive decomposition. However, several of Marcet-Marimon's restrictions can still be dispensed with. The requirements that constraints are linear in forward-looking state variables and that *every* continuation problem has a saddle can be dropped.

Moreover, when these theoretical conditions are not satisfied, we propose a numerical procedure for checking primal optimality of a dual solution. We also go beyond Marcet and Marimon (2011) in proving that the dual Bellman operator is a contraction and in describing how to numerically implement the recursive dual approach.[3]

Messner et al. (2012) consider the relationship between primal and dual Bellman operators. They restrict attention to concave problems without backward-looking state variables and with laws of motion that are linear in forward-looking ones. Thus, their setting is less general than the present one: it excludes problems such as default with capital accumulation, risk sharing with non-expected utility and optimal monetary policy. In addition, they do not provide contraction results or a numerical implementation. In an important extension of Marcet and Marimon (2011), Cole and Kubler (2012) show how recursive methods using dual variables can be adapted to give sufficient conditions for an optimal solution under weak concavity conditions. Their method expands the state space to incorporate realizations of an end-of-period lottery over the extreme points of flat regions of the continuation value function. This approach permits optimal policies to be isolated in the convexifications of otherwise non-concave problems. We leave integration of Cole and Kubler (2012)'s insights into our framework for future work. In a continuous time setting Miao and Zhang (2015) solve a limited commitment problem using a recursive dual approach. In this specific setting, Miao and Zhang (2015) obtain a sharp characterization of a solution to the problem. In contrast, we consider discrete time settings (where the mathematics is quite different) and develop a method applicable to a range of problems.

Abreu et al. (1990) provide a recursive characterization of payoff sets in repeated games. Applied to our setting, their formulation gives an alternative recursive primal approach to our recursive dual one. Numerical implementations of this approach have been provided by Chang (1998) and Judd et al. (2003). We discuss these in online Appendix F. The technical complications involved in the application of primal recursive methods to economics problems have prompted economists to adopt recursive formulations that replace or supplement standard "primal" state variables with "dual" ones. Examples include Kehoe and Perri (2002), Marimon and Quadrini (2006), Acemoğlu et al. (2010), Chien et al. (2011) and Aiyagari et al. (2002). Despite their widespread use, thorough analysis of these methods is limited and their application has often been ad hoc.

[3]For some problems, Marcet and Marimon (2011)'s recursive saddle Bellman operator is available and resembles our dual Bellman. However, in these cases, our Bellman operator implements a fairly straightforward inf-sup operation, whereas theirs involves a more difficult saddle point operation. In other cases, the recursive saddle Bellman operator is not available or is available, but is quite different from ours. In particular, several of the examples considered in this paper and its appendices either cannot be handled by Marcet and Marimon (2011)'s formulation or would be handled differently.

# 2 The Dual Recursive Method: An Example

This section uses a model of risk sharing with limited commitment and recursive preferences to introduce the dual recursive method. Limited commitment models with standard expected utility preferences have found application in macroeconomics, finance and development.[4] The extension to recursive preferences illustrates the flexibility of our method and is of independent interest. To maximize accessibility for practitioners our tone is heuristic throughout this section. Formal derivations of results in a more general setting are given in later sections.

## 2.1 A Limited Commitment Setting

Let $i \in \mathcal{I} = \{1, \ldots, n_I\}$ denote an agent and $s \in \mathcal{S} = \{1, \ldots, n_S\}$ a current shock. Let $s^t = (s_1, \ldots, s_t) \in \mathcal{S}^t$ be a shock history of length $t = 1, 2, \ldots, \infty$. Shocks are assumed to evolve according to a Markov process with transition matrix $\pi$. Let $\pi(s'|s)$ denote the probability of transitioning from shock $s$ in the current period to $s'$ in the next period and let $\pi^t(s^t|s_0)$ denote the induced probability of shock history $s^t$ conditional on initial shock $s_0$. Shocks impact the joint endowment of the agents and the default utility option obtained by an agent on leaving the group. Let $\gamma : \mathcal{S} \to \mathbb{R}_{++}$ give the endowment of the group after each shock and let $w : \mathcal{S} \to \mathbb{R}^{n_I}$, $w(s) = \{w^i(s)\}_{i \in \mathcal{I}}$, give agents' default utility options.

Agents derive utility from consumption. Let $\mathcal{A} := \times_{i \in \mathcal{I}} [\underline{a}^i, \overline{a}^i] \subset \mathbb{R}^{n_I}_+$ be a set of possible per period consumption levels for the group. Agents' choices at each history are collected into a *consumption plan*: $\mathbf{a} = \{a_t\}_{t=0}^{\infty}$, with $a_0 \in \mathcal{A}$ and for $t \geq 1$, $a_t : \mathcal{S}^t \to \mathcal{A}$. The $i$-th component of $a_t$ represents the consumption of the $i$-th agent at date $t$ and is denoted $a^i_t$; the $s^t$-continuation of a consumption plan $\mathbf{a}$ is denoted $\mathbf{a}|s^t = \{a_{t+\tau}(s^t, \cdot)\}_{\tau=0}^{\infty}$. Denote the set of such consumption plans by $\mathbf{A}$. The payoff to the $i$-th agent from consumption plan $\mathbf{a}$ given initial shock $s_0$ is given by the function $V^i(s_0, \mathbf{a})$, where $V^i$ satisfies the Kreps-Porteus/Epstein-Zin recursion:

$$V^i(s_t, \mathbf{a}|s^t) = \left[ (1-\delta)[a^i_t(s^t)]^{1-\sigma} + \delta \left\{ \sum_{s' \in \mathcal{S}} \left[ V^i(s', \mathbf{a}|(s^t, s')) \right]^{1-\rho} \pi(s'|s) \right\}^{\frac{1-\rho}{1-\sigma}} \right]^{\frac{1}{1-\sigma}}, \tag{U}$$

[4] Economists have studied risk sharing without commitment amongst family members, partners in businesses, villagers in developing economies, countries in sovereign debt markets and so on. Thomas and Worrall (1988) and Kocherlakota (1996) provide theoretical analyses of economies with two agents. Ligon et al. (2002) extend the analysis to a setting with finite (and more than two) agents and provide an empirical application. Numerical analysis, however, has been limited to games played by two or a continuum of agents. The latter reduce to one sided commitment problems at endogenous market clearing shadow prices. We are not aware of any analysis of the risk sharing model with limited commitment that uses preferences different from the standard expected utility.

with $0 < \sigma$, $0 < \rho$ and $0 < \delta < 1$. Here $1/\sigma$ is the intertemporal elasticity of substitution (IES), $\rho$ the coefficient of relative risk aversion (CRRA) and $\delta$ the intertemporal discount factor. The compactness of $\mathcal{A}$ ensures that $V = \{V^i\}_{i\in\mathcal{I}}$ takes its values in a bounded set $\mathcal{V}$. The planner attaches Pareto weight $\lambda^i \geq 0$ to the $i$-th agent's date zero payoff and evaluates consumption plans according to:

$$\sum_{i\in\mathcal{I}} \lambda^i V^i(s_0, \mathbf{a}). \tag{O}$$

To be feasible a plan $\mathbf{a}$ must satisfy for all $t \geq 0$, $s^t \in \mathcal{S}^t$, the *no default constraints*,

$$\text{for each } i \in \mathcal{I}, \qquad V^i(s_t, \mathbf{a}|s^t) - w^i(s_t) \geq 0, \tag{D}$$

and the *resource constraints*,

$$\gamma(s_t) - \sum_{i\in\mathcal{I}} a_t^i(s^t) \geq 0. \tag{R}$$

The no default constraints ensure that each agent is better off remaining in the risk sharing arrangement than taking her outside option. The planner must trade off her desire to smooth an agent's share of the endowment over time against the need to respond to large outside option shocks. The latter require raising a given agent's consumption in order to satisfy the no default constraint. The planner's problem is:

$$P_0^* = \sup_{\mathbf{a}\in\mathbf{A}} \sum_{i\in\mathcal{I}} \lambda^i V^i(s_0, \mathbf{a}) \quad \text{s.t. } \forall t \geq 0,\ s^t \in \mathcal{S}^t,\ \text{(D) and (R)}. \tag{PP}$$

## 2.2 The Dual Problem and the Recursive Dual Approach

Given a consumption plan $\mathbf{a}$, define a corresponding *promise plan* $\mathbf{v} = \{v_t\}_{t=0}^{\infty}$ according to $v_0 := V(s_0, \mathbf{a}) = \{V^i(s_0, \mathbf{a})\}_{i\in\mathcal{I}} \in \mathcal{V}$, and for all $t \geq 1$, $s^t \in \mathcal{S}^t$,

$$v_t(s^t) := V(s_t, \mathbf{a}|s^t) = \{V^i(s_t, \mathbf{a}|s^t)\}_{i\in\mathcal{I}} \in \mathcal{V}. \tag{1}$$

Let $\mathbf{p} := (\mathbf{a}, \mathbf{v})$ denote a plan for both consumption and utility promises. It is readily verified that $\mathbf{p}$ is feasible if and only if it is consistent with the utility recursion (U) and the no default (D) and resource (R) constraints, i.e. if $\forall t \geq 0$, $s^t \in \mathcal{S}^t$, $z_t^v(\mathbf{p})(s^t) = z^v(v_t(s^t), s_t, a_t(s^t), v_{t+1}(s^t, \cdot)) = 0$, where:

$$z^v(v, s, a, v') := \left\{ \left[ (1-\delta)[a^i(s)]^{1-\sigma} + \delta \left\{ \sum_{s'\in\mathcal{S}} \left[ v'^{,i}(s') \right]^{1-\rho} \pi(s'|s) \right\}^{\frac{1-\sigma}{1-\rho}} \right]^{\frac{1}{1-\sigma}} - v^i(s) \right\}_{i\in\mathcal{I}},$$

and $z^h_t(\mathbf{p})(s^t) = z^h(s_t, a_t(s^t), v_{t+1}(s^t, \cdot)) \geq 0$, where:

$$z^h(s,a,v') := \begin{pmatrix} \left\{ \left[ (1-\delta)[a^i]^{1-\sigma} + \delta \left\{ \sum_{s' \in \mathcal{S}} \left[ v'^{,i}(s') \right]^{1-\rho} \pi(s'|s) \right\}^{\frac{1-\sigma}{1-\rho}} \right]^{\frac{1}{1-\sigma}} - w^i(s) \right\}_{i \in \mathcal{I}} \\ \\ \gamma(s) - \sum_{i \in \mathcal{I}} a^i(s) \end{pmatrix}.$$

The utility recursion can be interpreted as a law of motion for utility promises. Let $\mathbf{q} := (\mathbf{m}, \mathbf{y})$ be a multiplier plan, where $\mathbf{m} = \{m_t\}_{t=0}^{\infty}$ are non-negative multipliers on the no default and resource constraints and $\mathbf{y} = \{y_t\}_{t=0}^{\infty}$ are non-negative multipliers on the law of motion for promises, i.e. are costates for promises. Define the Lagrangian:

$$\mathscr{L}(\mathbf{p}, \mathbf{q}) := \lambda \cdot v_0 + \sum_{t=0}^{\infty} \sum_{\mathcal{S}^t} \delta^t \{ y_t(s^t) \cdot z^v_t(\mathbf{p})(s^t) + m_t(s^t) \cdot z^h_t(\mathbf{p})(s^t) \} \pi^t(s^t|s_0).$$

The planner's problem may then be re-expressed as the sup-inf problem:[5]

$$P_0^* := \sup_{\mathbf{p}} \inf_{\mathbf{q}} \mathscr{L}(\mathbf{p}, \mathbf{q}). \tag{SI}$$

Problem (SI) delivers both the optimal value $P_0^*$ and the set of solutions to (PP). The planner's *dual problem* interchanges the inf and sup operations:

$$D_0^* := \inf_{\mathbf{q}} \sup_{\mathbf{p}} \mathscr{L}(\mathbf{p}, \mathbf{q}). \tag{IS}$$

Towards a recursive formulation of the dual problem (IS), define the *dual value function* $D^*$ for all possible weightings of initial utilities $y_0 \in \mathbb{R}^{n_I}_+$:

$$D^*(s_0, y_0) := \inf_{\mathbf{q}} \sup_{\mathbf{p}} \; y_0 \cdot v_0 + \sum_{t=0}^{\infty} \sum_{\mathcal{S}^t} \delta^t \{ y_t(s^t) \cdot z^v_t(\mathbf{p})(s^t) + m_t(s^t) \cdot z^h_t(\mathbf{p})(s^t) \} \pi^t(s^t|s_0). \tag{2}$$

Note that: $D_0^* = D^*(s_0, \lambda)$. The recursive dual formulation of (IS) decomposes (2) into sub-problems linked by shocks and costates on utility promises. Since the latter weight agent payoffs it is natural to interpret them as endogenously evolving Pareto weights. The recursive formulation is conveniently expressed in "time invariant"

[5] The domain of the Lagrangian is the product of a space of plans $\mathbf{P}$ (inclusive of promises) and non-negative multipliers $\mathbf{Q}$ (inclusive of costates on promises). Precise definitions are given in later sections. Note that although the law of motion for promises is an equality constraint, the monotonicity of the problem in these promises permits the relaxation of this constraint to an inequality. Consequently, the assumption that costates are non-negative is unrestrictive.

notation. Let $y \in \mathbb{R}_+^{n_I}$ be a costate for the current period and let $q = (m, y')$ be a *current dual choice*, where $m \in \mathbb{R}_+^{n_I+1}$ is a multiplier on the current constraints $z^h(s,p) \geq 0$ and $y' \in \mathbb{R}_+^{n_I \times n_S}$ is a state-contingent vector of costates for the next period. Let $p = (a, v')$ denote a *current primal choice*, where $a \in \mathcal{A}$ is current consumption and $v' \in \mathcal{V}^{n_S}$ is a state-contingent vector of continuation utility promises. The following dynamic programming result obtains.[6]

**Result 1 (Bellman).** *The continuation dual function $D^*$ satisfies the Bellman equation:*

$$\forall s, y) \in \mathcal{S} \times \mathbb{R}_+^{n_I}, \qquad D^*(s,y) = \inf_q \sup_p \; J(s,y;q,p) + \delta \sum_{s' \in \mathcal{S}} D^*\left(s', y'(s')\right) \pi(s'|s), \tag{3}$$

*where the current dual function $J$ is given by:*

$$J(s,y;q,p) := y \cdot \left\{ \left[ (1-\delta)[a^i]^{1-\sigma} + \delta \Big\{ \sum_{s' \in \mathcal{S}} \left[ v'^{,i}(s') \right]^{1-\rho} \pi(s'|s) \Big\}^{\frac{1-\sigma}{1-\rho}} \right]^{\frac{1}{1-\sigma}} \right\}_{i \in \mathcal{I}} + m \cdot z^h(s,p) - \delta \sum_{s' \in \mathcal{S}} y'(s') \cdot v'(s') \pi(s'|s). \tag{4}$$

The recursive dual problem (3) incorporates an inf-sup operation over current dual and primal choices. Its objective is the sum of a *current dual function* $J$ and a discounted expected *continuation dual function* $D^*$. The current dual function $J$ (defined in (4)) describes the shadow value of delivering utility to the agents inclusive of the benefit from relaxing the no default and resource constraints and net of the shadow cost of future utility promises. Heuristically, if agent $i$'s outside option $w^i(s)$ is large, the value of the element in $z^h(s,p)$ corresponding to this agent's no default constraint is reduced. The interaction of the inf and sup operations in (3) then lead to the choice of a higher multiplier $m^i$ on the $i$-th agent's no default constraint, higher continuation promises for agent $i$, $v'^{,i}(\cdot)$, and higher costate $y'^{,i}(\cdot)$ choices. The latter transmit the need for higher agent $i$ utility to the future period. As noted, they function as endogenously evolving Pareto weights.

Let $\mathscr{G}$ denote a candidate set of dual value functions containing $D^*$. The *dual Bellman operator* $\mathscr{B}$ implied by (3) is given by for $D \in \mathscr{G}$ and each $(s,y) \in \mathcal{S} \times \mathbb{R}_+^{n_I}$,

$$\mathscr{B}(D)(s,y) = \inf_q \sup_p \; J(s,y;q,p) + \delta \sum_{s' \in \mathcal{S}} D(s', y'(s')) \pi(s'|s). \tag{5}$$

Thus, from (3), $D^*$ is a fixed point of a Bellman operator: $D^* = \mathscr{B}(D^*)$. In addition, plans for consumption (and multipliers) that solve the dual problem (IS) may be recovered directly from the policy correspondences associated with $\mathscr{B}(D^*)$. Finally,

[6] The proof is a special case of Proposition 3 in Section 4.

the consumption plan consistent with these correspondences also attains the optimum in the original optimization (PP).[7] The recursive dual formulation has several convenient features from the point of view of computation.

1. *Contractivity.* Subject to the existence of certain bounding functions that define an appropriate domain $\mathscr{G}$ for the dual Bellman operator $\mathscr{B}$, this operator is a contraction.[8] $\mathscr{B}$ then has a unique fixed point, convergence of a value iteration from any $D_0$ in $\mathscr{G}$ to $D^*$ is ensured and both error bounds and rates of convergence are available.

2. *The state space is simple and exogenously given.* The value function $D^*$ is real-valued at all $(s, y) \in \mathcal{S} \times \mathbb{R}_+^{n_I}$.[9] Moreover, $D^*$ is positively homogenous implying that it is fully defined on $\mathcal{S} \times \mathcal{C}_+$, where $\mathcal{C}_+$ is the intersection of the unit sphere with $\mathbb{R}_+^{n_I}$. Thus, the dual value function is fully determined on a simple compact set that can serve as the dual state space. This contrasts with the situation under primal recursive approaches that use utility promises to keep track of histories. Under these approaches the state space is an endogenously determined set of promises.

3. *The* $\inf\sup$ *operation defining* $\mathscr{B}$ *may be reduced to an* $\inf$ *operation.* In general, additive separability in the current dual function with respect to the elements of the primal choice $p$ can be exploited to reduce the inner sup operation to a collection of simpler, parallelizable sup operations. Sometimes these simpler sup operations have analytical solutions that can be substituted into $J$ avoiding the need for any numerical maximization and reducing the application of $\mathscr{B}$ to a collection of minimizations over dual variables only. In the current setting, agent utilities may be (monotonically) transformed by applying the function $\tilde{v} := \frac{v^{1-\sigma}}{1-\sigma}$. The planner's problem can then be re-expressed in terms of transformed utilities. The law of motion for such utilities is:[10]

$$\tilde{v}^i = \frac{1-\delta}{1-\sigma}[a^i]^{1-\sigma} + \frac{\delta}{1-\sigma}\left\{\sum_{s' \in \mathcal{S}} \left[(1-\sigma)\tilde{v}'^{,i}(s')\right]^{\frac{1-\rho}{1-\sigma}} \pi(s'|s)\right\}^{\frac{1-\sigma}{1-\rho}}. \tag{6}$$

The (time) additive separability of (6) may be exploited to re-express the indirect

[7] These results follow from Propositions 4, 7 and 8 below.

[8] This result is proved in a general setting in Theorem 2. The proof specializes to the limited commitment setting. In the online appendices bounding functions for this setting are derived.

[9] We prove this in a general setting in Proposition 2 below.

[10] To ensure that the transformed problem is concave the restriction $\rho \geq \sigma$ is needed. To ensure that agent payoffs are bounded, if $\sigma < 1$, consumptions must be bounded below by an arbitrarily small $\underline{a} > 0$. In general no default constraints will ensure that this last restriction is non-binding.

current dual objective $J^*(s,y;q) := \sup_{p\in\mathcal{A}\times\mathcal{V}^{n_S}} J(s,y;q,p)$ as:

$$J^*(s,y;q) = m^{n_I+1}\gamma(s) - \sum_{i\in\mathcal{I}} m^i w^i(s) + \sup_{\mathcal{A}} \sum_{i\in\mathcal{I}} \left\{ \frac{1-\delta}{1-\sigma}(y^i+m^i)[a^i]^{1-\sigma} - m^{n_I+1}a^i \right\}$$
$$+ \sup_{\mathcal{V}^{n_S}} \sum_{i\in\mathcal{I}} \left\{ \frac{\delta}{1-\sigma}(y^i+m^i)\Big\{ \sum_{\mathcal{S}} \Big[(1-\sigma)\tilde{v}^{\prime,i}(s')\Big]^{\frac{1-\rho}{1-\sigma}} \pi(s'|s)\Big\}^{\frac{1-\sigma}{1-\rho}} - \delta \sum_{\mathcal{S}} y^{\prime,i}(s')\tilde{v}^{\prime,i}(s')\pi(s'|s) \right\}. \tag{7}$$

The component suprema in (7) are simple and easily calculated. When $(s,y;q)$ are such that the choices of $a$ and $v'$ are interior, then analytic solutions are available, the value of $J^*$ is:

$$J^*(s,y;q) = \frac{\sigma}{1-\sigma} \sum_{i\in\mathcal{I}} \Big[(1-\delta)(y^i+m^i)\Big]^{\frac{1}{\sigma}} (m^{n_I+1})^{\frac{\sigma-1}{\sigma}} - \sum_{i\in\mathcal{I}} m^i w^i(s) + m^{n_I+1}\gamma(s), \tag{8}$$

and the co-states satisfy:

$$\forall i \in \mathcal{I}, \quad y^i + m^i = \left\{ \sum_{s'\in\mathcal{S}} \{y^{\prime,i}(s')\}^{\frac{1-\rho}{\sigma-\rho}} \pi(s'|s) \right\}^{\frac{\sigma-\rho}{1-\rho}}.$$

For these $(s,y,q)$ values no explicit inner maximization is necessary.

## 2.3 Numerical Results

This section reports numerical results from a two agent version of the limited commitment problem.[11] Its goal is to highlight a few qualitative properties of the model. A complete quantitative analysis is deferred to later work. Given Result 1, approximations to $D^*$ and the associated policy functions are obtained via value iteration.[12]

To illustrate qualitative properties of the model, the discount factor is set to 0.8, $\sigma$ to 1.5 and $\rho$ to 5. Three shocks are assumed (in this two agent economy). In shock state $s$, agent $s$'s outside option is set equal to the utility obtained from a constant endowment stream equal to a high 52% of the aggregate endowment, while the other agent has an outside utility option equal to the utility from a constant endowment stream equal to a low 25% of the aggregate endowment. In the the third shock state both outside options are set to the low value. The aggregate endowment is held constant at 1.2 (i.e., $\gamma(s) = 1.2\ \forall s$). The Markov transition matrix takes values of 0.9

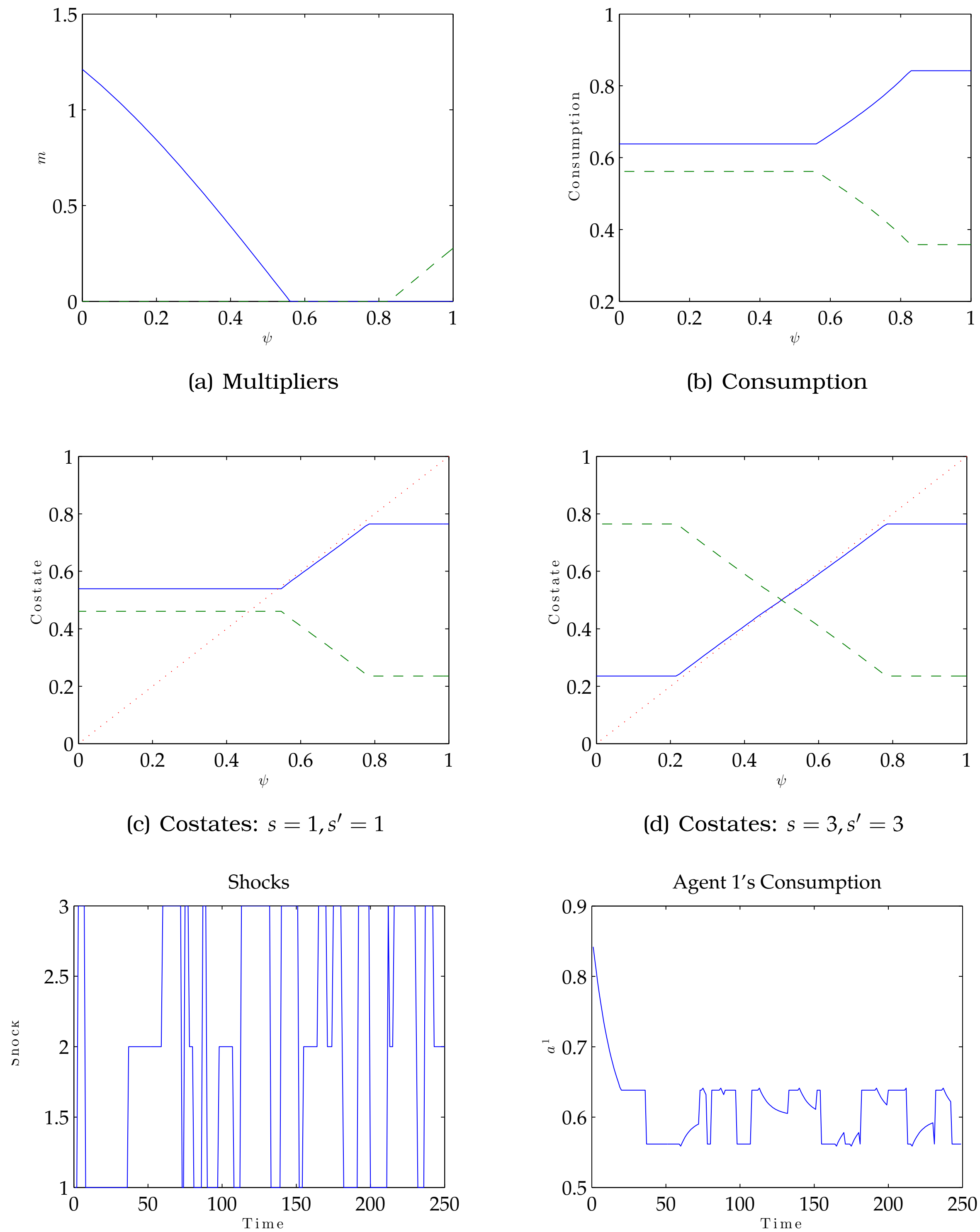
[11] Implications of the policy functions for the dynamics of the optimal allocation are most easily deduced in the two agent case. Results from a three agent version are given in online Appendix B.

[12] Details of the numerical implementation, including details of the approximation of the value functions, are given in online Appendix B.

on its leading diagonal; off diagonal elements are set to 0.05.

Figure 1 plots policies as a function of the normalized costate ("Pareto weight") for agent 1 $\psi = \frac{y^1}{(y^1+y^2)}$. Consider panel a. This shows both agents' no default multipliers $m^i$ in state 1 (in which agent 1's outside option is high and agent 2's low). There are three regions. In the first, agent 1's costate is relatively low (below 0.56) and only agent 1's no default constraint binds. Agent 1's multiplier (the solid line) is positive, while agent 2's (the dashed line) is zero. In the second region, agent 1's costate is intermediate (between 0.56 and 0.81), neither agents' no default constraint binds and both agents' multipliers are zero. In the third region, agent 1's costate is high (above 0.81) and only agent 2's no default constraint binds. In this region agent 1's multiplier is zero, while agent 2's is positive. Panel b shows agent consumption for $s = 1$; panel c shows the continuation costate if the economy remains in state $s = 1$. Each of these policy functions has a floor and ceiling structure: constant in regions 1 and 3 when a no default constraint binds and for agent 1 (resp. agent 2) increasing (resp. decreasing) otherwise. In region 1, agent 1's future costate is raised to 0.56. This translates into the future utility reward needed to maintain agent 1 in the risk sharing arrangement. In region 3, agent 1's costate is reduced to 0.81. This translates into the future utility reward needed to maintain agent 2 in the risk sharing arrangement. In region 2 (with no binding default constraint), agent 1's future costate is slightly reduced, while agent 2's is slightly raised by an equal amount. Hence, if shock $s = 1$ persists, agent 1's costate is gradually reduced to 0.56, while agent 2's is raised. This feature of the optimal risk sharing arrangement stems from the assumption that $\rho$ is greater than $\sigma$, which implies that agents have a preference for the early resolution of risk and a dislike of cross-state variation in utility. The presence of a no default constraint for agent 2 impedes full utility smoothing across states and ensures that there is always a chance that in the future agent 1's utility decreases. Consequently, absent binding no default constraints on agent 1, it is profitable to tilt agent 1's utility profile towards the present and away from future shock state $s' = 1$ and, hence, reduce future period utility dispersion. Agent 1's normalized costate in future state $s' = 1$ is correspondingly reduced. This feature is absent from limited commitment models under the standard assumption of expected utility[13] and reversed if $\rho < \sigma$. Panel d shows the response of costates to a repeated $s = 3$ shock. In the $s = 3$ state both agents' outside options are low. As in panel c, binding no default constraints introduce floors and ceilings into the updated costate function (although now, the floor for agent 1 is at a lower level). Between this

[13]See Alvarez and Jermann (2001) who find that policy functions for continuation utilities lie on the 45 degree line when no default constraints do not bind and agents have expected utility preferences.




(a) Multipliers

(b) Consumption

(c) Costates: $s = 1, s' = 1$

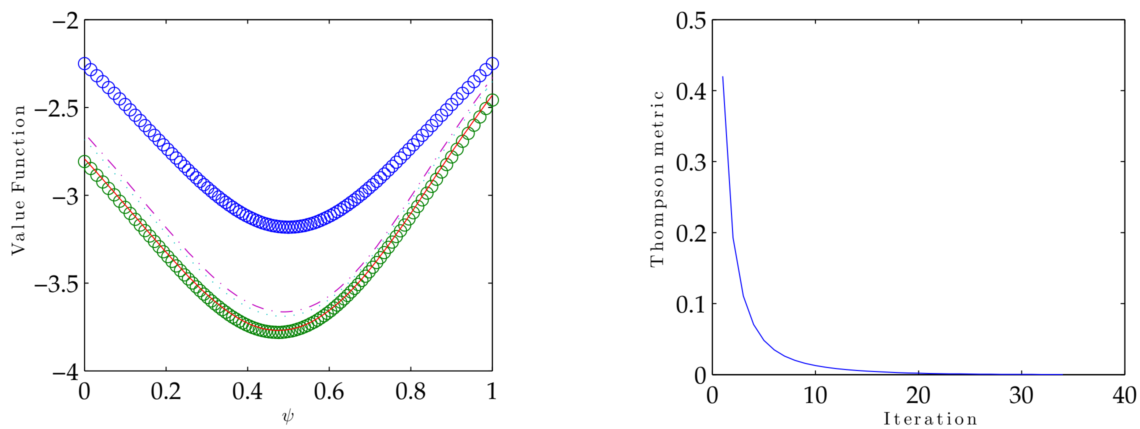
(d) Costates: $s = 3, s' = 3$

(e) Simulated values for state $s$

(f) Values for agent 1's consumption

Figure 1: *In panels a-d, solid lines give agent 1's policy, dashed lines agent 2's policy, the 45 degree line is indicated by dots. Policies are given as functions of agent 1's normalized costate. Panel a shows multipliers on the no default constraints if* $s = 1$*; panel b agent consumption in* $s = 1$*; panels c and d the costates associated with remaining in state 1 and state 3 respectively. Panels e and f show a 250 period simulation. Shocks are given in the left panel, agent 1's consumption in the right.*

floor and ceiling, agent 1's updated costate function crosses the 45 degree line and has a slope less than one. Again, absent binding no default constraints, recursive preferences introduce additional deterministic dynamics which, in this case, create a force for equality and equal division of the endowment. Panels e and f report a simulation of the optimal consumption policy. Initially, agent 1's costate is such that if shocks $s = 1$ and $s = 3$ are drawn then neither agents' no default constraint binds. Consistent with the properties of the policy functions described above, agent 1's consumption steadily falls towards 0.65 and remains there. After the draw of the first $s = 2$ shock, agent 2's no default constraint binds and agent 1's consumption is reduced to 0.55. Thereafter, agent 1's consumption jumps on entering state $s = 1$ or 2, while drifting towards 0.6 (an equal sharing of the aggregate endowment) in $s = 3$.




(a) Value Function Iterates

(b) Thompson Metric

Figure 2: *Value Function Convergence. Panel a shows the calculated value function at iterations 1 (solid line), 10 (dotted line) and 25 (dash dot line) at $D(\psi, 1-\psi)$. Bounding value functions $\underline{D}$ and $\overline{D}$ are shown with circles. Panel b reports Thompson metric distances between iterates.*

Figure 2 shows successive value functions calculated during the value function iteration. Results for other parameter values and for a three agent economy are reported in online Appendix B.

## 2.4 Primal Recursive Approach

We conclude this section by describing an alternative (primal) recursive approach to solving the limited commitment problem. Define $\tilde{\mathcal{I}} := \{2, 3, \ldots, n_I\}$ to exclude agent 1's label and let $\tilde{\mathcal{V}} := \times_{i \in \tilde{\mathcal{I}}} [\underline{a}^i, \overline{a}^i]$ be a set of possible promises for agents in $\tilde{\mathcal{I}}$. For each $s \in \mathcal{S}$, let $\tilde{\mathcal{X}}^*(s)$ be the set of feasible initial payoffs for these agents (i.e. the set of payoffs for agents in $\tilde{\mathcal{I}}$ that are initial elements of a feasible promise plan given initial shock $s$). Let $\tilde{P}^*(s_0, \tilde{v}_0)$ be the best possible feasible payoff to agent 1 given initial shock $s_0$ and feasible utility promise $\tilde{v}_0$ to the other agents. Problem (PP) can then be restated as:[14]

$$P_0^* = \sup_{\tilde{v}_0 \in \tilde{\mathcal{X}}^*(s_0)} \lambda^1 \tilde{P}^*(s_0, \tilde{v}_0) + \sum_{\tilde{\mathcal{I}}} \lambda^i \tilde{v}_0^i. \tag{9}$$

An optimal action and promise plan can be obtained by repeatedly solving:

$$\begin{aligned}
(a_t^*(s^t), \tilde{v}_{t+1}^*(s^t, \cdot)) = \operatorname*{argmax}_{(a, \tilde{v}')} & \left[(1-\delta)[a^1]^{1-\sigma} + \delta\Big\{\sum_{s' \in \mathcal{S}} \left[\tilde{P}^*(\tilde{v}(s'))\right]^{1-\rho} \pi(s'|s_t)\Big\}^{\frac{1-\sigma}{1-\rho}}\right]^{\frac{1}{1-\sigma}} \qquad (10) \\
\text{s.t.} \qquad & \gamma(s_t) - \sum_{i \in \mathcal{I}} a^i \geq 0; \qquad \forall s', \quad \tilde{v}'(s') \in \tilde{\mathcal{X}}^*(s'); \\
\forall i \in \tilde{\mathcal{I}}, \quad \tilde{v}_t^{i,*}(s^t) = & \left[(1-\delta)[a^i]^{1-\sigma} + \delta\Big\{\sum_{s' \in \mathcal{S}} [\tilde{v}'^{,i}(s')]^{1-\rho} \pi(s'|s_t)\Big\}^{\frac{1-\sigma}{1-\rho}}\right]^{\frac{1}{1-\sigma}} \geq w^i(s_t); \\
\text{and} \quad & \left[(1-\delta)[a^1]^{1-\sigma} + \delta\Big\{\sum_{s' \in \mathcal{S}} \left[\tilde{P}^*(\tilde{v}'(s'))\right]^{1-\rho} \pi(s'|s_t)\Big\}^{\frac{1-\sigma}{1-\rho}}\right]^{\frac{1}{1-\sigma}} \geq w^1(s_t),
\end{aligned}$$

starting from $(s_0, \tilde{v}_0^*)$. However, solution of (9) and (10) requires prior recovery of $(\tilde{\mathcal{X}}^*, \tilde{P}^*)$. One approach is to formulate $(\tilde{\mathcal{X}}^*, \tilde{P}^*)$ as the fixed point of a monotone (but not contractive) operator on a space of candidate value functions $P$ and state space correspondences $\mathcal{X}$. We describe this approach in detail in online Appendix B.4. This approach suggests an algorithm for computing $(\tilde{\mathcal{X}}^*, \tilde{P}^*)$ that relies on repeated application of the operator to a bounding pair $(\tilde{\mathcal{X}}_0, \tilde{P}_0)$ with $\tilde{\mathcal{X}}_0 \supset \tilde{\mathcal{X}}^*$ and $\tilde{P}_0 \geq P^*$. The algorithm is practical if there are two agents ($n_I = 2$), since then $\tilde{\mathcal{X}}^*$ is simply a family of intervals indexed by the shock. Thus, the iteration occurs on a space of value functions and end points for the intervals. For problems with $n_I > 2$, the iteration involves higher dimensional optimizations and approximation, repeated calculation and updating of the state spaces in $\tilde{\mathcal{X}}_n(s) \subset \mathbb{R}^{n_I - 1}$, $s = 1, \ldots, n_S$. In addition the operators used to update value functions and domains in these iterations, while monotone, are not contractions. Consequently, error bounds and convergence criteria are not available.[15]

[14]This approach is suggested by a formulation of Kocherlakota (1996), who considered the case with two agents ($n_I = 2$), i.i.d. shocks and expected utility preferences. See also Chapter 20 (Sections 20.7-20.10) in Ljungqvist and Sargent (2012) and the analysis of Alvarez and Jermann (2001). A related and more widely applicable approach is discussed in Appendix F.

[15]A recent promising approach is suggested by Cai et al. (2016) who, in a different setting, relax the incentive constraints with penalty functions and use adaptive splines to prevent penalties proliferating.

# 3 The Decision Maker's Problem

This section describes an abstract recursive choice problem that can be specialized to give many problems considered in the literature including the limited commitment problem of Section 2. Further examples, including optimal monetary policy and dynamic insurance with private information, are given in online Appendix E. Later sections formally develop the dual recursive method in this context.

**Shocks and Action Plans.** The process for shocks is denoted and defined as before. Transition matrix $\pi$ is assumed to be strictly positively valued: for all $s, s' \in \mathcal{S}$, $\pi(s'|s) > 0$.[16] A nonempty set $\mathcal{A} \subset \mathbb{R}^{n_A}$ contains actions potentially available to a decision-maker at each date. The notation for an *action plan* remains: $\mathbf{a} = \{a_t\}_{t=0}^{\infty}$, with $a_0 \in \mathcal{A}$ and, $\forall t \geq 1$, $a_t : \mathcal{S}^t \to \mathcal{A}$. Let $\mathbf{A} = \mathcal{A}^\infty$ denote the set of action plans and $\mathbf{a}|s^t$ the $s^t$-continuation of action plan $\mathbf{a}$.

**States and Constraints.** Let $\mathcal{V} \subset \mathbb{R}^{n_V}$ be a bounded set of "forward-looking" states similar to the utility promises of the last section and let $\mathbf{v} = \{v_t\}_{t=0}^{\infty}$, with $v_0 \in \mathcal{V}$ and for $t \geq 1$, $v_t : S^t \to \mathcal{V}$, be a plan for such states. To accommodate (amongst other things) agent utility promises consistent with non-expected utility, we assume that the law of motion for forward-looking states is constructed from a pair of functions $W^v : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^{n_V} \to \mathbb{R}^{n_V}$ and $M^v : \mathcal{S} \times \mathcal{V}^{n_S} \to \mathbb{R}^{n_V}$. Plans $\mathbf{a}$ and $\mathbf{v}$ satisfy the law of motion if for all $t \geq 0$ and $s^t \in \mathcal{S}^t$,

$$v_t(s^t) = W^v[s_t, a_t(s^t), M^v[s_t, v_{t+1}(s^t, \cdot)]]. \tag{11}$$

Further assume that for each $(s, a) \in \mathcal{S} \times \mathcal{A}$,[17]

$$W^v[s, a, M^v[s, \cdot]] : \mathcal{V}^{n_S} \to \mathcal{V}. \tag{12}$$

As in Section 2, forward-looking state variables often describe the payoffs of agents facing dynamic incentive constraints (with the decision-maker a planner maximizing a weighted sum of agent payoffs). $W^v$ then corresponds to a "time aggrega-

[16]The assumption of a finite number of shocks is maintained. This restriction and that of a positive transition avoid technical complications and streamline our presentation. Neither are essential for our main results.

[17]Many models incorporate stronger conditions that ensure a unique plan $\mathbf{v}$ can be associated with every $\mathbf{a}$. Most commonly, $W^v$ is assumed to be Lipschitz continuous with respect to its third argument and $M^v$ is assumed to be positively homogenous with respect to its second argument. If, in addition, $W^v$ is bounded with respect to $a \in \mathcal{A}$, then (12) holds and any plan $\mathbf{a}$ defines a unique plan $\mathbf{v}$. Marinacci and Montrucchio (2010) obtain a related result by assuming concavity of aggregators.

tor" that gives the current state as a function of current actions and a certainty equivalent of future states, while $M^v$ is a "stochastic aggregator" that generates the certainty equivalent. For example, if the problem involves a single agent with time additive-expected utility preferences, then $W^v[s, a, \mu] = (1-\delta)u(s, a) + \delta\mu$ and $M^v[s, v'] = \sum_{s' \in \mathcal{S}} v'(s')\pi(s'|s)$.

We enrich the environment of the previous section by allowing for "backward-looking" states that have an initial condition and that are determined prior to the realization of the current shock. This enables us to accommodate capital into our framework. Given a bounded set $\mathcal{K} \subset \mathbb{R}^{n_K}$, a plan for backward-looking states (with initial condition $\bar{k}$) is a sequence: $\mathbf{k} = \{k_t\}_{t=0}^{\infty}$, with $k_0 = \bar{k}$ and for $t \geq 1$, $k_t : S^t \to \mathcal{K}$. Let $W^k : \mathcal{K} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}^{n_K}$ define a law of motion for backward-looking states. An action plan $\mathbf{a}$ and a plan for backward-looking states $\mathbf{k}$ (with initial condition $\bar{k}$) satisfy this law of motion if for all $t \geq 0$ and $s^{t+1} \in \mathcal{S}^{t+1}$,[18]

$$k_{t+1}(s^{t+1}) = W^k[k_t(s^t), s_t, a_t(s^t)]. \tag{13}$$

The image of $W^k$, $W^k[\mathcal{K}, \mathcal{S}, \mathcal{A}]$, is assumed to be a bounded set.

Additional constraints are constructed from actions and state variables according to for all $t \geq 0$ and $s^t \in \mathcal{S}^t$,

$$H[k_t(s^t), s_t, a_t(s^t), \{v_{t+1}(s^t, s')\}_{s' \in \mathcal{S}}] \geq 0, \tag{14}$$

where $H : \mathcal{K} \times \mathcal{S} \times \mathcal{A} \times \mathcal{V}^{n_S} \to \mathbb{R}^{n_H}$ is bounded. In applications these inequalities capture incentive, resource and, perhaps, other constraints. Define $\mathcal{K}^*(s)$ to be the set of $\bar{k}$ such that $(s, \bar{k})$ is the initial condition for some triple of plans $(\mathbf{k}, \mathbf{a}, \mathbf{v})$ satisfying (13), (11) and (14).

**Assumption 1.** *For all $s \in \mathcal{S}$, $\mathcal{K}^*(s) \neq \emptyset$.*

*Remark* 1. Some problems such as those in Section 2 may lack backward-looking states. In these cases backward-looking states are dropped from $H$ and no law of motion for backward looking states is associated with the problem. Simplified versions of the arguments that follow go through.

**Decision maker's objective** The decision-maker maximizes a function of the initial values of the forward-looking states: $F : \mathcal{S} \times \mathcal{V} \to \mathbb{R}$. Typical examples include situ-

[18]The law of motion (13) constrains $k_{t+1}$ to be $s^t$-measurable. In our development of the dual recursive approach, it will be convenient to have this restriction explicit in the constraints, rather than implicit in the definition.

ations, as in Section 2, in which a planner maximizes a weighted sum of incentive constrained agent payoffs and $F(s, v_0) = \lambda \cdot v_0$, with $\lambda$ a vector of Pareto weights.

**Problem Statement.** Define a *primal plan* $\mathbf{p}$ to be an action plan $\mathbf{a}$ augmented with a pair of plans for backward and forward-looking states $\mathbf{k} = \{k_t\}_{t=0}^{\infty}$ and $\mathbf{v} = \{v_t\}_{t=0}^{\infty}$. The set of primal plans is given by:

$$\mathbf{P} = \left\{ \mathbf{p} = (\mathbf{a}, \mathbf{k}, \mathbf{v}) \;\middle|\; \begin{array}{l} a_0 \in \mathcal{A}, \quad \forall t \geq 1,\ a_t : \mathcal{S}^t \to \mathcal{A}, \\ k_0 \in \mathcal{K}, \quad \forall t \geq 1,\ k_t : \mathcal{S}^t \to \mathcal{K}, \\ v_0 \in \mathcal{V}, \quad \forall t \geq 1,\ v_t : \mathcal{S}^t \to \mathcal{V} \end{array} \right\}.$$

We consider decision-maker problems of (or that can be expressed in) the form:

$$P_0^* := \sup F[s_0, v_0] \tag{P}$$

subject to $\mathbf{p} \in \mathbf{P}$, $k_0 = \bar{k}$ and $\forall t$, $s^t$,

$$k_{t+1}(s^{t+1}) = W^k[k_t(s^t), s_t, a_t(s^t)], \tag{15}$$

$$v_t(s^t) = W^v[s_t, a_t(s^t), M^v[s_t, v_{t+1}(s^t, \cdot)]], \tag{16}$$

$$\text{and} \qquad H[k_t(s^t), s_t, a_t(s^t), v_{t+1}(s^t, \cdot)] \geq 0. \tag{17}$$

The limited commitment problem from Section 2 can be accommodated in this framework (by removing the backward-looking states and specializing the definitions of $F$, $W^v$ and $M^v$). Other examples are described in online Appendix E.

*Remark* 2. Classical problems in economic dynamics (e.g. those considered in Stokey et al. (1989)) are accommodated by the framework described above. In these the decision-maker's payoff is the only "forward-looking state variable", i.e. $F(s, v_0) = v_0 \in \mathbb{R}$, and this state variable does not appear in the function $H$. In addition, for all $(s, k) \in \mathcal{S} \times \mathcal{K}$ there is an $(a, k') \in \mathcal{A} \times \mathcal{K}$ such that $k' = W^k[k, s, a]$ and $H[k, s, a] \geq 0$. In such cases, repeated substitution of the law of motion of forward-looking variables into the objective eliminates these variables completely and standard dynamic programming arguments can be applied: the decision maker's optimal value function (on the exogenously *given* domain $\mathcal{S} \times \mathcal{K}$) satisfies a typical Bellman equation.

Assumption 2 provides sufficient conditions for the existence of an optimal plan.

**Assumption 2.** *(Continuity and Compactness) For all $s \in \mathcal{S}$, $W^v[s, \cdot, \cdot], M^v[s, \cdot]$ and $W^k[\cdot, s, \cdot]$ are continuous, and $H[\cdot, s, \cdot, \cdot]$ and $F[s, \cdot]$ are upper semicontinuous. In addition, $\mathcal{A}, \mathcal{K}$ and $\mathcal{V}$ are compact.*

**Proposition 1.** *Let Assumptions 1 and 2 hold and let $\bar{k}$ belong to $\mathcal{K}^*(s_0)$ in (P). Then $P_0^* > -\infty$ and an optimal solution $\mathbf{p}^*$ to (P) exists.*

*Proof.* See Appendix A. □

# 4 Recursive Dual

This section begins by defining a Lagrangian for (P). The Lagrangian involves the product of an infinite number of constraint values and multipliers. To compress notation, constraint values are collected into an object called a *constraint process* and multipliers into an object called a *dual plan*. Definitions of these follow. The recursive dual approach is developed.

## 4.1 Lagrangians and Dual Problems

**Notation for the Lagrangian.** A constraint process evaluates constraint functions inclusive of laws of motion in (15), (16), (17) at a given primal plan. For each primal plan $\mathbf{p}$, define $z_0^k(\mathbf{p}) := \bar{k} - k_0$ and, for all $t \geq 1$ and $s^t \in \mathcal{S}^t$, define:

$$z_t^k(\mathbf{p})(s^t) := W^k[k_{t-1}(s^{t-1}), s_{t-1}, a_{t-1}(s^{t-1})] - k_t(s^t).$$

Then $\mathbf{z}^k(\mathbf{p}) = \{z_t^k(\mathbf{p})\}_{t=0}^{\infty}$ gives the values of the backward-looking law of motion constraints (inclusive of the initial condition) at $\mathbf{p}$. Similarly, define for all $t \geq 0$, $s^t \in \mathcal{S}^t$,

$$z_t^v(\mathbf{p})(s^t) := W^v[s_t, a_t(s^t), M^v[s_t, v_{t+1}(s^t, \cdot)]] - v_t(s^t)$$

and $z_t^h(\mathbf{p})(s^t) := H[k_t(s^t), s_t, a_t(s^t), v_{t+1}(s^t, \cdot)]$. Then $\mathbf{z}^v(\mathbf{p}) = \{z_t^v(\mathbf{p})\}_{t=0}^{\infty}$ and $\mathbf{z}^h(\mathbf{p}) = \{z_t^h(\mathbf{p})\}_{t=0}^{\infty}$ give the values of the forward-looking law of motion and $H$ constraints at $\mathbf{p}$. These terms are collected into the *constraint process* $\mathbf{z}(\mathbf{p}) = \{\mathbf{z}^k(\mathbf{p}), \mathbf{z}^v(\mathbf{p}), \mathbf{z}^h(\mathbf{p})\}$. The boundedness assumptions placed on primitives and the countable number of constraints ensure that for all $\mathbf{p} \in \mathbf{P} \subset \ell^\infty$, $\mathbf{z}(\mathbf{p}) \in \ell^\infty$, where $\ell^\infty$ is the set of sup-norm bounded sequences: $\{\{x_n\}_{n=1}^{\infty} | x_n \in \mathbb{R}^{n_K + n_V + n_H}, \sup_{n \in \mathbb{N}} \|x_n\|_E < \infty\}$, with $\|\cdot\|_E$ the relevant Euclidean norm.

A dual plan contains summable multipliers for the various constraints facing the decision-maker. Let $\mathbf{y}^k = \{y_t^k\}_{t=0}^{\infty}$, with $y_t^k : \mathcal{S}^t \to \mathbb{R}^{n_K}$, denote multipliers (costates) for the backward-looking law of motion and $\mathbf{y}^v = \{y_t^v\}_{t=0}^{\infty}$, with $y_t^v : \mathcal{S}^t \to \mathbb{R}^{n_V}$, multipliers (costates) for the forward-looking law of motion. Let $\mathbf{m} = \{m_t\}_{t=0}^{\infty}$, with $m_t : \mathcal{S}^t \to \mathbb{R}_+^{n_H}$, denote multipliers for the $H$-constraints. Collect these various multipliers into a *dual* *plan* $\mathbf{q} = \{\mathbf{m}, \mathbf{y}^k, \mathbf{y}^v\}$ and define the set of dual plans:

$$\mathbf{Q} = \left\{ \mathbf{q} \left| \sum_{t=0}^{\infty} \sum_{\mathcal{S}^t} \delta^t \{ \|m_t(s^t)\|_E + \|y_t^k(s^t)\|_E + \|y_t^v(s^t)\|_E \} \pi^t(s^t|s_0) < \infty \right. \right\},$$

with $\delta \in (0, 1]$ a normalizing discount.[19] Define the Lagrangian:

$$\mathscr{L}(\mathbf{p}, \mathbf{q}) = F[s_0, v_0] + \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle,$$

where:

$$\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle = \sum_{t=0}^{\infty} \sum_{\mathcal{S}^t} \delta^t \{ m_t(s^t) \cdot z_t^h(\mathbf{p})(s^t) + y_t^k(s^t) \cdot z_t^k(\mathbf{p})(s^t) + y_t^v(s^t) \cdot z_t^v(\mathbf{p})(s^t) \} \pi^t(s^t|s_0)$$

is a weighted sum of constraint process terms. Since for each $\mathbf{p} \in \mathbf{P}$, $\mathbf{z}(\mathbf{p}) \in \ell^\infty$, it follows that for each $\mathbf{q} \in \mathbf{Q}$, $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle \in \mathbb{R}$.

**Primal and Dual Problems.** The decision-maker's primal problem (P) may be re-expressed as a sup-inf problem:

$$P_0^* := \sup_{\mathbf{p} \in \mathbf{P}} \inf_{\mathbf{q} \in \mathbf{Q}} \mathscr{L}(\mathbf{p}, \mathbf{q}). \tag{SI}$$

Its dual interchanges the inf and sup operations:

$$D_0^* := \inf_{\mathbf{q} \in \mathbf{Q}} \sup_{\mathbf{p} \in \mathbf{P}} \mathscr{L}(\mathbf{p}, \mathbf{q}). \tag{IS}$$

Discussion of the relation between these problems is deferred until Section 6. Instead in the remainder of this section a recursive formulation of (IS) is pursued.

## 4.2 Recursive Dual

The recursive dual approach decomposes (IS) into sub-problems linked by costates.

**Notation for the Recursive Dual.** Following the notation of Section 2, let $y = (y^k, y^v) \in \mathcal{Y} := \mathbb{R}^{n_K + n_V}$ denote a pair of costate variables on current laws of motion. Let $q = (m, y')$ be a *current dual choice* with $m \in \mathbb{R}_+^{n_H}$ a current $H$-constraint multiplier

[19] In most economic problems the aggregator over forward states incorporates discounting over time and a certainty equivalent operator over future states. For these problems it is convenient to normalize multipliers by the discount and by the probability distribution. We do so.

and $y' = (y^{k\prime}, y^{v\prime}) \in \mathcal{Y}^{n_S}$ a tuple of continuation costates for the next period's laws of motion. Denote the set of current dual choices by $\mathcal{Q} = \mathbb{R}_+^{n_H} \times \mathcal{Y}^{n_S}$. Let $p = (a, k, v')$ be a *current primal choice* with $a \in \mathcal{A}$ a current action, $k \in \mathcal{K}$ a *current* backwards-looking state and $v' \in \mathcal{V}^{n_S}$ a tuple of *continuation* forward-looking states, one for each future shock $s'$. Let $\mathcal{P} := \mathcal{A} \times \mathcal{K} \times \mathcal{V}^{n_S}$ denote the set of current primal choices.

Extracting the first three terms from $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle$, the Lagrangian in (IS) may be expanded as:

$$D_0^* = \inf_{\mathbf{q} \in \mathbf{Q}} \sup_{\mathbf{p} \in \mathbf{P}} F[s_0, v_0] - y_0^v \cdot \{v_0 - W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]]\} + y_0^k \cdot (\bar{k} - k_0) + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0), \tag{18}$$

with $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle$ the continuation of $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle$ from period 1 after the realization of $s_1$, i.e. $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle = m_0 \cdot z_0^h(\mathbf{p}) + y_0^k \cdot z_0^k(\mathbf{p}) + y_0^v \cdot z_0^v(\mathbf{p}) + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0)$.

Central to subsequent analysis is the following *continuation dual problem* which fixes the initial costates $y_0$ and removes the term $F[s_0, v_0] - y_0^v \cdot v + y^k \cdot \bar{k}$ from the objective in (18):

$$D^*(s_0, y_0) = \inf_{\mathbf{Q}(y_0)} \sup_{\mathbf{P}(v_0)} -y_0^k \cdot k_0 + y_0^v \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0). \tag{19}$$

In (19), $\mathbf{Q}(y_0)$ omits $y_0 = (y_0^k, y_0^v)$ from $\mathbf{Q}$ and $\mathbf{P}(v_0)$ omits $v_0$ from $\mathbf{P}$. We call $D^*$ the *dual value function*. In general, "inf-sup" operations on arbitrary functions can yield values of $-\infty$ or $+\infty$. We now show that, in fact, $D^*$ is typically real-valued. This has implications for the dual (co)state space which is described later in the section.

**Proposition 2.** *$D^*(s, y) < \infty$ for all $(s, y) \in \mathcal{S} \times \mathcal{Y}$. If, in addition, Assumption 1 holds, then $D^* : \mathcal{S} \times \mathcal{Y} \to \mathbb{R}$.*

*Proof.* See Appendix A. □

Collecting terms in (19) involving the initial current primal choice $p_0 = (k_0, a_0, v_1)$ gives the *current dual payoff* $J$:

$$J(s_0, y_0; q_0, p_0) = -y_0^k \cdot k_0 + y_0^v \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] - \delta \sum_{s_1 \in \mathcal{S}} y_1^v(s_1) \cdot v_1(s_1) \pi(s_1 | s_0) + \delta \sum_{s_1 \in \mathcal{S}} y_1^k(s_1) \pi(s_1 | s_0) \cdot W^k[k_0, s_0, a_0], \tag{20}$$

where the terms in the second line of (20) are extracted from $\delta \sum_{\mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0)$

in (19). Proposition 3 relates $D_0^*$, $D^*$ and $J$ and gives the key dynamic programming result for dual value functions.

**Proposition 3** (Value functions)**.** *The value $D_0^*$ satisfies:*

$$D_0^* = \inf_{y \in \mathcal{Y}} \sup_{v \in \mathcal{V}} F[s_0, v] - y^v \cdot v + y^k \cdot \bar{k} + D^*(s_0, y^k, y^v). \tag{21}$$

*Moreover, $D^*$ satisfies the functional equation:*

$$D^*(s, y) = \inf_{q \in \mathcal{Q}} \sup_{p \in \mathcal{P}} J(s, y; q, p) + \delta \sum_{s' \in \mathcal{S}} D^* \left(s', y'(s')\right) \pi(s'|s), \tag{22}$$

*where $y'(s') = (y^{k\prime}, y^{v\prime})(s')$ and*

$$\begin{aligned} J(s, y; q, p) = & - y^k \cdot k + y^v \cdot W^v[s, a, M^v[s, v']] + m \cdot H[k, s, a, v'] \\ & - \delta \sum_{s' \in \mathcal{S}} y^{v\prime}(s') \cdot v'(s') \pi(s'|s) + \delta \sum_{s' \in \mathcal{S}} y^{k\prime}(s') \pi(s'|s) \cdot W^k[k, s, a]. \end{aligned} \tag{23}$$

*Proof.* See Appendix A. □

The first stage problem (21) generates the initial costates; (22) then gives the dual Bellman equation. Moving from the dual problem (IS) to the recursive dual problems (21) and (22) involves interchanging an infimum operation over future dual variables with a supremum operation over current primal variables. The additive separability of the Lagrangian in these variables ensures that this interchange of operations does not alter the optimal value. See the proof of Proposition 3 for details.

*Remark* 3. Our recursive dual formulation relies entirely on dual costate variables $y_t$ to summarize the past. Note that this implies that the supremum component of the inf-sup operations in (22) is static, i.e. is of the form $J^*(s, y; q) := \sup_{\mathcal{P}} J(s, y; q, p)$, and is embedded into a rather standard looking Bellman equation:

$$D^*(s, y) = \inf_{q \in \mathcal{Q}} J^*(s, y; q) + \delta \sum_{s' \in \mathcal{S}} D^* \left(s', y'(s')\right) \pi(s'|s). \tag{24}$$

*Remark* 4. The primal "state" variables $k$ and $v$ continue to appear in the recursive dual problem. However, these variables are no longer passed between sub-problems and in this sense no longer function as state variables. They are used to “penalize” choices of continuation costates and, hence, align them with current costates and multipliers. Notice that in the recursive dual framework $k$ and $v$ are restricted to the exogenous set $\mathcal{K} \times \mathcal{V}^{n_S}$ and not the endogenous $\mathcal{X}^*$.

**Definition 1.** *Let* $\mathscr{F}$ *denote the set of functions* $D : \mathcal{S} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$*. Define the dual Bellman operator* $\mathscr{B} : \mathscr{F} \to \mathscr{F}$, $\forall (s,y) \in \mathcal{S} \times \mathcal{Y}$,

$$\mathscr{B}(D)(s,y) = \inf_{q \in \mathcal{Q}} \sup_{p \in \mathcal{P}} J(s,y;q,p) + \delta \sum_{s' \in \mathcal{S}} D\left(s', y'(s')\right) \pi(s'|s).$$

Theorem 1 recasts $D^*$ as a fixed point of $\mathscr{B}$. It is a corollary of Proposition 3.

**Theorem 1.** $D^* = \mathscr{B}(D^*)$.

The following lemma is a straightforward consequence of the definition of $\mathscr{B}$.

**Lemma 1** (Monotonicity)**.** *If* $D_1$, $D_2 \in \mathscr{F}$ *and for all* $(s,y) \in \mathcal{S} \times \mathcal{Y}$, $D_1(s,y) \geq D_2(s,y)$, *then for all* $(s,y) \in \mathcal{S} \times \mathcal{Y}$, $\mathscr{B}(D_1)(s,y) \geq \mathscr{B}(D_2)(s,y)$.

**State Spaces.** Theorem 1 provides a dynamic programming formulation of the dual of the decision-maker's problem. It locates this dynamic programming problem on a state space of dual costate variables. Proposition 2 below shows that in the dual setting (with bounded primal variables and a non-empty feasibility set), the dual value function $D^*$ is finite-valued on all of $\mathcal{S} \times \mathcal{Y}$ $(= \mathcal{S} \times \mathbb{R}^{n_K + n_V})$. Thus, the effective dual state space on which value functions are finite is immediately determined. In addition, as shown in Lemma 2 below, $D^*(s, \cdot)$ is positively homogenous of degree one. This has the advantage that once the dual value functions $D^*(s, \cdot)$ are determined on the unit sphere $\mathcal{C} = \{y \in \mathcal{Y} \mid \|y\| = 1\}$, then they are determined everywhere via positive scaling. From a practical point of view, the state space may be identified with $\mathcal{S} \times \mathcal{C}$. In some problems further refinement of the state space is natural and convenient. For example, in Section 2 and in many Paretian problems, it is natural to consider only non-negative initial values for Pareto weights. It is then usual for costates to remain non-negative and for the state space to be correspondingly restricted. Thus, in Section 2, the state space was set equal to $\mathcal{S} \times \mathcal{C}_+$ with $\mathcal{C}_+ = \{y \in \mathbb{R}_+^{n_K + n_V} \mid \|y\|_E = 1\}$. Approximation of value functions on $\mathcal{C}$ (or $\mathcal{C}_+$) is discussed in online Appendix A. However, the homogeneity of candidate value functions combined with the unboundedness of the current dual set $\mathcal{Q}$ disrupts the conventional approach to proving that $\mathscr{B}$ is a contraction. This is addressed in Section 5.

**Calculations** The dual Bellman operator involves an outer minimization over dual variables $q \in \mathcal{Q} = \mathbb{R}_+^{n_H} \times \mathcal{Y}^{n_S}$ and an inner maximization over primal variables $p \in \mathcal{P} = \mathcal{A} \times \mathcal{K} \times \mathcal{V}^{n_S}$ parameterized by $q$. These operations involve only simple constraints defined by product sets (i.e. $p \in \mathcal{P}$ and $q \in \mathcal{Q}$). Moreover, any additive separability in the functions $W^k$, $W^v$ and $H$ can be exploited to decompose the inner maximization into component maximizations that can be run in parallel. In some cases some or all of these component maximizations are sufficiently simple that they can be solved analytically. Thus, less and sometimes no inner step numerical maximization is needed. Additive separability may be across different pairs of actions and forward-looking states $(a^i, v'^{,i}(\cdot))$, $i \in \mathcal{I}$, (for example corresponding to the actions and continuation payoffs of different agents); it may be across actions $a$ and forward-looking states $v'(\cdot)$ (corresponding to time additive separability in payoffs) or across the components of $v'(s')$, $s' \in \mathcal{S}$ (corresponding to state additive separability in payoffs, i.e. expected utility). There is also often additive separability between the backward-looking state and other actions. The examples in Section 2 and online Appendix E and the quasilinear set up discussed below in Subsection 4.3 have different degrees of separability.

**Policies.** For arbitrary sets $C$ and $E$ and function $r : E \times C \to \mathbb{R} \cup \{-\infty, \infty\}$, define the argminmax operation:

$$\underset{C|E}{\operatorname{argminmax}}\, r(e,c) = \left\{ (c^*, e^*) \,\middle|\, c^* \in \underset{c \in C}{\operatorname{argmin}} \sup_{e \in E} r(e,c) \text{ and } e^* \in \underset{e \in E}{\operatorname{argmax}}\, r(e, c^*) \right\}.$$

The solution to the dual (IS) is given by:

$$\mathbf{\Lambda} := \underset{\mathbf{Q}|\mathbf{P}}{\operatorname{argminmax}}\, \mathscr{L}(\mathbf{p}, \mathbf{q}).$$

On the other hand, the solution to the recursive dual is described by a policy set:

$$G_0 = \underset{\mathcal{Y}|\mathcal{V}}{\operatorname{argminmax}}\, F[s_0, v] - y^v \cdot v + y^k \cdot \bar{k} + D^*(s_0, y^k, y^v)$$

and a policy correspondence

$$G(s,y) = \underset{\mathcal{Q}|\mathcal{P}}{\operatorname{argminmax}}\ J(s,y;m,y',p) + \delta \sum_{s' \in \mathcal{S}} D^* \left(s', y'(s')\right) \pi(s'|s).$$

The next proposition relates policies from the dual and the recursive dual.

**Proposition 4** (Policies). $(\mathbf{q}^*, \mathbf{p}^*) \in \mathbf{\Lambda}$ *only if* $(y_0^*, v_0^*) \in G_0$ *and for each* $t \geq 1$, $s^t \in \mathcal{S}^t$, $(m_t^*(s^t), y_{t+1}^*(s^t), p_t^*(s^t)) \in G(s_t, y_t^*(s^t))$. *Conversely,* $(\mathbf{q}^*, \mathbf{p}^*) \in \mathbf{\Lambda}$ *if* $(y_0^*, v_0^*) \in G_0$, *for each* $t \geq 1$, $s^t \in \mathcal{S}^t$, $(m_t^*(s^t), y_{t+1}^*(s^t), p_t^*(s^t)) \in G(s_t, y_t^*(s^t))$ *and:*

$$\liminf_{T \to \infty} \delta^{T+1} \sum_{\mathcal{S}^{T+1}} D^*(s_{T+1}, y_{T+1}^*(s^{T+1})) \pi^{T+1}(s^{T+1}|s_0) \geq 0. \tag{T}$$

*Proof.* Appendix A. □

### 4.3 The Quasilinear Case

Many problems have constraint functions that are quasilinear in $k$ and $v$, i.e.

$$W^k[k,s,a] = B^k(s)k - c(s,a),$$
$$W^v[s,a,M^v[s,v']] = (1-\delta)u(s,a) + \delta \sum_{s'\in\mathcal{S}} B^v(s,s')v'(s')\pi(s'|s),$$
and
$$H[k,s,a,v'] = N^k(s)k + h(s,a) + \delta \sum_{s'\in\mathcal{S}} N^v(s,s')v'(s')\pi(s'|s),$$

for $\delta \in [0,1)$, matrices $B^k(s)$, $N^k(s)$, $B^v(s,s')$ and $N^v(s,s')$, and functions $c : \mathcal{S}\times\mathcal{A} \to \mathbb{R}^{n_K}$, $u : \mathcal{S}\times\mathcal{A} \to \mathbb{R}^{n_I}$ and $h : \mathcal{S}\times\mathcal{A} \to \mathbb{R}^{n_H}$. Examples include the limited commitment problem discussed in Section 2 specialized to the case of expected utility preferences (i.e., $\rho = \sigma$) and all of the problems considered in Messner et al. (2012) (each of which abstracts from backward-looking state variables). A Lagrangian of the form $\mathscr{L}$ (inclusive of laws of motion for states) may be associated with such problems and their recursive duals derived as in Proposition 3. However, directly exploiting quasilinear structure prior to the formulation of the Lagrangian leads to simplification. In particular, when the matrices $B^r, r \in \{k,v\}$, satisfy appropriate bounding conditions, the primal states $k_t$ and $v_t$ can be substituted out of the problem using their laws of motion. This leads to a modified Lagrangian and a simplified dual Bellman operator. In models without backward-looking primal states this operator is:

$$\mathscr{B}(D)(s,y) = \inf_{m\in\mathbb{R}^H_+} \sup_{a\in\mathcal{A}} J(s,y;m,a) + \delta \sum_{s'\in\mathcal{S}} D(s',\phi(s,y,m,s'))\pi(s'|s),$$

with $\phi(s,y,m,s') = y\cdot B^v(s,s') + m\cdot N^v(s,s')$ and $J(s,y;m,a) = y\cdot u(s,a) + m\cdot h(s,a)$. Note that here continuation costates (for forward-looking primal states) are determined as a function of past costates and multipliers on current states. The dual value function $D^*$ is a fixed point of this (modified) dual Bellman (by arguments essentially identical to those in Proposition 3). A detailed treatment of the quasilinear case is given in online Appendix D.

# 5 Contraction

This section establishes sufficient conditions for $\mathscr{B}$ to be contractive on an appropriate space of functions. The combination of an unbounded dual value function and an unbounded dual constraint set is an obstacle to conventional approaches to proving contractivity.[20] Following Marinacci and Montrucchio (2010) and especially Rinćon-Zapatero and Rodríguez-Palmero (2003), we pursue a different approach. The idea is to restrict attention to spaces of scaleable real-valued (in practice positive real-valued) functions such that for each function pair $g_1$ and $g_2$ in the space there is a scaling factor $b \in \mathbb{R}_{++}$ satisfying $bg_1 \geq g_2$ and $bg_2 \geq g_1$. The distance between a scaleable function pair $(g_1, g_2)$ is identified with the log of the smallest scaling factor. Scaleability of a set of candidate value functions is ensured via a renormalization involving bounding value functions. Since the optimal dual value function is sub-linear (i.e. convex and positively homogenous) in costates, attention may be restricted to candidate value functions that are sub-linear. Consequently, it is sufficient to have scaleability on the unit sphere in the costate space (i.e. on a compact set) and to define distance measures accordingly. The interval of sub-linear functions between the bounding value functions is a complete metric space. If $\mathscr{B}$ is a self-map on this interval, then contractivity follows from monotonicity and concavity of $\mathscr{B}$, (plus the properties of the bounding value functions and the homogeneity of candidate value functions). Thus, Blackwell's Theorem is avoided.

The formal definition of sub-linearity follows.

**Definition 2.** *A function $D : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ is sub-linear if (i) $D(\cdot)$ is convex and (ii) $D(\cdot)$ is positively homogeneous of degree 1. A function $D : \mathcal{S} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ is sub-linear if each $D(s, \cdot)$ is sub-linear.*

Lemma 2 indicates the importance of the previous definition for our setting.

**Lemma 2.** *(i) $D^*$ is sub-linear. (ii) If $D : \mathcal{S} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ is sub-linear, then $\mathscr{B}(D)$ is sub-linear.*

*Proof.* See Appendix B. □

The key assumption ensuring contractivity is the following. As before, let $\mathcal{C} = \{y \in \mathcal{Y} \mid \|y\|_E = 1\}$ denote the unit sphere in $\mathbb{R}^{n_K + n_V}$.

**Assumption 3.** *There is a triple of bounding functions $\underline{\underline{D}} : \mathcal{S} \times \mathcal{Y} \to \mathbb{R}$, $\underline{D} : \mathcal{S} \times \mathcal{Y} \to \mathbb{R}$ and $\overline{D} : \mathcal{S} \times \mathcal{Y} \to \mathbb{R}$ and a number $\varepsilon > 0$ such that for all $s$, $\underline{D}(s, \cdot)$ is continuous and positively homogeneous of degree 1, $\overline{D}(s, \cdot)$ is continuous, and for all $(s, y) \in \mathcal{S} \times \mathcal{C}$, (i) $\underline{\underline{D}}(s, y) + \varepsilon \leq \underline{D}(s, y) \leq \overline{D}(s, y)$, (ii) $\underline{D}(s, y) \leq \mathscr{B}(\underline{D})(s, y)$ and $\mathscr{B}(\overline{D})(s, y) \leq \overline{D}(s, y)$ and (iii) $\underline{\underline{D}}(s, y) + \varepsilon < \mathscr{B}(\underline{\underline{D}})(s, y)$.*

[20]When the optimal value function is unbounded and the constraint correspondence compact-valued it is often possible to prove contractivity on a space of weight norm bounded functions. In the dual setting, this approach is disrupted by the unboundedness of the constraint correspondence (for multipliers and costates).

We discuss the selection of bounding functions in the context of specific examples in online-Appendices E and B.2. However, if $\underline{\underline{D}}$ satisfies Assumption 3 (iii) and $\underline{\underline{D}} \leq D^* \leq \overline{D}$, then, from the monotonicity of $\mathscr{B}$ and Theorem 1, for all $(s,y) \in \mathcal{S} \times \mathcal{C}$,

$$\underline{\underline{D}}(s,y) + \varepsilon < \mathscr{B}(\underline{\underline{D}})(s,y) \leq \mathscr{B}(\mathscr{B}(\underline{\underline{D}}))(s,y) \leq \mathscr{B}(\mathscr{B}(D^*))(s,y) = D^*(s,y) \leq \overline{D}(s,y),$$

and $\underline{D}$ may be set equal to $\mathscr{B}(\underline{\underline{D}})$. Given a triple of functions $\underline{\underline{D}}$, $\underline{D}$ and $\overline{D}$ satisfying Assumption 3, let:

$$\mathscr{G} = \{D : \mathcal{S} \times \mathcal{Y} \to \mathbb{R} | D \text{ is sub-linear and } \underline{D} \leq D \leq \overline{D}\}.$$

Define the "Thompson-like" metric $d : \mathscr{G} \times \mathscr{G} \to \mathbb{R}_+$ according to for each $D_1, D_2 \in \mathscr{G}$:

$$\begin{aligned} d(D_1, D_2) &= \sup_{\mathcal{S}\times\mathcal{C}} \left| \ln\left(\frac{D_1(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) - \ln\left(\frac{D_2(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) \right| \\ &\leq \sup_{\mathcal{S}\times\mathcal{C}} \ln\left(\frac{\overline{D}(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) \leq \sup_{\mathcal{S}\times\mathcal{C}} \ln\left(\frac{\overline{D}(s,y) - \underline{\underline{D}}(s,y)}{\varepsilon}\right) < \infty, \end{aligned}$$

where the penultimate inequality and finiteness stem from Assumption 3.[21] That $(\mathscr{G}, d)$ is complete metric space is shown next.

**Lemma 3.** *$(\mathscr{G}, d)$ is a complete metric space.*

*Proof.* See Appendix B. □

Proposition 5 verifies that $\mathscr{B}$ is a contraction on $\mathscr{G}$. It relies on the concavity (and monotonicity) of $\mathscr{B}$ rather than any discounting-type conditions. This makes it well suited to the present setting where concavity of $\mathscr{B}$ is easy to show, but discounting (with respect to a suitable bounding norm) is not.

**Proposition 5.** *Let Assumption 3 hold. There is a $\rho \in [0,1)$ such that for all $D_1, D_2 \in \mathscr{G}$, $d(\mathscr{B}(D_1), \mathscr{B}(D_2)) \leq \rho d(D_1, D_2)$, i.e. $\mathscr{B}$ is a contraction on $(\mathscr{G}, d)$ with modulus of contraction $\rho$.*

*Proof.* See Appendix B. □

Application of the contraction mapping theorem implies that $\mathscr{B}$ has a unique fixed point in $\mathscr{G}$.

**Theorem 2.** *Let Assumption 3 hold and assume that $\underline{D} \leq D^* \leq \overline{D}$. $D^*$ is the unique fixed point of $\mathscr{B}$ in $\mathscr{G}$. Also, there is a $\rho \in [0,1)$ such that for any $D_0 \in \mathscr{G}$, $\mathscr{B}^n(D_0) \xrightarrow{d} D$ with $d(\mathscr{B}^n(D_0), D^*) \leq \rho^n d(D_0, D^*) \leq \rho^n d(\overline{D}, \underline{D})$.*

[21] In particular, $\underline{D}(s,y) - \underline{\underline{D}}(s,y) \geq \varepsilon > 0$ for all $(s,y)$.

*Proof.* From Lemma 2, $D^*$ is sub-linear and by assumption it is bounded below by $\underline{D}$ and above by $\overline{D}$. Thus, $D^* \in \mathscr{G} \neq \emptyset$. Also by Lemma 2, if $D \in \mathscr{G}$, then $\mathscr{B}(D)$ is sub-linear and by the montonicity of $\mathscr{B}$ and Assumption 3 it is bounded below by $\underline{D}$ and above by $\overline{D}$. Thus, $\mathscr{B} : \mathscr{G} \to \mathscr{G}$. By Proposition 5, it is contractive on $\mathscr{G}$. The desired results then stem from the contraction mapping theorem. □

*Remark* 5. An immediate consequence of the previous (uniqueness) result is the following 'Verification Theorem': *Under the assumptions of Theorem 2, if $\hat{D}$ is a fixed point of $\mathscr{B}$ in the space of sub-linear functions bounded below by $\underline{D}$ and above by $\overline{D}$, then $\hat{D} = D^*$.*

*Remark* 6. For application of Theorem 2, it is sufficient to know (i) that bounding functions satisfying Assumption 3 and $\underline{D} \leq D^* \leq \overline{D}$ exist and (ii) that a given sub-linear function $D_0$ lies between $\underline{D}$ and $\overline{D}$ and can thus serve as an initial condition in a value iteration. Explicit calculation of the bounding functions is unnecessary. This contrasts with results relying on monotone (not contractive) operators, which require an upper or lower bound to the true value function as an initial condition. In addition, as always, the contraction result allows us to calculate error bounds and rates of convergence and is, thus, an improvement on results relying only on monotone iterations and pointwise convergence of iterates.

# 6 Relating Primal and Dual

The preceding sections established that the recursive dual supplies the optimal value and solutions for the dual problem (IS). Consequently, *if* the dual problem supplies the optimal value and an optimal plan for the original primal problem (P), then the recursive dual does as well and the primal may be solved via dual value iteration. This section discusses conditions for the dual and primal problems to have common values and policies. It begins by recalling the classical weak duality inequality which requires no additional assumptions. It then gives conditions in terms of the existence and properties of saddle points of the Lagrangian $\mathscr{L}$. Next it gives sufficient conditions for saddle existence (and, hence, equality of optimal values and necessity of primal solutions for dual problems) in terms of primitives. Finally, a numerical procedure is described for checking whether a primal plan obtained via the recursive dual approach solves the primal problem.

## 6.1 Saddles and Recursive Dual Policies

Without further restriction, classical weak duality implies that the optimal dual value bounds the optimal primal value: $D_0^* \geq P_0^*$. Thus, with no further assumptions the recursive dual gives welfare bounds for optimal policies or policy improvements.

A well known sufficient condition for equality of optimal values, albeit not on primitives, is that the Lagrangian admits a saddle point, i.e. there is a pair $(\mathbf{p}^*, \mathbf{q}^*) \in \mathbf{P} \times \mathbf{Q}$ such that for all $\mathbf{p} \in \mathbf{P}$ and $\mathbf{q} \in \mathbf{Q}$, $\mathscr{L}(\mathbf{p}, \mathbf{q}^*) \leq \mathscr{L}(\mathbf{p}^*, \mathbf{q}^*) \leq \mathscr{L}(\mathbf{p}^*, \mathbf{q})$. Saddle existence also ensures that if $\mathbf{p}^*$ solves (P) and $\mathbf{q}^*$ solves the dual (i.e. attains the minimum in (IS)), then $(\mathbf{q}^*, \mathbf{p}^*) \in \Lambda$. The following proposition summarizes the situation.

**Definition 3.** *A primal plan $\hat{\mathbf{p}}$ is consistent with the recursive dual policy if there is a $\hat{\mathbf{q}}$ such that $(\hat{y}_0, \hat{v}_0) \in G_0$ and for $t \geq 1$, $s^t \in \mathcal{S}^t$, $(\hat{m}_t(s^t), \hat{y}_{t+1}(s^t), \hat{p}_t(s^t)) \in G(s_t, \hat{y}_t(s^t))$.*

**Proposition 6.** *Assume that $\mathscr{L}$ admits a saddle point. Then the optimal dual and primal values are equal: $D_0^* = P_0^*$. In addition, if $\mathbf{p}^*$ solves (P), then it is consistent with the recursive dual policy.*

*Proof.* See Appendix C. □

Proposition 6 only requires that $\mathscr{L}$ admits a saddle point. It does *not* require that the Lagrangian associated with every $(s^t, y_t(s^t))$-continuation problem has a saddle point, as is the case in Marcet and Marimon (2011). Proving, or numerically checking, the existence of a saddle point for $\mathscr{L}$, while non-trivial, is less demanding than doing so for all possible histories.

From Proposition 6 if a saddle exists, then the dual recursive problem gives the optimal primal value and optimal primal plans are consistent with the recursive dual policy. However, other (suboptimal) plans may also be consistent with the recursive dual policy. Stronger conditions are needed to ensure that the "finite penalization" implicit in the dual problem is "sharp enough" to pin down only primal solutions.[22] A simple way to obtain a sufficiency result for policies is to require that the set of plans consistent with the recursive dual policy is unique.[23]

**Proposition 7.** *Assume that $\mathscr{L}$ has a saddle point. Let $\hat{\mathbf{P}}$ be the set of primal plans that are consistent with the recursive dual policy. If $\hat{\mathbf{p}}$ is the unique element of $\hat{\mathbf{P}}$, then $\hat{\mathbf{p}}$ is the unique solution of (P).*

*Proof.* See Appendix C. □

[22]This issue was emphasized by Messner and Pavoni (2015). Cole and Kubler (2012) describe a procedure for augmenting a recursive saddle point problem with lotteries over the extreme points of flat regions of the continuation value function that permits recovery of an optimal primal solution. It remains to integrate their approach into our formulation.

[23]Note that uniqueness of multipliers is not needed.

## 6.2 Concave Problems

For problems with only inequality constraints, a well known sufficient condition for equality of optimal values and a minimizing dual multiplier (so called "strong duality") is that (i) the objective and constraint functions are defined on a convex domain and are concave and (ii) the evaluation of the constraints at some primal choice lies in the interior of the constraint space's closed non-negative cone (a Slater condition). If, in addition, a solution to the optimization exists then it and the minimizing multiplier constitute a saddle point.

There are two difficulties in applying these results to our setting. First, (P) incorporates equality constraints describing the laws of motion for states. Thus, standard conditions for saddle existence are not directly applicable. We deal with this below by imposing monotonicity assumptions on nonlinear laws of motion for states and relaxing the corresponding constraints. In addition, we substitute forward-looking state variables with quasilinear laws of motion from the problem (along with their law of motion constraints). The classical sufficient conditions given above are then applicable. A second difficulty stems from the fact that these conditions ensure multiplier existence in the dual of the constraint space. In our setting this is $\ell^{\infty,\star}$, the set of all continuous linear functionals on $\ell^{\infty}$, not the more convenient space of summable sequences $\ell^1$ on which $\mathscr{L}$ is defined.[24] However, as we show below our constraint structure ensures that if a minimizing multiplier exists in $\ell^{\infty,\star}$, then one exists in $\ell^1$ as well.

We partition the elements of the forward-looking state $v$ into two groups: $v = (v^c, v^l)$ and assume that $\mathcal{V} = \mathcal{V}^c \times \mathcal{V}^l$, with $v^r \in \mathcal{V}^r \subset \mathbb{R}^{n_r}$, $r = c, l$, and each $\mathcal{V}^r$ bounded. In what follows, $v^c$ corresponds to a forward-looking state with a (fully) nonlinear law of motion and $v^l$ corresponds to a forward-looking state with a quasilinear law of motion. As indicated above, these different types of forward-looking states are handled differently in the analysis below. Denote the aggregators describing laws of motion for them by $W^r$ and $M^r$, for $r \in \{c, l\}$. Corresponding to the earlier requirement that $W^v[s, a, M^v[s, \cdot]] : \mathcal{V}^{n_S} \to \mathcal{V}$, assume $W^r[s, a, M^r[s, \cdot]] : (\mathcal{V}^r)^{n_S} \to \mathcal{V}^r$, $r = c, l$. Finally, for $r = c, l$, let $v^{r\prime} = \{v^{r\prime}(s)\}_{s \in \mathcal{S}} \in (\mathcal{V}^r)^{n_S}$. The following monotonicity, concavity and quasilinearity assumptions are imposed upon the problem.

**Assumption 4** (Monotonicity). *(i) $F[s_0, \cdot]$ is increasing in $v^c$. (ii) $\forall (s, a) \in \mathcal{S} \times \mathcal{A}$, $W^c[s, a, \cdot]$, $M^c[s, \cdot]$ and $W^k[\cdot, s, a]$ are increasing. In addition, if $W^k[k, s, a] \in \mathcal{K}$, then for all $k' \in \mathcal{K}$ with $k' > k$, $W^k[k', s, a] \in \mathcal{K}$. (iii) $\forall (s, a, v^{l\prime}) \in \mathcal{S} \times \mathcal{A} \times (\mathcal{V}^l)^{n_S}$, $H[\cdot, s, a, \cdot, v^{l\prime}]$ is increasing in $(k, v^{c\prime})$.*

[24] Strictly, $\mathscr{L}$ is defined on a set of discounted and weighted multipliers. However, the unnormalized multipliers are in $\ell^1$ and $\mathscr{L}$ can be redefined to have $\ell^1 \times \mathbf{P}$ as its domain.

**Assumption 5** (Concavity and Quasilinearity). *(i) $F[s_0,\cdot]$ is concave in all arguments and linear in $v_0^l \in \mathbb{R}^{n_l}$. (ii) $\forall s \in \mathcal{S}$, $H[\cdot,s,\cdot]$ is concave in all arguments and linear in $v^{l\prime}$. (iii) $\forall s \in \mathcal{S}$, $W^c[s,\cdot], M^c[s,\cdot]$ and $W^k[s,\cdot]$ are concave in all arguments. (iv) $\forall(s,a,v^{l\prime})$, $W^l[s,a,M^l[s,v^{l\prime}]] = (1-\delta)u^l(s,a) + \delta\sum_{s'\in\mathcal{S}} B^l(s,s')v^{l\prime}(s')\pi(s'|s)$, with $\delta \in [0,1)$, $u^l : \mathcal{S}\times\mathcal{A} \to \mathcal{V}^l$ a concave function, $\mathcal{V}^l \subset \mathbb{R}^{n_l}$ a bounded set and each $B^l(s,s')$ a diagonal matrix of size $n_l$ with all elements less than or equal to 1 in absolute value.*

We also impose a Slater condition. Let $\mathbf{z}^j(\mathbf{p}) = \{z_t^j(\mathbf{p})\}_{t=0}^{\infty}$, $j = k,h,l,c$, denote the constraint process terms, with for $j = l,c$, $z_t^j(\mathbf{p})(s^t) = -v_t^j(s^t) + W^j[s_t, a_t(s^t), M^j[s_t, v_{t+1}^j(s^t,\cdot)]]$.

**Assumption 6** (Slater). *There is a $\hat{\mathbf{p}} \in \mathbf{P}$ such that $\mathbf{z}^l(\hat{\mathbf{p}}) = 0$ and $\inf_{t,s^t} z_t^j(\hat{\mathbf{p}})(s^t) > 0$ for $j = k,c,h$.*

Proposition 8 establishes saddle point existence given a solution to the original problem (P) and Assumptions 4 to 6. Note that the multiplier component of the saddle point is in $\ell^1$ rather than the larger and less convenient set $\ell^{\infty,\star}$.

**Proposition 8.** *Let Assumptions 2 and 4 to 6 hold and let $\bar{k} \in \mathcal{K}^*(s_0)$. Then $\mathscr{L}$ admits a saddle point. If $\mathbf{p}^*$ is the primal component of a saddle point and $\mathbf{q}^* \in \ell^1$ solves (IS), then $\mathscr{L}(\mathbf{p}^*,\mathbf{q}^*) = D_0^* = P_0^*$ and the elements of $(\mathbf{p}^*,\mathbf{q}^*)$ satisfy $(y_0^{k*}, y_0^{v*}, v_0^*) \in G_0$ and for all $t \geq 1$ and $s^t \in \mathcal{S}^t$, $(m_t^*(s^t), y_{t+1}^*(s^t), p_t^*(s^t)) \in G(s_t, y_t^*(s^t))$.*

*Proof.* See Appendix C. □

*Remark* 7. Proposition 8 applies to problems with a mixture of non-linear forward and backward-looking states and quasilinear forward-looking states. Assumptions 4 and 5 may be reduced for simpler problems. For example, if $\rho \neq \sigma$, then the limited commitment problem in Section 2 involves only nonlinear forward-looking state variables and those parts of Assumptions 4 and 5 relating to backward-looking or quasilinear forward-looking constraints may be dropped. If $\rho = \sigma$, then preferences are specialized to the expected utility form and, hence, the limited commitment problem involves only quasilinear forward-looking constraints. Then Assumption 4 and parts of Assumption 5 are not needed.

The role of Assumption 4 is to permit a relaxation of the nonlinear law of motion constraints. Assumption 4 (i) and (ii) are satisfied in all settings in which the forward-looking states are payoffs and the backward-looking states are capital (and output is increasing in capital). Hence, they are quite natural. Assumption 4 (iii) is more restrictive. An alternative is to supplement Assumption 4(ii) with the requirement that for all $(k,s)$, $W^k[k,s,\cdot]$ is decreasing in $a$ and $W^c[s,\cdot,m]$ is increasing in $a$, to modify Assumption 4 (iii) to require that $H[k,s,\cdot,\cdot,\cdot]$ is increasing in $a$, $v^{c\prime}$ and $v^{l\prime}$, and to strengthen Assumption 5 by assuming that $u^l(s, \cdot)$ is increasing in $a$ and the matrix $B^l$ is positive. This configuration of assumptions is appropriate for models of limited commitment with capital accumulation. In such models additional capital both increases production opportunities and the default values of uncommitted agents. The net effect is to reduce (some elements of) $H$ and to make it potentially undesirable to take additional capital into a subsequent period. However, if each agent's utility is increasing in her consumption, then it is never desirable to discard output and select an allocation in which the law of motion for capital is relaxed, i.e. an allocation in which: $k_{t+1}(s^t) < W^k[k_t(s^{t-1}), s_t, a_t(s^t)] := f(k_t(s^{t-1})) - \sum_{i \in \mathcal{I}} a^i(s_t)$, with $f$ the production function. Instead, consumption will be raised, so relaxing both current and past no default constraints. Thus, in this setting relaxation of the laws of motion for capital and (if agents have non-standard preferences) agent utility does not raise the optimal payoff or expand the solution set. Proposition 8 is applicable to this setting after modification of Assumptions 4 and 5 along the lines described.[25]

## 6.3 Ex Post Check

The following proposition describes a numerical procedure for checking whether a primal plan obtained via the recursive dual approach solves the primal problem. The procedure does not require any concavity assumptions on primitives. We call a pair $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ *a candidate plan* if it is obtained from the recursive dual policy correspondence: $(\hat{q}_0^k, \hat{q}_0^v, \hat{v}_0) \in G_0$, and $\forall t \geq 1$ and $s^t \in \mathcal{S}^t$, $(\hat{q}_t(s^t), \hat{p}_t(s^t)) \in G(s_t, \hat{y}_t(s^t))$.

**Proposition 9.** *If $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ is a candidate plan satisfying (i) Condition (T), (ii) $-\hat{y}_0^v \cdot \hat{v}_0 + \hat{y}_0^k \cdot \bar{k} + D^*(s_0, \hat{y}_0^k, \hat{y}_0^v) \leq 0$ and (iii) feasibility of $\hat{\mathbf{p}}$ for (P), then $\hat{\mathbf{p}}$ is optimal for (P). In addition, $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ is a saddle point for $\mathscr{L}$.*

*Proof.* See Appendix C. □

Despite its simplicity, Proposition 9 is the basis of a useful ex post check of primal optimality. Suppose the recursive dual problem has been solved and a fixed point $\hat{D}$ of the operator $\mathscr{B}$ obtained. If Assumption 3 and the condition $\underline{D} \leq D^* \leq \overline{D}$ are satisfied and $\hat{D}$ lies between the bounding functions $\underline{D}$ and $\overline{D}$, then by Theorem 2, $\hat{D} = D^*$. If the solution of the recursive dual delivers a candidate plan $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, then Proposition 9 provides sufficient conditions for $\hat{\mathbf{p}}$ to be a solution to (P) and for the existence of a saddle point of the associated Lagrangian. In practice, the value function $D^*$ and the policy correspondence $G$ must be approximated via, say, a numerical

[25] Specifically, Proposition 8 is then applicable to the contracting problem in Cooley et al. (2004). This is a limited commitment problem with default in which the production function is strictly concave, but the outside option affine in capital (e.g. the entrepreneur can sell off capital after default).

implementation of the value iteration described in Theorem 2, and the conditions in Proposition 9 checked numerically to within some acceptable level of tolerance.

## 7 Conclusion

In many settings the (primal) state space of a dynamic economic problem is defined implicitly and must be recovered as part of the solution to the problem. This complicates the application of recursive methods. Associated dual problems have recursive formulations in which costates are used to keep track of histories of past or feasible future actions. The dual (co-)state space is immediately determined as $\mathbb{R}^N$ (or, perhaps, $\mathbb{R}^N_+$). Despite the unboundedness of the dual value functions and the lack of a bounded constraint correspondence, contractivity of the dual Bellman operator (with respect to the modified Thompson metric) may be established if suitable bounding functions are available. In many problems they are.

## A Proofs for Section 3 and Section 4

*Proof of Proposition 1.* Endow $\mathbf{P}$ with the product topology. Given $s_0$, for all $k \in \mathcal{K}$ let: $\mathbf{C}(s_0, k) := \{\mathbf{p} \in \mathbf{P} \mid k_0 = k, v_0 \in \mathcal{V}, \text{ and } \forall t \geq 1, \ldots, s^t \in \mathcal{S}^t \text{ (15), (16), and (17) hold }\}$.

**Lemma 4.** *Given Assumption 2, $\forall k \in \mathcal{K}$, $\mathbf{C}(s_0, k)$ is compact in the product topology.*

*Proof.* Under Assumption 2, $\mathcal{A} \times \mathcal{K} \times \mathcal{V}^{n_S}$ is compact. Hence, by the Tychonoff theorem, $\mathcal{V} \times \mathcal{A} \times (\mathcal{A} \times \mathcal{K} \times \mathcal{V}^{n_S})^\infty$ is compact in the product topology. Since $\mathbf{C}(s_0, k) \subset \mathcal{V} \times \mathcal{A} \times (\mathcal{A} \times \mathcal{K} \times \mathcal{V}^{n_S})^\infty$ and closed subsets of compact spaces are compact, it suffices to show that $\mathbf{C}(s_0, k)$ is closed. Let $\{\mathbf{p}^n\}_{n=1}^\infty$ denote a sequence with each $\mathbf{p}^n \in \mathbf{C}(s_0, k)$ and such that $\mathbf{p}^n$ converges in the product topology to $\mathbf{p}$. Then for all $t$ and $s^t$ the corresponding components of $\mathbf{p}^n$ converge to those of $\mathbf{p}$. The compactness of $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{V}$, continuity of $W^k$ and $W^v$ and upper semicontinuity of $H$ then ensure that $\mathbf{p}$ is in $\mathbf{P}$ and satisfies (15), (16), and (17). Thus, $\mathbf{p} \in \mathbf{C}(s_0, k)$. □

The planner's problem is: $P_0^* = \sup_{\mathbf{p} \in \mathbf{C}(s_0, \bar{k})} F(s_0, v_0)$. By assumption $F(s_0, \cdot)$ is upper semicontinuous. Since $\bar{k} \in \mathcal{K}^*(s_0)$ and by Assumption 1, $\mathbf{C}(s_0, \bar{k}) \neq \emptyset$. By Lemma 4, $\mathbf{C}(s_0, \bar{k})$ is compact. The desired result follows from Weierstrass' theorem. □

*Proof of Proposition 2.* To show $D^* < \infty$, choose an arbitrary $(s, y) \in \mathcal{S} \times \mathcal{Y}$. Then:

$$\begin{aligned} D^*(s, y) = \inf_{\mathbf{Q}(y)} \sup_{\mathbf{P}(v_0)} & -y^k \cdot k_0 + y^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] \\ & + m_0 \cdot H[k_0, s, a_0, v_1] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s) \\ \leq \sup_{p \in \mathcal{P}} & -y^k \cdot k_0 + y^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] < \infty, \end{aligned} \tag{25}$$

where the first inequality uses $0 \in \mathbf{Q}(y)$ (and, hence, its feasibility for the infimum) and the second inequality uses the boundedness of $\mathcal{K}$ and $\mathcal{V}$ and $W^v[s, \cdot, M^v[s, \cdot]] : \mathcal{A} \times \mathcal{V}^{n_S} \to \mathcal{V}$. Since $(s, y)$ was arbitrary, $D^* < \infty$.

To show that $D^* > -\infty$, note that Assumption 1 guarantees that for all $s$ there is a $\hat{k}_0 \in \mathcal{K}^*(s)$ and, so, a feasible primal plan $\hat{\mathbf{p}} \in \mathbf{P}$ starting at $(s, \hat{k}_0)$. Hence, for arbitrary $\mathbf{q} \in \mathbf{Q}(y)$,

$$\sup_{\mathbf{P}} \; -y^k \cdot k_0 + y^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] + m \cdot H[k_0, s, a_0, v_1(\cdot)] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1|s)$$
$$\geq -y^k \cdot \hat{k}_0 + y^v \cdot W^v[s, \hat{a}_0, M^v[s, \hat{v}_1(\cdot)]] > -\infty,$$

where the first inequality uses the fact that $\hat{\mathbf{p}}$ is feasible, but not necessarily maximal at $\mathbf{q}$. Since $\mathbf{q}$ was arbitrary, $-y^k \cdot \hat{k}_0 + y^v \cdot W^v[s, \hat{a}_0, M^v[s, \hat{v}_1(\cdot)]]$ is a finite lower bound for $\sup_{\mathbf{P}} \; -y^k \cdot k_0 + y^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] + m \cdot H[k_0, s, a_0, v_1(\cdot)] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1|s)$ and, so, for $D^*(s, y)$. Since $(s, y)$ was arbitrary, $D^* > -\infty$. □

*Proof of Proposition 3.* We have:

$$\begin{aligned} D_0^* &= \inf_{\mathbf{Q}} \sup_{\mathbf{P}} \mathscr{L}(\mathbf{p}, \mathbf{q}) = \inf_{\mathcal{Y}} \inf_{\mathbf{Q}(y_0)} \sup_{\mathcal{V}} \sup_{\mathbf{P}(v_0)} \; F[s_0, v_0] + y_0^k \cdot (\bar{k} - k_0) \\ &\quad + y_0^v \cdot (W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] - v_0) + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1|s_0) \\ &= \inf_{\mathcal{Y}} \sup_{\mathcal{V}} \; y_0^k \cdot \bar{k} + F[s_0, v_0] - y_0^v \cdot v_0 + \inf_{\mathbf{Q}(y_0)} \sup_{\mathbf{P}(v_0)} \; -y_0^k \cdot k_0 + y_0^v \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] \\ &\quad + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1|s_0), \end{aligned} \tag{26}$$

where the second equality uses the definitions of $\mathscr{L}$, $\mathcal{Y}$, $\mathbf{Q}(\cdot)$, $\mathcal{V}$ and $\mathbf{P}(\cdot)$ and the decomposition property of inf and sup operations over product sets. The third equality uses the additive separability of $\mathscr{L}$. Once $y_0^k$ and $y_0^v$ are chosen, the Lagrangian is additively separable in $v_0$ and all other variables, including the remaining dual variables. Hence, given $(y_0^k, y_0^v)$, the inf over these remaining dual variables and the sup over $v_0$ do not affect each other and may be interchanged. Combining (26) with the definition of $D^*$ gives (21) in the proposition. For each $(s, y) = (s, y^k, y^v) \in \mathcal{S} \times \mathcal{Y}$,

$$\begin{aligned} D^*(s, y) &= \inf_{\mathbf{Q}(y)} \sup_{\mathbf{P}(v_0)} \; -y^k \cdot k_0 + y^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] \\ &\qquad\qquad + m_0 \cdot H[k_0, s, a_0, v_1] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1|s_0) \\ &= \inf_{\mathbf{Q}(y)} \sup_{\mathbf{P}(v_0)} \; -y^k \cdot k_0 + y^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] + m_0 \cdot H[k_0, s, a_0, v_1(\cdot)] \\ &\qquad - \delta \sum_{s_1 \in \mathcal{S}} y_1^v(s_1) \cdot v_1(s_1) \pi(s_1|s) + \delta \sum_{s_1 \in \mathcal{S}} y_1^k(s_1) \cdot \{W^k[k_0, s, a_0] - k_1(s_1)\} \pi(s_1|s) \\ &\qquad + \delta \sum_{s_1 \in \mathcal{S}} \Big\{ y_1^v(s_1) \cdot W^v[s_1, a_1(s_1), M^v[s_1, v_2(s_1, \cdot)]] \\ &\qquad\quad + m_1(s_1) \cdot H[k_1, s_1, a_1(s_1), v_2(s_1, \cdot)] + \delta \sum_{s_2 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1, s_2 \rangle \pi(s_2|s_1) \Big\} \pi(s_1|s), \end{aligned}$$

where the second equality expands $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle$. Once $q_0 = (m_0, y_1^k, y_1^v)$ is chosen, $p_0 = (k_0, a_0, v_1)$ is independent of the remaining dual variables. Hence, given $q_0$, the inf over these dual variables and the sup over $p_0$ may be interchanged to give:

$$
\begin{aligned}
D^*(s,y) = \inf_{\mathcal{Q}} \sup_{\mathcal{P}} \; & -y_0^k \cdot k_0 + y_0^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] + m_0 \cdot H[k_0, s, a_0, v_1(\cdot)] \\
& - \delta \sum_{s_1 \in \mathcal{S}} y_1^v(s_1) v_1(s_1) \pi(s_1|s) + \sum_{s_1 \in \mathcal{S}} y_1^k(s_1) \cdot W^k[k_0, s, a_0] \pi(s_1|s) \\
& + \delta \inf_{\tilde{\mathbf{Q}}(q_0)} \sup_{\tilde{\mathbf{P}}(p_0)} \sum_{s_1 \in \mathcal{S}} \Big\{ - y_1^k(s_1) \cdot k_1(s_1) + y_1^v(s_1) \cdot W^v[s_1, a_1(s_1), M^v[s_1, v_2(s_1, \cdot)]] \\
& + m_1(s_1) \cdot H[k_1, s_1, a_1(s_1), v_2(s_1, \cdot)] + \delta \sum_{s_2 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1, s_2 \rangle \pi(s_2|s_1) \Big\} \pi(s_1|s),
\end{aligned}
\tag{27}
$$

where $\tilde{\mathbf{Q}}(q_0)$ denotes the continuation of dual plans in $\mathbf{Q}$ after the exclusion of $q_0$ and $\tilde{\mathbf{P}}(p_0)$ denotes the continuation of primal plans in $\mathbf{P}$ after the exclusion of $p_0$. Then:

$$
\begin{aligned}
D^*(s,y) = & \inf_{\mathcal{Q}} \sup_{\mathcal{P}} \; -y_0^k \cdot k_0 + y_0^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] + m_0 \cdot H[k_0, s, a_0, v_1(\cdot)] \\
& - \delta \sum_{s_1 \in \mathcal{S}} y_1^v(s_1) v_1(s_1) \pi(s_1|s) + \sum_{s_1 \in \mathcal{S}} y_1^k(s_1) \cdot W^k[k_0, s, a_0] \pi(s_1|s) \\
& + \delta \sum_{s_1 \in \mathcal{S}} \Big\{ \inf_{\mathbf{Q}(y_1(s_1))} \sup_{\mathbf{P}(v_1(s_1))} \; -y_1^k(s_1) \cdot k_1(s_1) + y_1^v(s_1) \cdot W^v[s_1, a_1(s_1), M^v[s_1, v_2(s_1, \cdot)]] \\
& + m_1(s_1) \cdot H[k_1, s_1, a_1(s_1), v_2(s_1, \cdot)] + \delta \sum_{s_2 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1, s_2 \rangle \pi(s_2|s_1) \Big\} \pi(s_1|s). \\
= & \inf_{\mathcal{Q}} \sup_{\mathcal{P}} \; -y_0^k \cdot k_0 + y_0^v \cdot W^v[s, a_0, M^v[s, v_1(\cdot)]] + m_0 \cdot H[k_0, s, a_0, v_1(\cdot)] \\
& - \delta \sum_{s_1 \in \mathcal{S}} y_1^v(s_1) \cdot v_1(s_1) \pi(s_1|s) + \delta \sum_{s_1 \in \mathcal{S}} y_1^k(s_1) \cdot W^k[k_0, s, a_0] \pi(s_1|s) + \delta \sum_{s_1 \in \mathcal{S}} D^*\left(s_1, y_1(s_1)\right) \pi(s_1|s),
\end{aligned}
\tag{28}
$$

where the first equality uses (27), the additive separability in $s_1$ of the terms in the last row of (27) and the fact that, following the argument in Proposition 2, each inf-sup in the curly brackets of (28) is bounded above and, hence, the inf-sup of the sum equals the sum of the inf-sup's (although some may equal $-\infty$). The second equality then follows from the definition of $D^*$. Combining the last equality with the definition of $J$ gives the second equality (22) in the proposition. □

*Proof of Proposition 4. (Only if)* Let $J_0(y_0, v_0) = F[s_0, v_0] - y_0^v \cdot v_0 + y_0^k \cdot \bar{k}$. Then from the definition of the Lagrangian for all $(\mathbf{p}, \mathbf{q}) \in \mathbf{P} \times \mathbf{Q}$,

$$
\begin{aligned}
\mathscr{L}(\mathbf{p}, \mathbf{q}) = J_0(y_0, v_0) - y_0^k \cdot k_0 + y_0^v \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0 \cdot H[k, s_0, a_0, v_1(\cdot)] \\
+ \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1|s_0).
\end{aligned}
$$

To show $D_0^* \geq J_0(y_0^*, v_0^*) + D^*(s_0, y_0^*)$, note that by definition, $D_0^* = \mathscr{L}(\mathbf{p}^*, \mathbf{q}^*)$ and

$$
\begin{aligned}
\mathscr{L}(\mathbf{p}^*, \mathbf{q}^*) =& \sup_{\mathbf{P}} J_0(y_0^*, v_0) - y_0^{k*} \cdot k_0 + y_0^{v*} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0^* \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}^*, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0) \\
=& \sup_{\mathcal{V}} J_0(y_0^*, v_0) + \sup_{\mathbf{P}(v_0)} -y_0^{k*} \cdot k_0 + y_0^{v*} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0^* \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}^*, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0) \\
\geq& J_0(y_0^*, v_0^*) + \inf_{\mathbf{Q}(y_0^*)} \sup_{\mathbf{P}(v_0^*)} -y_0^{k*} \cdot k_0 + y_0^{v*} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0) \\
=& J_0(y_0^*, v_0^*) + D^*(s_0, y_0^*), \qquad (29)
\end{aligned}
$$

where the first equality uses the definition of $\mathbf{p}^*$, the second uses the decomposition and moving forward over constants properties of supremum operations, the inequality uses the feasibility of $v_0^*$ for the supremum over $\mathcal{V}$ and the feasibility of the continuation of $\mathbf{q}^*$ for the continuation minimization described in this line, and the final equality uses the definition of $D^*$. For the reverse inequality, note first that:

$$
\begin{aligned}
\mathscr{L}(\mathbf{p}^*, \mathbf{q}^*) =& \inf_{\mathbf{Q}} \sup_{\mathbf{P}} J_0(y_0, v_0) - y_0^{k} \cdot k_0 + y_0^{v} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0), \\
\leq& \sup_{\mathbf{P}} J_0(y_0^*, v_0) - y_0^{k*} \cdot k_0 + y_0^{v*} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0 \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0) \qquad \forall \mathbf{q} \in \mathbf{Q}(y_0^*), \\
=& J_0(y_0^*, v_0^*) + \sup_{\mathbf{P}(v_0)} -y_0^{k*} \cdot k_0 + y_0^{v*} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0^* \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0), \qquad \forall \mathbf{q} \in \mathbf{Q}(y_0^*), \qquad (30)
\end{aligned}
$$

where the first inequality uses the feasibility of $\mathbf{q} \in \mathbf{Q}(y_0^*)$ for the inf problems, the second equality uses the decomposition property of the sup operator and the maximality of $v_0^*$ in $\sup_{\mathcal{V}} J_0^*(y_0^*, v_0)$. Since the last equality in (30) holds $\forall \mathbf{q} \in \mathbf{Q}(y_0^*)$:

$$
\begin{aligned}
D_0^* \leq& J_0(y_0^*, v_0^*) + \inf_{\mathbf{Q}(y_0^*)} \sup_{\mathbf{P}(v_0)} -y_0^{k*} \cdot k_0 + y_0^{v*} \cdot W^v[s_0, a_0, M^v[s_0, v_1(\cdot)]] + m_0^* \cdot H[k_0, s_0, a_0, v_1(\cdot)] \\
&\qquad + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s_1 \rangle \pi(s_1 | s_0) = J_0(y_0^*, v_0^*) + D^*(s_0, y_0^*), \qquad (31)
\end{aligned}
$$

where the last equality uses the definition of $D^*$. Combining (29) and (31) gives: $D_0^* = J_0(y_0^*, v_0^*) + D^*(s_0, y_0^*)$. Thus, $y_0^*$ attains the minimum in (21) and since $v_0^*$ attains the maximum in $\sup_{\mathcal{V}} J_0(y_0^*, v_0)$, $(y_0^*, v_0^*) \in G_0$. Repeating the argument at successive histories implies that $q_t^*(s^t) = (m_t^*(s^t), y_{t+1}^*(s^t))$ attains the minimum in (22) at $(s_t, y_t^*(s^t))$ and so $(q_t^*(s^t), p_t^*(s^t)) \in G(s_t, y_t^*(s^t))$.

*(If)* The remainder of the proof uses the following lemma.

**Lemma 5.** *Let $J_0(y_0, v_0) = F[s_0, v_0] - y_0^v \cdot v_0 + y_0^k \cdot \bar{k}$. For all $(\mathbf{p}, \mathbf{q}) \in \mathbf{P} \times \mathbf{Q}$, $\mathscr{L}(\mathbf{p}, \mathbf{q}) = J_0(y_0, v_0) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), p_t(s^t))\pi^t(s^t|s_0)$.*

*Proof.* Expanding $\langle \mathbf{q}, \mathbf{z}(\mathbf{p}) \rangle$, the Lagrangian $\mathscr{L}$ is:

$$\mathscr{L}(\mathbf{p}, \mathbf{q}) = F[s_0, v_0] + \sum_{t=0}^{\infty} \sum_{\mathcal{S}^t} \delta^t \{m_t(s^t) \cdot z_t^h(\mathbf{p})(s^t) + y_t^k(s^t) \cdot z_t^k(\mathbf{p})(s^t) + y_t^v(s^t) \cdot z_t^v(\mathbf{p})(s^t)\} \pi^t(s^t|s_0).$$

From the definition of $J$, we have for each $T$:

$$\begin{aligned} \mathscr{L}(\mathbf{p}, \mathbf{q}) = & J_0(y_0, v_0) + \sum_{t=0}^{T} \delta^t \sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), p_t(s^t))\pi^t(s^t|s_0) \\ & + \delta^T \sum_{\mathcal{S}^T} \left[ y_T^v(s^T) v_T(s^T) + m_T(s^T) H[k_T(s^T), s_T, v_{T+1}(s^T, \cdot)] \right] \pi^T(s^T|s_0) \\ & - \delta^T \sum_{\mathcal{S}^T} y_T^k(s^T) k_T(s^T) \pi^T(s^T|s_0) + \delta^{T+1} \sum_{\mathcal{S}^{T+1}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p}) | s^T \rangle \pi^{T+1}(s^{T+1}|s_0). \end{aligned} \tag{32}$$

Since $\mathbf{q} \in \mathbf{Q}$ and since $\mathbf{v}$, $\mathbf{k}$ and $\mathbf{z}(\mathbf{p})$ belong to $\ell^\infty$, the infinite sums in the second and third rows of (32) converge to zero as $T \to \infty$ giving the result. □

Let $(\mathbf{p}^*, \mathbf{q}^*)$ satisfy (i) $(y_0^{k*}, y_0^{v*}, v_0^*) \in G_0$, (ii) $\forall t \geq 1$, $s^t \in \mathcal{S}^t$, $(m_t^*(s^t), y_{t+1}^*(s^t), p_t^*(s^t)) \in G(s_t, y_t^*(s^t))$ and (iii) $\mathbf{q}^* \in \mathbf{Q}$. From the definitions of $G_0$ and $G$, $J_0(y_0^*, v_0^*) = \sup_{\mathcal{V}} J_0(y_0^*, v_0)$ and $J(s_t, y_t^*(s^t); q_t^*(s^t), p_t^*(s^t)) = \sup_{\mathcal{P}} J(s_t, y_t^*(s^t); q_t^*(s^t), p_t(s^t))$. So, by Lemma 5, $\forall \mathbf{p} \in \mathbf{P}$,

$$\begin{aligned} \mathscr{L}(\mathbf{p}^*, \mathbf{q}^*) &= J_0(y_0^*, v_0^*) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J(s_t, y_t^*(s^t); q_t^*(s^t), p_t^*(s^t))\pi^t(s^t|s_0) \\ &\geq J_0(y_0^*, v_0) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J(s_t, y_t^*(s^t); q_t^*(s^t), p_t(s^t))\pi^t(s^t|s_0), \end{aligned}$$

and $\mathbf{p}^* \in \operatorname{argmax}_{\mathbf{P}} \mathscr{L}(\mathbf{p}, \mathbf{q}^*)$. Let $J_0^*(y) = \sup_{\mathcal{V}} J_0(y, v_0)$ and let $J^*(s_t, y_t(s^t); q_t(s^t)) = \sup_{\mathcal{P}} J(s_t, y_t(s^t); q_t(s^t), p)$. For all $\mathbf{q} \in \mathbf{Q}$, $\mathbf{p} \in \mathbf{P}$: $J_0^*(y_0) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J^*(s_t, y_t(s^t); q_t(s^t))\pi^t(s^t|s_0) \geq J_0(y_0, v_0) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), p_t(s^t))\pi^t(s^t|s_0)$. Hence,

$$\begin{aligned} & J_0^*(y_0) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J^*(s_t, y_t(s^t); q_t(s^t))\pi^t(s^t|s_0) \geq \\ & \qquad \sup_{\mathbf{P}} J_0(y_0, v_0) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), p_t(s^t))\pi^t(s^t|s_0). \end{aligned} \tag{33}$$

On the other hand, for small $\varepsilon > 0$, let $\hat{\mathbf{p}} \in \mathbf{P}$ be such that $J_0(y_0, \hat{v}_0) > J^*(y_0) - \frac{\varepsilon}{2}$ and for each $t$ and $s^t$, $J(s_t, y_t(s^t); q_t(s^t), \hat{p}_t(s^t)) > J^*(s_t, y_t(s^t); q_t(s^t)) - (1-\delta)\frac{\varepsilon}{2}$ , then $J_0^*(y_0) +$

$\sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J^*(s_t, y_t(s^t); q_t(s^t))\pi^t(s^t|s_0) < J_0(y_0, \hat{v}_0) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), \hat{p}_t(s^t))\pi^t(s^t|s_0)$ $+\ \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$J_0^*(y_0) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J^*(s_t, y_t(s^t); q_t(s^t))\pi^t(s^t|s_0) \\ \leq \sup_{\mathbf{P}} J_0(y_0, v_0) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), p_t(s^t))\pi^t(s^t|s_0). \tag{34}$$

Combining the definition of $D_0^*$, Lemma 5 and (33)-(34) and using $\mathbf{q}^* \in \mathbf{Q}$ gives:

$$\begin{aligned} D_0^* = \inf_{\mathbf{Q}}\sup_{\mathbf{P}} \mathscr{L}(\mathbf{p}, \mathbf{q}) &= \inf_{\mathbf{Q}}\sup_{\mathbf{P}} J_0(y_0, v_0) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J(s_t, y_t(s^t); q_t(s^t), p_t(s^t))\pi^t(s^t|s_0) \\ &= \inf_{\mathbf{Q}} J_0^*(y_0) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J^*(s_t, y_t(s^t); q_t(s^t))\pi^t(s^t|s_0) \\ &\leq J_0^*(y_0^*) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J^*(s_t, y_t^*(s^t); q_t^*(s^t))\pi^t(s^t|s_0). \end{aligned} \tag{35}$$

The definitions of $G_0$ and $G$ and the dual Bellman equation imply $D_0^* = J_0^*(y_0^*) + \sum_{t=0}^{T}\delta^t\sum_{\mathcal{S}^t} J^*(s_t, y_t^*(s^t); q_t^*(s^t))\pi^t(s^t|s_0) + \delta^{T+1}\sum_{\mathcal{S}^{T+1}} D^*(s_{T+1}, y_{T+1}^*(s^{T+1}))\pi(s^{T+1}|s_0)$. Taking the limit as $T$ goes to infinity and using condition (T) implies:

$$D_0^* \geq J_0^*(y_0^*) + \sum_{t=0}^{\infty}\delta^t\sum_{\mathcal{S}^t} J^*(s_t, y_t^*(s^t); q_t^*(s^t))\pi^t(s^t|s_0). \tag{36}$$

Combining (35) with (36) implies that $\mathbf{q}^*$ attains the minimum as required. □

# B Proofs for Section 5

*Proof of Lemma 2.* Let $\Psi$, $\Phi$ and $\Omega$ denote vector spaces and let $L : \Psi \times \Phi \times \Omega \to \mathbb{R} \cup \{-\infty\}$. Assume that for each $\omega \in \Omega$, $L(\cdot;\cdot,\omega)$ is sub-linear. For $\psi \in \Psi$, let: $\mathscr{T}(\psi) = \inf_\Phi \sup_\Omega L(\psi;\phi,\omega)$. Assume that $\mathscr{T} < \infty$. We prove that $\mathscr{T}$ is sub-linear. We first show that $\mathscr{T}$ is convex. Let $\psi^1$ and $\psi^2$ be elements of $\Psi$ and $\lambda \in [0,1]$. Let $\psi^\lambda = \lambda\psi^1 + (1-\lambda)\psi^2$. For $i = 1, 2$, let $\{\phi_n^i\}$ denote sequences such that: $\mathscr{T}(\psi^i) = \inf_{\Phi}\sup_{\Omega} L(\psi^i;\phi,\omega) = \lim_{n\to\infty}\sup_\Omega L(\psi^i;\phi_n^i,\omega)$. Also let $\phi_n^\lambda = \lambda\phi_n^1 + (1-\lambda)\phi_n^2$. Then:

$$\begin{aligned} \lambda\mathscr{T}(\psi^1) + (1-\lambda)\mathscr{T}(\psi^2) &= \lambda \inf_{\Phi}\sup_{\Omega} L(\psi^1;\phi,\omega) + (1-\lambda)\inf_{\Phi}\sup_{\Omega} L(\psi^2;\phi,\omega) \\ &= \lambda \lim_{n\to\infty}\sup_{\Omega} L(\psi^1;\phi_n^1,\omega) + (1-\lambda)\lim_{n\to\infty}\sup_{\Omega} L(\psi^2;\phi_n^2,\omega) \\ &\geq \lim_{n\to\infty}\sup_{\Omega}\{\lambda L(\psi^1;\phi_n^1,\omega) + (1-\lambda)L(\psi^2;\phi_n^2,\omega)\} \\ &\geq \lim_{n\to\infty}\sup_{\Omega} L(\psi^\lambda;\phi_n^\lambda,\omega) \geq \inf_{\Phi}\sup_{\Omega} L(\psi^\lambda;\phi,\omega) = \mathscr{T}(\psi^\lambda), \end{aligned}$$

where the sum on the left hand side of the first line is well defined because $\mathscr{T} < \infty$, the first equality follows from the definition of $\mathscr{T}$, the second equality uses the definition of the sequences $\{\phi_n^i\}$, $i = 1, 2$, the first inequality uses the fact that the sup of a sum is less than or equal to the sum of sups and the second inequality uses the convexity of $L(\cdot;\cdot,\omega)$. Thus, $\mathscr{T}$ is convex. Next we show positive homogeneity. Suppose that $\psi \in \Psi$ and $\lambda > 0$. Then: $\mathscr{T}(\lambda\psi) = \inf_{\Phi} \sup_{\Omega} L(\lambda\psi;\phi,\omega) = \lambda \inf_{\Phi} \sup_{\Omega} L(\psi;\phi/\lambda,\omega) = \lambda\mathscr{T}(\psi)$, where the second equality uses the positive homogeneity of $L(\cdot,\cdot,\omega)$. Thus, $\mathscr{T}$ is positively homogenous of degree 1 and, combining results, sub-linear.

(i) For fixed $s_0 \in \mathcal{S}$ and a pair $(\mathbf{p}, \mathbf{q}) \in \mathbf{P} \times \mathbf{Q}$, let:

$$L(y_0; (m_0, \{\mathbf{q}|s_1\}), (p, \{\mathbf{z}(\mathbf{p})|s_1\})) := -y_0^k \cdot k_0 + y_0^v \cdot W^v[s_0, a_0, M^v[s_0, v_1]] + m_0 \cdot H[k_0, s_0, a_0, v_1] + \delta \sum_{s_1 \in \mathcal{S}} \langle \mathbf{q}, \mathbf{z}(\mathbf{p})|s_1 \rangle \pi(s_1|s_0).$$

Then: $D^*(s_0, y_0) = \inf_{\mathbf{Q}(y_0)} \sup_{\mathbf{P}(v_0)} L(y_0; (m_0, \{\mathbf{q}|s_1\}), (p, \{\mathbf{z}(\mathbf{p})|s_1\}))$. By Proposition 2, $D^*(s_0, \cdot) < \infty$. Also, for each $\omega = (p, \{\mathbf{z}(\mathbf{p})|s_1)$, the function $L(\cdot;\cdot,\omega)$ is linear and, hence, sub-linear. Applying the general result from the first part of the proof, $D^*(s_0, \cdot)$ is sub-linear. Since $s_0$ was arbitrary in $\mathcal{S}$, $D^*$ is sub-linear. (ii) Recall that:

$$\mathscr{B}(D)(s, y) = \inf_{\mathcal{Q}} \sup_{\mathcal{P}} J(s, y; q, p) + \delta \sum_{s' \in \mathcal{S}} D\left(s', y'(s')\right) \pi(s'|s).$$

For $q = 0$, $\sup_{\mathcal{P}} J(s, y; 0, p) + \delta \sum_{s' \in \mathcal{S}} D\left(s', 0\right) \pi(s'|s) < \infty$. Hence, $\mathscr{B}(D)(s, y) < \infty$. Also, for each $(s, p)$, $J(s, \cdot; \cdot, p)$ is linear and $D$ is sub-linear by assumption. Thus, the function $J(s, y; q, p) + \delta \sum_{s' \in \mathcal{S}} D\left(s', y'(s')\right) \pi(s'|s)$ is sublinear. Thus, again by the general result, $\mathscr{B}(D)(s, \cdot)$ is sub-linear and, hence, $\mathscr{B}(D)$ is sub linear. □

*Proof of Lemma 3.* Evidently, $(\mathscr{G}, d)$ is a metric space. Let $\{D_n\}$ be a Cauchy sequence in $\mathscr{G}$. Thus, as $n, m \to \infty$,

$$d(D_n, D_m) = \sup_{\mathcal{S} \times \mathcal{C}} \left| \ln\left(\frac{D_n(s, y) - \underline{\underline{D}}(s, y)}{\underline{D}(s, y) - \underline{\underline{D}}(s, y)}\right) - \ln\left(\frac{D_m(s, y) - \underline{\underline{D}}(s, y)}{\underline{D}(s, y) - \underline{\underline{D}}(s, y)}\right) \right| \to 0.$$

For each $n = 1, 2, \ldots$, define $g_n : \mathcal{S} \times \mathcal{C} \to \mathbb{R}$ according to: $g_n(s, y) = \ln\left(\frac{D_n(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right)$, $(s, y) \in \mathcal{S} \times \mathcal{C}$. Let $\underline{g} = 0$ and $\overline{g} = \ln\left(\frac{\overline{D} - \underline{\underline{D}}}{\varepsilon}\right) \geq \ln\left(\frac{\overline{D} - \underline{\underline{D}}}{\underline{D} - \underline{\underline{D}}}\right)$. Note that $\overline{g}$ is bounded in $\mathcal{S} \times \mathcal{C}$ by the continuity of each $\overline{D}(s, \cdot) - \underline{\underline{D}}(s, \cdot)$ and the compactness of $\mathcal{C}$. The sequence of functions $\{g_n\}$ is Cauchy with respect to the sup-norm and that for each $n$, $\underline{g} \leq g_n \leq \overline{g}$. By the completeness (with respect to the sup-norm) of the bounded functions from $\mathcal{C}$ to $\mathbb{R}$,$\{g_n\}$ converges in the sup-norm to a function $g_\infty$, with each $g_\infty(s, \cdot)$ bounded and $\underline{g} \leq g_\infty \leq \overline{g}$. Use $g_\infty$ to define the homogeneous function $D_\infty$ as:

$$D_\infty(s, y) = \|y\|_E \left\{ \underline{\underline{D}}\left(s, \frac{y}{\|y\|_E}\right) + \exp\left\{ g_\infty\left(s, \frac{y}{\|y\|_E}\right) \right\} \left( \underline{D}\left(s, \frac{y}{\|y\|_E}\right) - \underline{\underline{D}}\left(s, \frac{y}{\|y\|_E}\right) \right) \right\}.$$

By construction $\underline{D} \leq D_\infty \leq \overline{D}$ and $D_n \xrightarrow{d} D_\infty$. Since $D_\infty$ is the pointwise limit of a sequence of sub-linear and, so, convex functions, it is convex. Thus, it is in $\mathscr{G}$. $\square$

*Proof of Proposition 5.* Let $\mathscr{G}_0$ denote the interval of real-valued functions between $\underline{\underline{D}}$ and $\overline{D}$. Let $D_1$ and $D_2$ be any pair of functions in $\mathscr{G}_0$ and let $\lambda \in [0,1]$. Define for each $(s,y,q) \in \mathcal{S} \times \mathcal{Y} \times \mathcal{Q}$, $J^*(s,y,q) = \sup_{\mathcal{P}} J(s,y;q,p) \in \mathbb{R}$. Then, for each $(s,y) \in \mathcal{S} \times \mathcal{Y}$,

$$
\begin{aligned}
&\mathscr{B}(\lambda D_1 + (1-\lambda) D_2)(s,y) \\
&\quad = \inf_{\mathcal{Q}} \quad J^*(s,y,m,y') + \delta \sum_{s' \in \mathcal{S}} \{\lambda D_1\left(s', y'(s')\right) + (1-\lambda) D_2\left(s', y'(s')\right)\} \pi(s'|s) \\
&\quad = \inf_{\mathcal{Q}} \quad \lambda \Big\{ J^*(s,y,m,y') + \delta \sum_{s' \in \mathcal{S}} D_1\left(s', y'(s')\right) \pi(s'|s) \Big\} \\
&\qquad\qquad + (1-\lambda) \Big\{ J^*(s,y,m,y') + \delta \sum_{s' \in \mathcal{S}} D_2\left(s', y'(s')\right) \pi(s'|s) \Big\} \\
&\quad \geq \lambda \inf_{\mathcal{Q}} \Big\{ J^*(s,y,m,y') + \delta \sum_{s' \in \mathcal{S}} D_1\left(s', y'(s')\right) \pi(s'|s) \Big\} \\
&\qquad\qquad + (1-\lambda) \inf_{\mathcal{Q}} \Big\{ J^*(s,y,m,y') + \delta \sum_{s' \in \mathcal{S}} D_2\left(s', y'(s')\right) \pi(s'|s) \Big\} \\
&\quad = \lambda \mathscr{B}(D_1)(s,y) + (1-\lambda) \mathscr{B}(D_2)(s,y),
\end{aligned}
$$

where, recall, $\mathscr{B}(D) < \infty$ hence the sums in the last two rows are well defined. Thus, $\mathscr{B}$ is concave on $\mathscr{G}_0$. Let $D_1, D_2 \in \mathscr{G} \subset \mathscr{G}_0$. By definition of $d$, for each $(s,y) \in \mathcal{S} \times \mathcal{C}$,

$$
\ln\left(\frac{D_2(s,y) - \underline{\underline{D}}(s,y)}{\underline{D} - \underline{\underline{D}}}\right) \leq \ln\left(\frac{D_1(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) + d(D_1, D_2).
$$

Taking the exponential of each side and rearranging gives:

$$
\exp\{-d(D_1, D_2)\} \left(\frac{D_2(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) \leq \left(\frac{D_1(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right).
$$

But, by Assumption 3 (i), $\underline{D} - \underline{\underline{D}} > 0$ and so, after rearrangement,

$$
D_1(s,y) \geq \exp\{-d(D_1, D_2)\} D_2(s,y) + (1 - \exp\{-d(D_1, D_2)\}) \underline{\underline{D}}(s,y).
$$

Since $D_1$, $D_2$ and $\underline{\underline{D}}$ are positively homogeneous of degree 1, this inequality holds at all $(s,y) \in \mathcal{S} \times \mathcal{Y}$. Then, by monotonicity and concavity of $\mathscr{B}$ (on $\mathscr{G}_0$),

$$
\begin{aligned}
\mathscr{B}(D_1) &\geq \mathscr{B}(\exp\{-d(D_1, D_2)\} D_2 + (1 - \exp\{-d(D_1, D_2)\}) \underline{\underline{D}}) \\
&\geq \exp\{-d(D_1, D_2)\} \mathscr{B}(D_2) + (1 - \exp\{-d(D_1, D_2)\}) \mathscr{B}(\underline{\underline{D}}).
\end{aligned} \tag{D1}
$$

By assumption there is an $\varepsilon > 0$ such that for each $(s,y) \in \mathcal{S} \times \mathcal{C}$, $\mathscr{B}(\underline{\underline{D}})(s,y) > \underline{\underline{D}}(s,y) + \varepsilon$. For $(s,y) \in \mathcal{S} \times \mathcal{C}$, define:

$$
\lambda(s,y) := \frac{\varepsilon}{\overline{D}(s,y) - \underline{\underline{D}}(s,y)}.
$$

Since $\overline{D}(s,y) \geq \mathscr{B}(\overline{D})(s,y) \geq \mathscr{B}(\underline{\underline{D}})(s,y) > \underline{\underline{D}}(s,y) + \varepsilon$, $\lambda(s,y) \in (0,1)$. Now, for each $s \in \mathcal{S}$, $\underline{\underline{D}}(s,\cdot)$ and $\overline{D}(s,\cdot)$ are continuous. Thus, $\lambda(s,\cdot)$ is continuous and since $\mathcal{C}$ is compact, there is a $\lambda^* = \min_{\mathcal{S}\times\mathcal{C}} \lambda(s,y) \in (0,1)$. Then, for all $(s,y) \in \mathcal{S} \times \mathcal{C}$,

$$\begin{aligned}
\mathscr{B}(\underline{\underline{D}})(s,y) &> \underline{\underline{D}}(s,y) + \varepsilon = \lambda(s,y)\overline{D}(s,y) + (1-\lambda(s,y))\underline{\underline{D}}(s,y) \\
&\geq \lambda^*\overline{D}(s,y) + (1-\lambda^*)\underline{\underline{D}}(s,y) \\
&\geq \lambda^*\mathscr{B}(D_2)(s,y) + (1-\lambda^*)\underline{\underline{D}}(s,y), \qquad \text{(D2)}
\end{aligned}$$

where the first inequality is by assumption, the first equality uses the definition of $\lambda(s,y)$, the second inequality uses the definition of $\lambda^*$ and $\overline{D} \geq \underline{\underline{D}}$ and the final inequality uses $\overline{D} \geq \mathscr{B}(\overline{D}) \geq \mathscr{B}(D_2)$. Combining (D1) with (D2) gives for all $(s,y) \in \mathcal{C}$,

$$\begin{aligned}
\mathscr{B}(D_1)(s,y) \geq & \exp\{-d(D_1,D_2)\}\mathscr{B}(D_2)(s,y) + (1-\exp\{-d(D_1,D_2)\}) \\
& \times [\lambda^*\mathscr{B}(D_2)(s,y) + (1-\lambda^*)\underline{\underline{D}}(s,y)].
\end{aligned}$$

Letting $r := \exp\{-d(D_1,D_2)\} + (1-\exp\{-d(D_1,D_2)\})\lambda^*$, then gives for $(s,y) \in \mathcal{S} \times \mathcal{C}$:

$$\frac{\mathscr{B}(D_1)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)} \geq r \frac{\mathscr{B}(D_2)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}.$$

Hence, taking logs, for $(s,y) \in \mathcal{S} \times \mathcal{C}$,

$$\ln\left(\frac{\mathscr{B}(D_1)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) \geq \ln r + \ln\left(\frac{\mathscr{B}(D_2)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right).$$

But from the definition of $r$ and Jensen's inequality:

$$\ln r \geq (1-\lambda^*)\ln\exp\{-d(D_1,D_2)\} + \lambda^*\ln 1 = -(1-\lambda^*)d(D_1,D_2).$$

Thus, for $(s,y) \in \mathcal{S} \times \mathcal{C}$,

$$(1-\lambda^*)d(D_1,D_2) \geq -\ln r \geq \ln\left(\frac{\mathscr{B}(D_2)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) - \ln\left(\frac{\mathscr{B}(D_1)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right). \qquad \text{(D3)}$$

Repeating the argument with $D_1$ and $D_2$ interchanged and combining with (D3) implies that for all $(s,y) \in \mathcal{S} \times \mathcal{C}$,

$$(1-\lambda^*)d(D_1,D_2) \geq \left|\ln\left(\frac{\mathscr{B}(D_2)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) - \ln\left(\frac{\mathscr{B}(D_1)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right)\right|.$$

Consequently, letting $\rho := (1-\lambda^*) \in (0,1)$,

$$\begin{aligned}
\rho d(D_1,D_2) &\geq \sup_{\mathcal{S}\times\mathcal{C}}\left|\ln\left(\frac{\mathscr{B}(D_2)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right) - \ln\left(\frac{\mathscr{B}(D_1)(s,y) - \underline{\underline{D}}(s,y)}{\underline{D}(s,y) - \underline{\underline{D}}(s,y)}\right)\right| \\
&= d(\mathscr{B}(D_1),\mathscr{B}(D_2))
\end{aligned}$$

as desired. □

We now show that the Bellman operator's iterates converge uniformly to $D^*$ on $\mathcal{S} \times \mathcal{C}$. For sublinear functions $D_1, D_2$ define $\|D_1 - D_2\| := \sup_{\mathcal{S}\times\mathcal{C}} |D_1(s,y) - D_2(s,y)|$.

**Proposition 10.** *Let Assumption 3 hold. (i) For any two functions $D_1, D_2 \in \mathscr{G}$ we have $\|D_1 - \underline{\underline{D}}\|, \|D_2 - \underline{\underline{D}}\| \leq \|\overline{D} - \underline{\underline{D}}\| < \infty$, and $\|D_1 - D_2\| \leq \|\overline{D} - \underline{\underline{D}}\|(\exp d(D_1, D_2) - 1)$. (ii) Let $\underline{D} \leq D^* \leq \overline{D}$. For any initial function $D_0 \in \mathscr{G}$ we have $\lim_{n\to\infty} \|\mathscr{B}^n(D_0) - D^*\| = 0$, that is, the dual Bellman operator's iterates converge uniformly to $D^*$ in $\mathcal{S} \times \mathcal{C}$.*

*Proof.* (i) Since $D_1, D_2 \in \mathscr{G}$ the inequalities $\|D_1 - \underline{\underline{D}}\|, \|D_2 - \underline{\underline{D}}\| \leq \|\overline{D} - \underline{\underline{D}}\|$ are immediate, the boundedness comes from the continuity of $\overline{D}$ and $\underline{\underline{D}}$ and compactness of $\mathcal{S} \times \mathcal{C}$. Now, for any two distinct functions $D_1$ and $D_2$ we have $d(D_1, D_2) > 0$ hence $\mu := \exp d(D_1, D_2) > 1$. From the definition of the Thompson distance, we have $\mu = \exp d(D_1, D_2) \geq \left|\frac{D_1(s,y) - \underline{\underline{D}}(s,y)}{D_2(s,y) - \underline{\underline{D}}(s,y)}\right|$ for all $(s, y)$. We hence have (pointwise) that both $D_1 - \underline{\underline{D}} \leq \mu(D_2 - \underline{\underline{D}})$ and $D_2 - \underline{\underline{D}} \leq \mu(D_1 - \underline{\underline{D}})$. Therefore $D_1 - D_2 \leq (\mu - 1)(D_2 - \underline{\underline{D}})$ and $D_2 - D_1 \leq (\mu - 1)(D_1 - \underline{\underline{D}})$. This implies that (again pointwise) $-(\mu - 1)|D_2 - \underline{\underline{D}}| \leq D_2 - D_1 \leq (\mu - 1)|D_1 - \underline{\underline{D}}|$. From this, we have $|D_2 - D_1| \leq (\mu - 1)\max\{|D_2 - \underline{\underline{D}}|, |D_1 - \underline{\underline{D}}|\} \leq (\mu - 1)\|\overline{D} - \underline{\underline{D}}\|$, delivering $\|D_1 - D_2\| \leq \|\overline{D} - \underline{\underline{D}}\|(\mu - 1)$ as we stated in the proposition. (ii) Immediate from point (i) and Theorem 2 which implies $\exp d(\mathscr{B}^n(D), D^*) \to 1$. □

# C Proofs for Section 6

*Proof of Proposition 6.* Equality of values follows from standard arguments (e.g., Rockafellar (1974), Theorem 2). Saddle point existence implies that both the dual and primal problems have solutions (again see Rockafellar (1974), Theorem 2). If $\mathbf{q}^*$ solves the dual problem (i.e. $\mathbf{q}^*$ attains the minimum in (IS)) and $\mathbf{p}^*$ solves the primal problem, then from the definitions of these problems: $D_0^* = \sup_{\mathbf{p}\in\mathbf{P}} \mathscr{L}(\mathbf{p}, \mathbf{q}^*) \geq \mathscr{L}(\mathbf{p}^*, \mathbf{q}^*) \geq \inf_{\mathbf{q}\in\mathbf{Q}} \mathscr{L}(\mathbf{p}^*, \mathbf{q}) = P_0^*$. Hence, equality of values $D_0^* = P_0^*$, implies that $\mathbf{p}^*$ solves the inner supremum in the dual problem and $(\mathbf{q}^*, \mathbf{p}^*) \in \mathbf{\Lambda}$. The desired result then follows from Proposition 4. □

*Proof of Proposition 7.* Since a saddle point exists, the set of primal plans $\mathbf{P}^*$ that solve (P) is nonempty. From Proposition 6, whenever $\mathscr{L}$ admits a saddle, $\mathbf{P}^* \subset \hat{\mathbf{P}}$. By assumption, $\hat{\mathbf{P}}$ only contains the element $\hat{\mathbf{p}}$. Hence $\mathbf{P}^* = \{\hat{\mathbf{p}}\}$. □

*Proof of Proposition 8.* This proof is long and is reported in Online Appendix C. It is structured as follows. First, existence of a saddle point with summable multipliers is established for an abstract problem with inequality constraints. Next this problem is related to a modified version of (P) (called (MP)). A Lagrangian is associated with (MP) and it is shown that each primal plan solving (MP) is part of a saddle point with a minimizing summable multiplier. Finally, it is shown that each solution to (P) defines a solution to (MP) and the minimizing multiplier from (MP) is used to construct a minimizing multiplier and, hence, saddle point for (P). □

*Proof of Proposition 9.* If $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ is a candidate plan satisfying Condition (T), then from Proposition 4, it is a solution to the dual problem (IS). Moreover, from the dual Bellman equation (21) in Proposition 3, $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ solves: $D_0^* = F[s_0, \hat{v}_0] - \hat{y}_0^v \cdot \hat{v}_0 + \hat{y}_0^k \cdot \bar{k} + D^*(s_0, \hat{y}_0^k, \hat{y}_0^v)$. Condition (ii) implies $F[s_0, \hat{v}_0] \geq D_0^*$, and hence, from the weak duality inequality, $F[s_0, \hat{v}_0] \geq P_0^*$. The feasibility condition (iii) then implies that $P_0^* \geq F[s_0, \hat{v}_0]$. Combining inequalities $F[s_0, \hat{v}_0] = P_0^*$ and $\hat{\mathbf{p}}$ solves (P). In addition, $\mathscr{L}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \sup_{\mathbf{p}} \mathscr{L}(\mathbf{p}, \hat{\mathbf{q}}) = D_0^* = P_0^* = \inf_{\mathbf{q}} \mathscr{L}(\hat{\mathbf{p}}, \mathbf{q})$, where the first and second inequalities use the fact that $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ solves the dual, the third uses the result above that $D_0^* = P_0^*$ and the fourth the fact that $\hat{\mathbf{p}}$ solves (P) and, so, maximizes $\inf_{\mathbf{q}} \mathscr{L}(\mathbf{p}, \mathbf{q})$ and attains $P_0^*$. Thus, $\hat{\mathbf{p}}$ solves $\max \mathscr{L}(\mathbf{p}, \hat{\mathbf{q}})$ and $\hat{\mathbf{q}}$ solves $\min \mathscr{L}(\hat{\mathbf{p}}, \mathbf{q})$ and $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ is a saddle for $\mathscr{L}$. □

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# Appendices for Online Publication

# A Numerical Implementation

This section describes how to implement the recursive dual approach numerically. Under the conditions of Theorem 2, the dual Bellman operator is a contraction and, consequently, it is natural to calculate $D^*$ via value iteration. Numerical approximation of candidate dual value functions is facilitated by their sub linearity and the simplicity of their domain. The dual Bellman involves an (outer) minimization over a set of multipliers; these multipliers are passed to (and "coordinate") a family of simple (inner) maximizations over current actions and states. Additive separability in the objective may be exploited to decompose the inner maximizations into a family of simpler maximizations that in parametric settings often have analytical solutions.

**Dual Value Function Approximation** Numerical implementation of a value function iteration algorithm requires approximations to candidate value functions. Our implementation exploits the sublinearity of dual value functions and use a piecewise linear approximation (on the spherical domain $\mathcal{C}$). Piecewise linear approximations to value functions defined on spheres were first applied in economics by Judd et al. (2003). We apply their approximation procedure to our setting.[26] Recall that under the conditions of Theorem 2, the domain for the dual Bellman operator may be identified with an interval of functions $D : \mathcal{S} \times \mathcal{Y} \to \mathbb{R}$ each of which is sub-linear in its second argument. As noted, these functions are fully determined on $\mathcal{S} \times \mathcal{C}$ (or a subset thereof). Moreover, their sub-linearity implies that for all $y \in \mathcal{C}$,[27]

$$D(s, y) = \max_{r \in \mathcal{V}} \{r \cdot y \mid \forall y' \in \mathcal{C},\ r \cdot y' \leq D(s, y')\}. \tag{A.1}$$

Given such a function $D$ and a set of $N$ distinct points $\hat{\mathcal{C}}^N := \{y_n\}_{n=1}^N \subset \mathcal{C}$, define the approximation $\hat{D}^N$ as, for each $(s, y) \in \mathcal{S} \times \mathcal{C}$,

$$\hat{D}^N(s, y) := \max_{r \in \mathcal{V}} \{r \cdot y \mid \forall y_n \in \hat{\mathcal{C}}^N,\ r \cdot y_n \leq D(s, y_n)\}. \tag{A.2}$$

Since the constraint set in problem (A.2) is less restrictive than that in (A.1):

$$D \leq \hat{D}^N,$$

with equality at each $(s, y_n) \in \mathcal{S} \times \hat{\mathcal{C}}^N$. In addition, the approximation $\hat{D}^N$ remains sub-linear, is summarized by $\{s, y_n, \hat{D}^N(s, y_n)\}_{n=1}^N$ and is easily evaluated by solving the simple linear programming problem in (A.2). Let $\{\hat{\mathcal{C}}^N\}$, $N \geq 1$, be a sequence of subsets of $\mathcal{C}$ such that (i) for all $N$, $\hat{\mathcal{C}}^N \subset \hat{\mathcal{C}}^{N+1}$ and (ii) $\hat{\mathcal{C}}^\infty = \cup_N \hat{\mathcal{C}}^N$ is dense in $\mathcal{C}$.[28] It is readily verified that the corresponding sequence of approximating functions $\hat{D}^N(s, \cdot)$ converges pointwise to $D(s, \cdot)$

[26] Judd et al. (2003) use this approach to approximate the support function of a payoff set in a repeated game; we use it to approximate the recursive dual value function. In other aspects our (recursive dual) formulation is different from that of Judd et al. (2003). Alternative approaches to approximation on spherical domains are described in Sloan and Womersley (2000).

[27] For this and other properties of sublinear functions used below, see Florenzano and Van (2001).

[28] For example, the set of points in $\mathcal{C}$ with rational coordinates is dense in $\mathcal{C}$, see Schmutz (2008).

from above.[29] Moreover, by Dini's theorem[30] it converges uniformly on $\mathcal{C}$ and, hence, in the Thompson-like metric $d$ to $D(s,\cdot)$. In practical applications we use (hyper)spherical coordinates to represent points in $\hat{\mathcal{C}}^N$. The corresponding Cartesian coordinates of points in this grid are recovered from spherical coordinates $\{\phi_n^r\}$ according to the formulas $y_n^1 = \cos(\phi_n^1)$, for $j = 2, \dots, n_K + n_V - 1$, $y_n^j = \cos(\phi_n^j) \prod_{r=1}^{j-1} \sin(\phi_n^r)$ and $y_n^{n_K+n_V} = \prod_{r=1}^{n_K+n_V-1} \sin(\phi_n^r)$.

The approximation procedure described above may be integrated into the dual value iteration to give the $\nu+1$-iteration step:[31]

$$\forall (s, y_n) \in \mathcal{S} \times \hat{\mathcal{C}}^N, \quad \hat{\mathscr{B}}(\hat{D}_\nu^N)(s, y_n) = \inf_{q \in \mathcal{Q}} \sup_{p \in \mathcal{P}} J(s, y_n; q, p) + \delta \sum_{s' \in \mathcal{S}} D_\nu^N(s', y'(s'))\pi(s'|s). \tag{A.3}$$

**Optimization** The inner supremum operation in (A.3) results in the indirect current dual function $J^*(s, y_n; q) = \sup_{p \in \mathcal{P}} J(s, y_n; q, p)$. Additive separability of the function $J$ across different components of $p$ can often be exploited to break the supremum down into separate optimizations over the components of $p$ which can be run in parallel or in some cases solved analytically. In these latter cases no explicit numerical maximization over primal choices is needed. Once the inner suprema are solved, an indirect objective over multipliers is obtained and (A.3) becomes:

$$\forall (s, y_n) \in \mathcal{S} \times \hat{\mathcal{C}}^N, \qquad \hat{\mathscr{B}}(\hat{D}_\nu^N)(s, y_n) = \inf_{q \in \mathcal{Q}} J^*(s, y_n; q) + \delta \sum_{s' \in \mathcal{S}} D_\nu^N(s', y'(s'))\pi(s'|s). \tag{A.4}$$

The objective in (A.4) is convex (even if the underlying problem is not), but it is not smooth.[32] There are many optimization procedures for non-smooth, convex dual problems (e.g. subgradient algorithms, cutting plane algorithms and so forth[33]). These may be used to solve the problems (A.4). An alternative approach developed by Necoara and Suykens (2008) is to smooth the dual problem through the addition of strongly concave (prox) functions to the objective in (A.2) (and, if necessary, the objective $J$ in the inner sup problems). In our calculations, we follow Necoara and Suykens (2008) by adding terms $c_v\|r\|^2$ to the objective in (A.2) and allowing $c_v \to 0$ with successive iterations. We use the optimizer `SNOPT` to solve these (smoothed) optimizations.

# B The Limited Commitment Example

This section collects details of and extensions to the results of Section 2.

[29] It clearly converges at all points in $\hat{\mathcal{C}}^\infty$. Choose a point $y \in \mathcal{C}$. Let $\{y_r^1\}$ and $\{y_r^2\}$ be two sequences in $\cup_N \hat{\mathcal{C}}^N$ converging to $y$ and such that $y = \lambda_r a_r y_r^1 + (1-\lambda_r) b_r y_r^2$, with $\lambda_r \in (0,1)$, $a_r, b_r \in \mathbb{R}_+$ and $a_r, b_r \downarrow 1$, i.e. $a_r y_r^1$ and $b_r y_r^2$ lie either side of $y$ on the tangent to $\mathcal{C}$ passing through $y$. There is a sequence $\{N_r\}$ such that $\hat{D}^{N_r}(s, y_r^1) = D(s, y_r^1)$ and $\hat{D}^{N_r}(s, y_r^2) = D(s, y_r^2)$. By the sub-linearity of $D(s,\cdot)$ and each $\hat{D}^{N_r}(s,\cdot)$, we have $D(s,y) \leq D^{N_r}(s,y) \leq \lambda_r \hat{D}^{N_r}(s, a_r y_r^1) + (1-\lambda_r)\hat{D}^{N_r}(s, b_r y_r^2) = \lambda_r a_r \hat{D}^{N_r}(s, y_r^1) + (1-\lambda_r) b_r \hat{D}^{N_r}(s, y_r^2) = \lambda_r a_r D(s, y_r^1) + (1-\lambda_r) b_r D(s, y_r^2)$. Since $D(s,\cdot)$ is real-valued and convex, it is continuous at all interior points; by linear homogeneity, $D(s,\cdot)$ is continuous throughout $\mathcal{Y}$, hence $y_r^1 \to y$, $y_r^2 \to y$ and $a_r, b_r \downarrow 1$, it follows that the last term in the string of inequalities converges to $D(s,y)$. Thus, the sequence of functions converges pointwise on $\mathcal{C}$ and by the positive homogeneity of the functions on $\mathcal{Y}$ as well.

[30] See Chapter 2, Aliprantis and Border (2006) for a statement and proof of Dini's theorem.

[31] With some simplification if the problem is quasilinear.

[32] $J^*$ may be smooth if it is obtained from component problems with strictly concave objectives and concave constraint functions. However, our approximation procedure implies that $\hat{D}^N$ is non-smooth.

[33] Good references for such methods include Bertsekas (2003) and Ruszczyński (2006).

## B.1 Numerical Method

The numerical approach is outlined in Appendix A. We apply this approach to the transformed version of the problem described in Section 2. Inner maximizations are solved analytically when possible and the indirect payoffs $J^*(s,y;q)$ are substituted directly into (5). On the $\nu$-th application of the Bellman operator the following family of minimizations is solved, for $s \in \mathcal{S}$ and $y_n \in \hat{\mathcal{C}}^N_+$,

$$D^N_{\nu+1}(s,y_n) = \inf_q J^*(s,y_n;q) + \delta \sum_{s' \in \mathcal{S}} D^N_\nu(s',y'(s'))\pi(s'|s), \tag{B.1}$$

where $J^*$ is defined as in (7) and $\hat{\mathcal{C}}^N_+$ is a finite subset of $\mathcal{C}_+ = \mathcal{C} \cap \mathbb{R}^{n_I}_+$ and is represented as a grid of points in spherical coordinates (either $\{\phi_n\} \subset [0,\pi)$ if $n_I = 2$ or $\{\phi^1_n, \phi^2_n\} \in [0,\pi)^2$ if $n_I = 3$). In the $n_I = 2$ case, the spherical coordinate gives the (Pareto) weights on agents one and two according to $y^1 = \cos\phi$ and $y^2 = \sin\phi$); in the $n_I = 3$ case, $\phi^1$ gives the weight on agent 1 relative to agents 2 and 3, while $\phi^2$ gives the weight on agent 2 relative to 3.

## B.2 Construction of Bounding Functions

Application of Theorem 2 (and the proof that $\mathscr{B}$ is a contraction) requires the definition of bounding value functions $\underline{\underline{D}}$, $\underline{D}$ and $\overline{D}$. In this subsection bounding functions satisfying Assumption 3 are obtained for the transformed limited commitment problem. Let $\overline{v} := \max_{\mathcal{S}} \frac{\gamma(s)^{1-\sigma}}{1-\sigma}$ and $\underline{v} = \frac{(\underline{a}/2)^{1-\sigma}}{1-\sigma}$, where $\underline{a}$ is the non-negative (positive if $\sigma > 1$) lower bound on agent consumptions. Assume an $\tilde{a} \in \mathcal{A}^{n_S}$ and let $\xi > 0$ be such that for each $s \in \mathcal{S}$, $\gamma(s) > \sum_{i \in \mathcal{I}} \tilde{a}^i(s)$, and for each $s \in \mathcal{S}$ and $i \in \mathcal{I}$,

$$\overline{v} - \xi \geq \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \delta\overline{v} > \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma}\Big\{\sum_{s' \in S}[(1-\sigma)w^i(s')]^\theta \pi(s'|s)\Big\}^{\frac{1}{\theta}} > w^i(s) + \xi, \tag{B.2}$$

where $\theta := \frac{1-\rho}{1-\sigma}$. Set:

$$\overline{D}(s,y) = \sum_{i \in \mathcal{I}} y^i \varphi^i(y^i,s), \qquad \varphi^i(y^i,s) = \begin{cases} \overline{v} & \text{if } \quad y^i \geq 0 \\ \underline{v} & \text{else if } y^i < 0, \end{cases}$$

and

$$\underline{\underline{D}}(s,y) = \sum_{i \in \mathcal{I}} \{y^i \psi^i(y^i,s) + |y^i|\xi\}, \qquad \psi^i(y^i,s) = \begin{cases} w^i(s) & \text{if } \quad y^i \geq 0 \\ \overline{v} & \text{else if } y^i < 0. \end{cases}$$

It is immediate that $\overline{D}$ is continuous and that $\underline{\underline{D}}$ is continuous and positively homogeneous. It is also easy to see that $\overline{D} \geq D^*$: while the supremum operations defining $D^*$ are restricted by feasibility and default constraints, $\overline{D}$ gives the maximal weighted payoff subject only to the restriction that payoffs remain within $\mathcal{V}$. In addition, it follows from (B.2) that $\underline{\underline{D}} \leq D^*$.

We verify that for $\varepsilon > 0$, $\mathscr{B}(\overline{D}) \leq \overline{D}$ and $\mathscr{B}(\underline{\underline{D}}) > \underline{\underline{D}} + \varepsilon$ on $\mathcal{S} \times \mathcal{C}$. $\mathscr{B}(D)$ is given by, for all

$(s,y) \in \mathcal{S} \times \mathcal{Y}$,

$$\mathscr{B}(D)(s,y) = \inf_{\mathcal{Q}} \sup_{\mathcal{P}} \sum_{i\in\mathcal{I}} (y^i + m^i) \left\{ \frac{1-\delta}{1-\sigma}[a^i]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s'\in\mathcal{S}} [(1-\sigma)v'^{,i}(s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \right\} - \sum_{i\in\mathcal{I}} m^i w^i(s)$$
$$- m^{n_I+1} \left( \sum_{i\in\mathcal{I}} a^i - \gamma(s) \right) - \delta \sum_{s'\in\mathcal{S}} \sum_{i\in\mathcal{I}} y'^{,i}(s') v'^{,i}(s') \pi(s'|s) + \delta \sum_{s'\in\mathcal{S}} D(s', y'(s')) \pi(s'|s). \tag{B.3}$$

Setting $D = \overline{D}$, using the definition of $\underline{v}$ and $\overline{v}$ and noting that the dual variables $(m, y')$ can always be chosen equal to 0 in the infimum and $\overline{D}(s,0) = 0$, we have $\mathscr{B}(\overline{D})(s,y) \leq \overline{D}(s,y)$. Finally, we show $\mathscr{B}(\underline{\underline{D}}) > \underline{\underline{D}} + \varepsilon$ on $\mathcal{S} \times \mathcal{C}$. Given $y' = \{y'^{,i}(s')\}$, define $\psi(y') = \{\psi^i(y'^{,i}(s'), s')\}_{(i,s')\in\mathcal{I}\times\mathcal{S}}$. Setting $D = \underline{\underline{D}}$ and noting that for any $s$ and choice of $(m, y')$, the pair $(\tilde{a}(s), \psi(y'))$ is a feasible choice for the supremum with respect to both the resource and no default constraints, we have:

$$\mathscr{B}(\underline{\underline{D}})(s,y) \geq \inf_{q\in\mathcal{Q}} \sum_{i\in\mathcal{I}} (y^i + m^i) \left\{ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s'\in\mathcal{S}} [(1-\sigma)\psi^i(y'^{,i}(s'), s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \right\}$$
$$- \sum_{i\in\mathcal{I}} m^i w^i(s) - m^{n_I+1} \left( \sum_{i\in\mathcal{I}} \tilde{a}^i(s) - \gamma(s) \right)$$
$$\geq \inf_{y'} \sum_{i\in\mathcal{I}} y^i \left\{ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s'\in\mathcal{S}} [(1-\sigma)\psi^i(y'^{,i}(s'), s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \right\}$$

where the first inequality follows from the replacement of the sup with the choices $(\tilde{a}(s), \psi(y'))$. The second inequality uses the feasibility of these choices, i.e $\gamma(s) - \sum_{i\in\mathcal{I}} \tilde{a}^i(s) \geq 0$ and $\frac{1-\delta}{1-\sigma}(\tilde{a}^i(s))^{1-\sigma} + \frac{\delta}{1-\sigma} \left( \sum_{s'\in\mathcal{S}} [(1-\sigma)\psi^i(y'^{,i}(s'), s')]^{\theta} \pi(s'|s) \right)^{\frac{1}{\theta}} \geq w^i(s)$, and thus the fact that $m = 0$ is minimising. Now, using the additive separability across agents, each agent $i$ can be analyzed separately. If $y^i \geq 0$, then

$$\inf_{y'^{,i}(\cdot)} y^i \left\{ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s'\in\mathcal{S}} [(1-\sigma)\psi^i(y'^{,i}(s'), s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \right\}$$
$$\geq y^i \left\{ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s'\in\mathcal{S}} [(1-\sigma)w^i(s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \right\} \geq y^i(w^i(s) + \xi),$$

with the inequality strict if $y^i > 0$. Note that if $\sigma > 1$, then $0 \leq (1-\sigma)\psi^i(y'^{,i}(s'), s') \leq (1-\sigma)w^i(s')$, $\forall\, i, s$, and hence $\left\{ \sum_{\mathcal{S}} [(1-\sigma)\psi^i(y'^{,i}(s'), s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \leq \left\{ \sum_{\mathcal{S}} [(1-\sigma)w^i(s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}}$ which implies the above inequality when both sides are multiplied by the negative number $\frac{\delta}{1-\sigma}$. The last inequality holds by our assumption on $\tilde{a}^i(s)$. Similarly, if $y^i < 0$, then

$$\inf_{y'^{,i}} y^i \left\{ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s'\in\mathcal{S}} [(1-\sigma)\psi^i(y'^{,i}(s'), s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} \right\}$$
$$\geq y^i \left\{ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \delta\overline{v} \right\} > y^i(\overline{v} - \xi).$$

Consider now the auxiliary function:

$$\tilde{D}(s,y) = \sum_{i \in \mathcal{I}} \{ y^i \tilde{\psi}^i(y^i, s) + |y^i| \xi \},$$

where:

$$\tilde{\psi}^i(y^i, s) = \begin{cases} \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \frac{\delta}{1-\sigma} \left\{ \sum_{s' \in \mathcal{S}} [(1-\sigma) w^i(s')]^{\theta} \pi(s'|s) \right\}^{\frac{1}{\theta}} & \text{if} \quad y^i \geq 0 \\ \frac{1-\delta}{1-\sigma}[\tilde{a}^i(s)]^{1-\sigma} + \delta \overline{v} & \text{else if} \;\; y^i < 0. \end{cases}$$

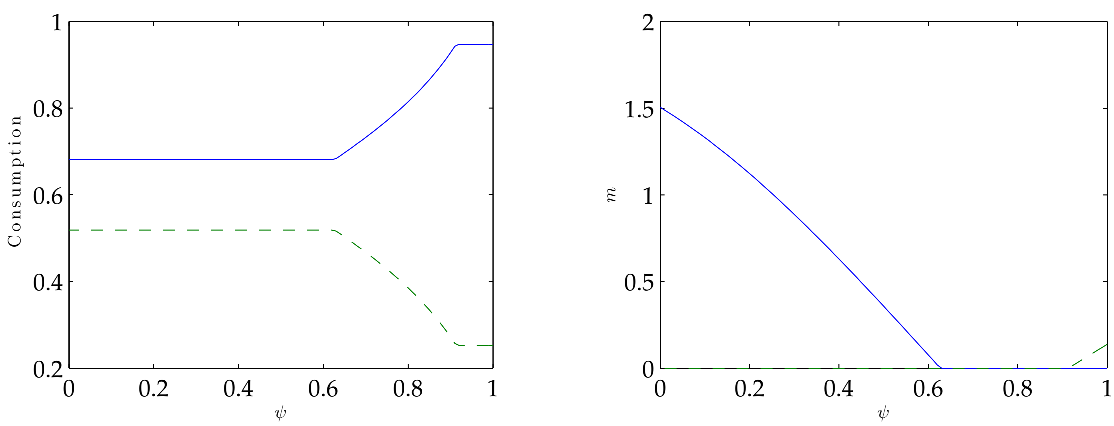
$\tilde{D}(s,y)$ is clearly a continuous function. Our preceding derivations show that for all $(s,y) \in \mathcal{S} \times \mathcal{C}$, $\mathscr{B}(\underline{\underline{D}})(s,y) \geq \tilde{D}(s,y) > \underline{\underline{D}}(s,y)$. The continuity of each $\underline{\underline{D}}(s,\cdot)$ and $\tilde{D}(s,\cdot)$ and the compactness of $\mathcal{C}$ then implies that there is a $\varepsilon > 0$ such that for all $(s,y) \in \mathcal{S} \times \mathcal{C}$, $\mathscr{B}(\underline{\underline{D}})(s,y) \geq \tilde{D}(s,y) > \underline{\underline{D}}(s,y) + \varepsilon$, and hence $\mathscr{B}(\underline{\underline{D}})(s,y) \geq \underline{\underline{D}}(s,y) + \varepsilon$ as required.

## B.3 Numerical Calculations

This subsection reports additional calculations for the two and three agent cases.

### B.3.1 Two Agents

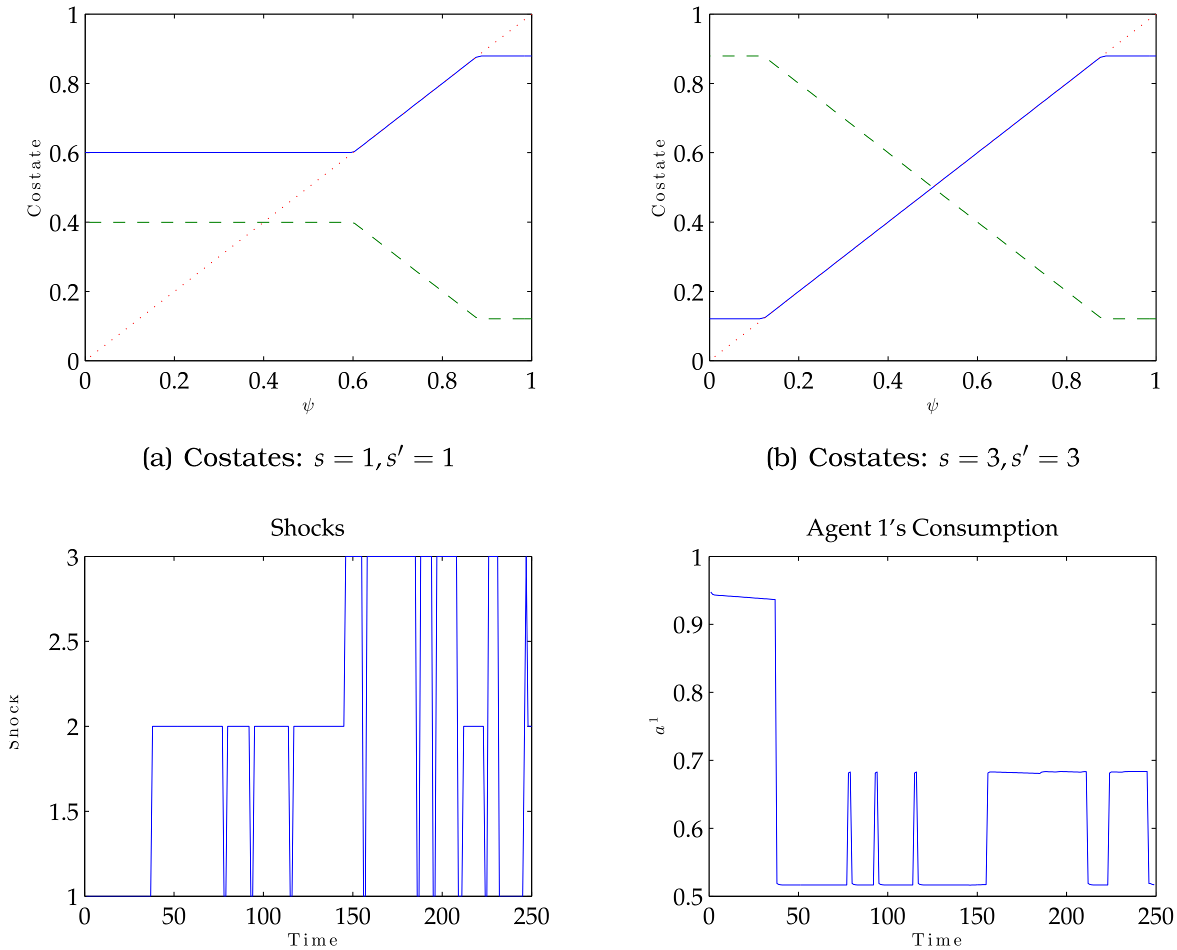
In the main text results for the case $\sigma = 1.5$ and $\rho = 5$ are reported. Values of $\rho$ that are smaller and closer to $\sigma$ result in more volatility in consumption and a muting of the dynamics that occur when none of the no default constraints are binding. In the limiting case of expected utility preferences $\rho = \sigma$ these dynamics disappear completely. The following figures illustrate for the case $\sigma = \rho = 1.5$.




(a) Consumption

(b) Multipliers

Figure 3: *Solid lines give agent 1's policy, dashed lines agent 2's policy. Policies are given as functions of agent 1's normalized costate and for* $s = 1$. *Panel a shows agent consumption; panel b multipliers on the commitment constraints.*




(a) Costates: $s = 1, s' = 1$

(b) Costates: $s = 3, s' = 3$

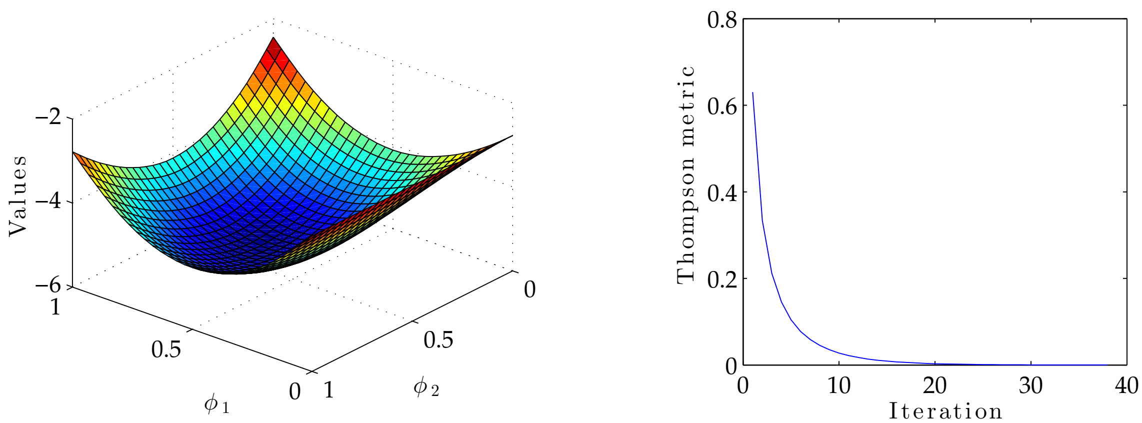
(c) Simulated state shocks

(d) Agent 1's simulated consumption

Figure 4: *Panels a and b show the (normalized) costates associated with remaining in state 1 and state 3, respectively, as a function of the initial value of the costate $\psi$. Solid lines give agent 1's policy, dashed lines agent 2's policy. The 45 degree line is illustrated with dots. Policies are given as functions of agent 1's normalized costate and for $s = 1$. Panels c and d illustate a 250 period simulation of agent 1's consumption which displays the usual 'memoryless' property. When a change of state $s$ leads a different agent's no default constraint to bind, an adjustment in agent 1's consumption occurs. Thus, this agent's consumption bounces between two (history independent) levels; it remains constant whenever the economy remains in the same state or transitions into $s = 3$.*

### B.3.2 Three Agents

We now show computed results from a 3 agent economy. The preference parameters are set to $\sigma = 1.5$, $\rho = 5$ and $\delta = 0.8$. In shock state $s$ agent $s$ has an outside option equal to the utility from a steady endowment stream of 40% of the total endowment, agents $s' \neq s$ have outside options equal to the utility from a steady endowment stream of 10% of the total endowment. These values preclude full risk sharing. It is convenient to plot policies as functions of spherical coordinates $(\phi_1, \phi_2)$. The corresponding costates or "Pareto weights" on agents are $y^1 = \cos\phi_1$, $y^2 = \sin\phi_1 \cos\phi_2$ and $y^3 = \sin\phi_1 \sin\phi_2$. Thus, higher values of $\phi_1$ imply less weight on agent 1's utility and more on agent 2 and 3's, while higher values of $\phi_2$ imply less weight on agent 2's utility and more on agent 3's (with no change in the weight on agent 1's). Figure 5 shows the computed optimal dual value function and the Thompson




(a) Value function

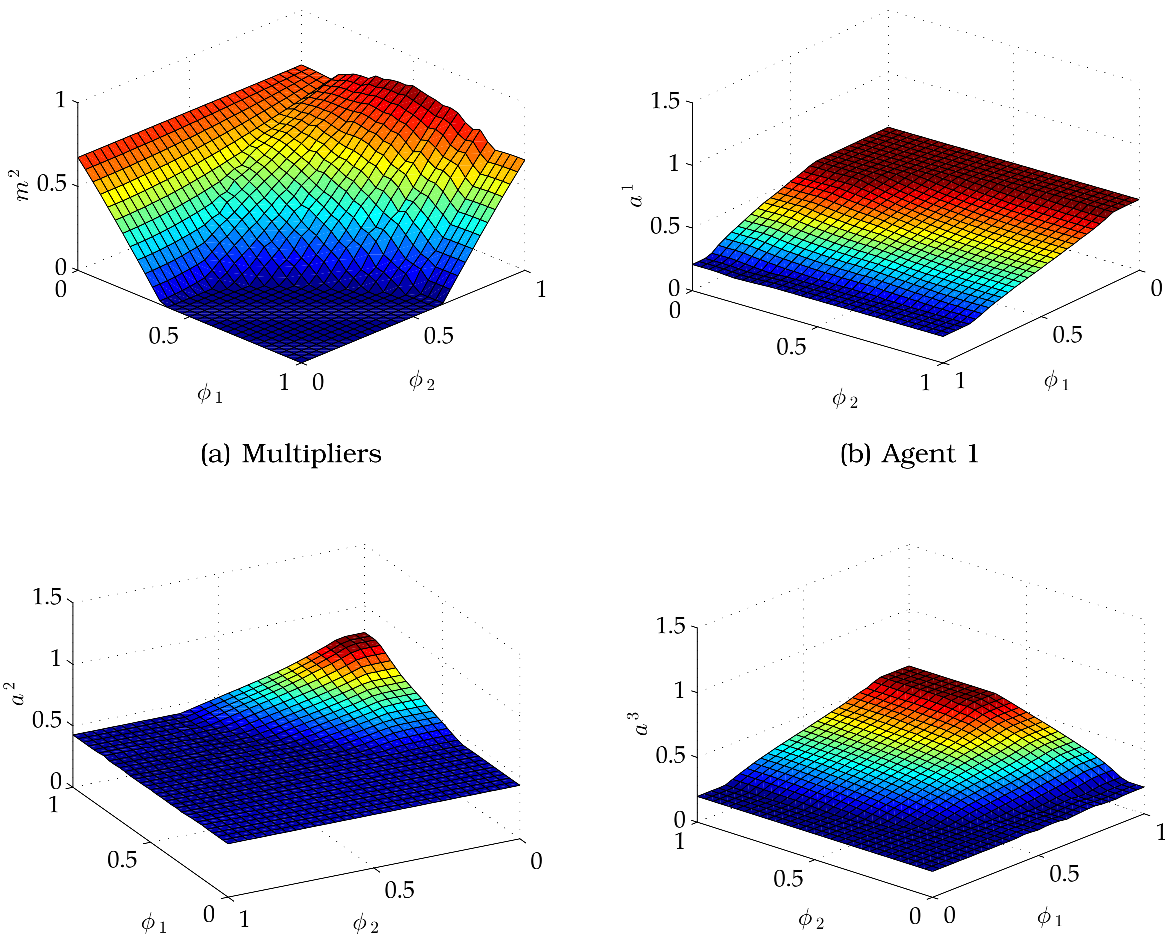
(b) Convergence

Figure 5: *Panel a shows the planner's dual value function in shock state* $s = 1$ *at the terminal iteration. Panel b shows the Thompson metric distance between iterates.*

metric distance between successive iterates - illustrating the geometric convergence of the value iteration.

Figure 6 shows calculated policy functions. Panel a of the figure displays the optimal multiplier $m^2$ for agent 2 in shock state 2 (in which agent 2 has a high outside option and agents 1 and 3 low ones). The weight $y^2$ is small and agent 2's incentive multiplier correspondingly large when the spherical coordinates $\phi_1$ and $\phi_2$ are, respectively, small and large. Then, the combination of a low costate and a high outside option imply that additional consumption must be given to agent 2 now and in the future to keep her inside the risk sharing arrangement. Panels (b), (c), and (d) of Figure 6 show the consumption of agents 1, 2 and 3 as functions of the costates again given $s = 2$. Each agent's consumption rises in areas of the state space corresponding to a higher (Pareto) weighting. Agent 2's consumption is sustained above 0.5 by her binding incentive constraint, while the consumption of agents 1 and 3 decreases towards 0.2 as their (Pareto) weights decrease to zero.

Panels a and b of Figure 7 show "quiver plots" indicating the direction in which the spherical coordinates describing costates are updated if the economy remains in shock state $s = 1$ (panel a) and $s = 2$ (panel b). Consider panel a. Recall that high values of $\phi_1$ imply that the costate on agent 1 is low. Agent 1's no default constraint is then binding, her multiplier is positive and her costate is raised. The spherical coordinate $\phi_1$ is correspondingly reduced




(a) Multipliers

(b) Agent 1

(c) Agent 2

(d) Agent 3

Figure 6: *Figures are drawn on a domain of costates expressed in spherical coordinates. Higher values of $\phi_1$ place more weight on agents 2 and 3, higher values of $\phi_2$ place more weight on agent 3. Panel a displays the incentive multiplier for agent 2 in shock $s = 2$. The remaining 3 panels display the consumption of agents in shock $s = 2$.*

(placing less weight on agents 2 and 3 and more on agent 1). This is indicated by a left pointing arrow at high $\phi_1$ values (on the right hand side of the plot). Low values for $\phi_1$ and high values for $\phi_2$ imply that the costate on agent 2 is low. If it is low enough, then even in state 1 (when agent 2's outside option is low), agent 2's outside option binds. Hence, agent 2's costate is increased. Spherical coordinate $\phi_1$ is correspondingly increased and $\phi_2$ decreased and the arrows in the top left hand corner of the plot point down and inwards. Similar reasoning holds with respect to the bottom left hand corner of the plot, where agent 3's costate is low and outside option binds and the arrows point up and inwards. In the dotted region in the center left of the plot no incentive constraints bind. Here very small adjustments to costates occur that stem from the early resolution of uncertainty structure of preferences and the force for equality that it imparts. Panel b shows the adjustments in spherical coordinates when the economy remains in shock state $s = 2$ and agent 2's outside option is large. It has an analogous interpretation to panel a.

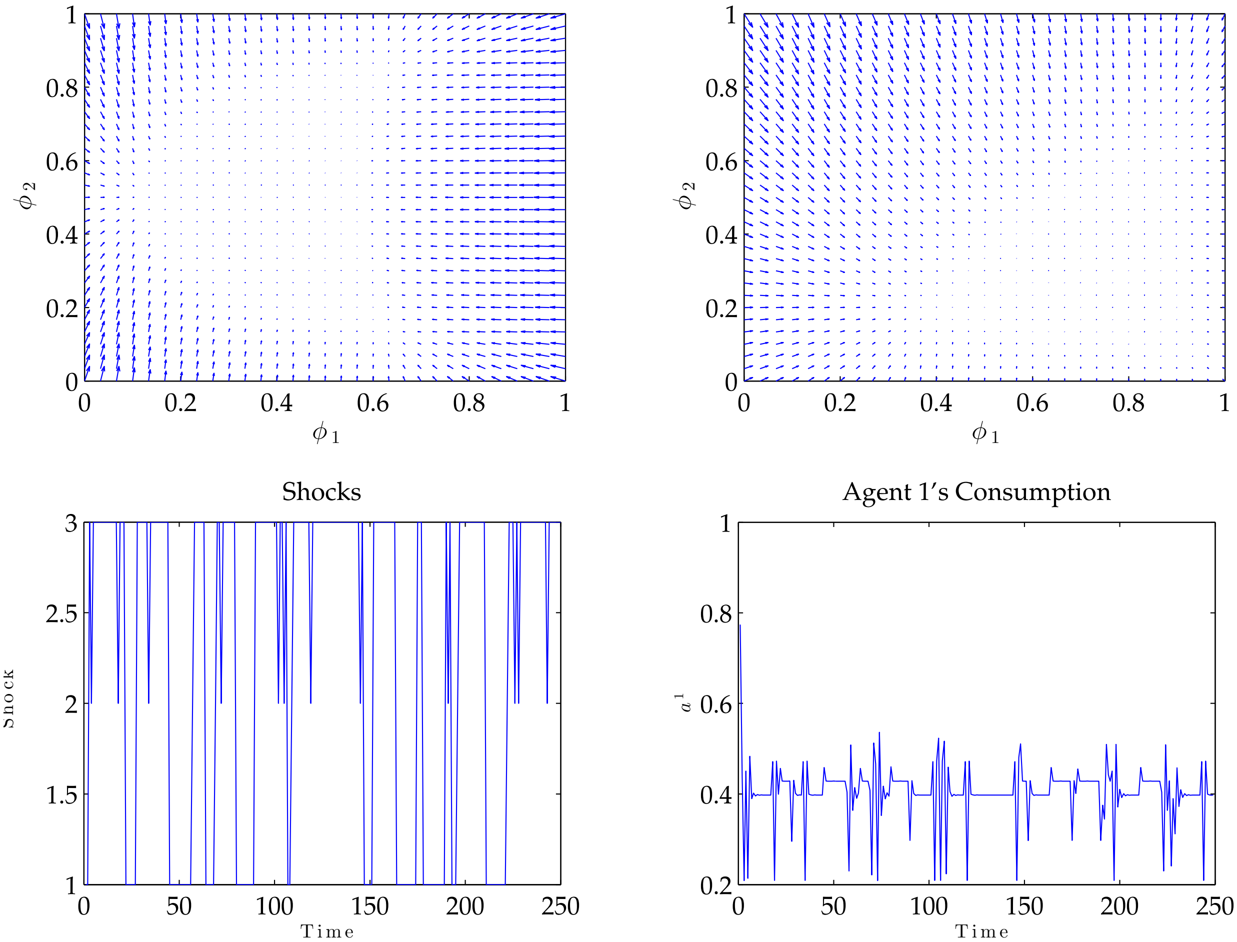



Figure 7: *Policy functions and simulations. Panels a and b show a quiver plot for the costate policy functions associated with remaining in states* $s = 1$ *and* $s = 2$ *respectively. The arrows in the plots indicate the direction in which the spherical coordinates describing costates are updated. Panels c and d show simulations of shocks and of the consumption of agent 1.*

Panel c and d show a simulation of shocks and agent 1's consumption. Consumption remains relatively stable if the economy persists in a given shock state. But large adjustments occur when a transition into or out of shock $s = 1$ occurs (and agent 1's outside option abruptly changes relative to the other agents' outside options).

## B.4 Primal Formulation

This section considers the primal approach to solving the limited commitment problem in Section 2 (with untransformed utilities). Assume that $\lambda^1 > 0$. Recall the definitions in the main text of the following objects: $\tilde{\mathcal{I}} := \{2, 3, \ldots, n_I\}$, $\tilde{\mathcal{V}} := \times_{i \in \tilde{\mathcal{I}}} [\underline{a}^i, \overline{a}^i]$, $\tilde{\mathcal{X}}^*(s) = \{\tilde{v} : \exists v^1 \text{ with } (v^1, \tilde{v}) \in \mathcal{X}^*(s)\}$ and $\tilde{P}^*(s, \tilde{v}) = \sup\{v^1 : (v^1, \tilde{v}) \in \mathcal{X}^*(s)\}$. Suppose the sets $\{\mathcal{X}^*(s)\}_{s \in \mathcal{S}}$ of feasible continuation promise values are unknown. The pair $(\tilde{\mathcal{X}}^*, \tilde{P}^*)$ can be computed as a fixed point of the following Bellman-like relation:

$$\forall s \in \mathcal{S}, \quad \tilde{\mathcal{X}}^*(s) = \{\tilde{v} \in \tilde{\mathcal{V}} : \tilde{\Gamma}(s, \tilde{v}; \tilde{\mathcal{X}}^*, \tilde{P}^*) \neq \emptyset\} \tag{B.4}$$

$$\forall (s, \tilde{v}) \in \text{Graph } \tilde{\mathcal{X}}^*, \quad \tilde{P}^*(s, \tilde{v}) = \sup_{\tilde{\Gamma}(s, \tilde{v}; \tilde{\mathcal{X}}^*, \tilde{P}^*)} \left\{ (1-\delta)[a^1]^{1-\sigma} + \delta \Big\{ \sum_{s' \in \mathcal{S}} \left[\tilde{P}^*(s', \tilde{v}'(s'))\right]^{1-\rho} \pi(s'|s) \Big\}^{\frac{1-\sigma}{1-\rho}} \right\}^{\frac{1}{1-\sigma}} \tag{B.5}$$

where:

$$\tilde{\Gamma}(s, \tilde{v}; \tilde{\mathcal{X}}^*, \tilde{P}^*) := \left\{ (a, \tilde{v}') \in \mathcal{A} \times \tilde{\mathcal{V}}^{n_S} \;\middle|\; \begin{array}{l} \forall i \in \tilde{\mathcal{I}},\ \tilde{v}^i(s) = \left[(1-\delta)[a^i]^{1-\sigma} + \delta \Big\{ \sum_{\mathcal{S}} \left[\tilde{v}'^{,i}(s')\right]^{1-\rho} \pi(s'|s) \Big\}^{\frac{1-\sigma}{1-\rho}}\right]^{\frac{1}{1-\sigma}} \\ H[s, a, \{\tilde{P}^*(s', \tilde{v}'(s')), \tilde{v}(s')\}_{s' \in \mathcal{S}}] \geq 0 \\ \text{and } \forall s', \quad \tilde{v}'(s') \in \tilde{\mathcal{X}}^*(s'). \end{array} \right\}$$

and

$$H[s, a, \{\tilde{P}^*(s', \tilde{v}'(s')), \tilde{v}(s')\}_{s' \in \mathcal{S}}] = \begin{bmatrix} \left[(1-\delta)[a^1]^{1-\sigma} + \delta \Big\{ \sum_{\mathcal{S}} \left[\tilde{P}^*(s', \tilde{v}'(s'))\right]^{1-\rho} \pi(s'|s) \Big\}^{\frac{1-\sigma}{1-\rho}}\right]^{\frac{1}{1-\sigma}} - w^1(s) \\ \left\{ \left[(1-\delta)[a^i]^{1-\sigma} + \delta \Big\{ \sum_{\mathcal{S}} \left[\tilde{v}'^{,i}(s')\right]^{1-\rho} \pi(s'|s) \Big\}^{\frac{1-\sigma}{1-\rho}}\right]^{\frac{1}{1-\sigma}} - w^i(s) \right\}_{i \in \tilde{\mathcal{I}}} \\ \gamma(s) - \sum_{i \in \mathcal{I}} a^i \end{bmatrix}.$$

The pair $(\tilde{\mathcal{X}}^*, \tilde{P}^*)$ may be calculated via a monotone iteration that jointly updates a state space-value function pair at each step. Specifically, this involves selecting a domain $\tilde{\mathcal{X}}_1 \supset \tilde{\mathcal{X}}^*$ and a value function $\tilde{P}_1 : \text{Graph } \tilde{\mathcal{X}}_1 \to \mathbb{R}$, such that for all $(s, \tilde{v}) \in \text{Graph } \tilde{\mathcal{X}}^*$, $\tilde{P}_1(s, \tilde{v}) \geq \tilde{P}^*(s, \tilde{v})$ and successively updating the pair according to, $\forall s \in \mathcal{S}$, $\tilde{\mathcal{X}}_{n+1}(s) = \{\tilde{v} \in \tilde{\mathcal{V}} : \tilde{\Gamma}(s, \tilde{v}; \tilde{\mathcal{X}}_n, \tilde{P}_n) \neq \emptyset\}$ and $\forall (s, \tilde{v}) \in \text{Graph } \tilde{\mathcal{X}}_{n+1}$,

$$\tilde{P}_{n+1}(s, \tilde{v}) = \sup_{\tilde{\Gamma}(s, \tilde{v}; \tilde{\mathcal{X}}_n, \tilde{P}_n)} \left\{ (1-\delta)[a^1]^{1-\sigma} + \delta \Big\{ \sum_{s' \in \mathcal{S}} \left[\tilde{P}_n(s', \tilde{v}'(s'))\right]^{1-\rho} \pi(s'|s) \Big\}^{\frac{1-\sigma}{1-\rho}} \right\}^{\frac{1}{1-\sigma}}.$$

Rustichini (1998a) gives sufficient conditions for monotone convergence of the sequence $\{\tilde{\mathcal{X}}_n, \tilde{P}_n\}$ to $(\tilde{\mathcal{X}}^*, \tilde{P}^*)$ in a related problem. Numerical implementation of the procedure requires approximation of the sequence of correspondences $\{\tilde{\mathcal{X}}_n\}$. When $n_I = 2$ this is practical as each $\tilde{\mathcal{X}}_n(s)$ is an interval (for a given $s$) and is, thus, summarized by its two end points. More generally, for $n_I > 2$, it remains difficult. A recent promising approach is suggested by Cai et. al. (2016) who, in a different setting, relax the incentive constraints with penalty functions and use adaptive splines to prevent penalties proliferating.

# C Proof of Proposition 8

We first establish existence of a saddle point with summable multipliers for an abstract problem with inequality constraints. We then relate this problem to a modified version of (P) (called (MP)). We associate a Lagrangian with (MP) and show that each primal plan solving (MP) is part of a saddle point with a minimizing summable multiplier. Finally, we show that each solution to (P) defines a solution to (MP) and use the minimizing multiplier from (MP) to construct a minimizing multiplier and, hence, saddle point for (P).

**The Abstract Problem** Consider:

$$\sup\ f(x)\ \text{ s.t. }\ g(x) \geq 0, \tag{AP}$$

where $f : \ell^\infty \to \mathbb{R}$ and $g : \ell^\infty \to \ell^\infty$, with $g(x) = \{g_r(x)\}_{r=1}^\infty$ and each $g_r : \ell^\infty \to \mathbb{R}$. Associate the Lagrangian $\mathcal{L} : \ell^\infty \times \ell_+^{\infty,\star} \to \mathbb{R}$ with (AP), where:[34]

$$\mathcal{L}(x, \lambda) := f(x) + \langle \lambda, g(x) \rangle.$$

Given Assumption 7 below and the existence of a solution $x^*$ to (AP), Theorem C.1 establishes the existence of a saddle point.

**Assumption 7.** *(C) Concavity: $f$ and $g$ are concave. (S) Slater Condition: There is an $\hat{x}$ such that* $\inf_r g_r(\hat{x}) > 0$.

**Theorem C.1** (Saddle point existence for the abstract problem)**.**

*(i) If $x^*$ is feasible for* (AP) *and solves* $\max_{x \in \ell^\infty} \mathcal{L}(x, \lambda^*)$ *with $\lambda^* \in \ell^{\infty,\star}$ such that $\lambda^* \geq 0$ and $\langle \lambda^*, g(x^*) \rangle = 0$, then $x^*$ solves* (AP).

*(ii) If Assumption 7 holds and $x^*$ solves* (AP)*, then there is a $\lambda^* \in \ell^{\infty,\star}$ such that $\lambda^* \geq 0$ and $\langle \lambda^*, g(x^*) \rangle = 0$. Moreover, $x^*$ solves* $\max_{x \in \ell^\infty} \mathcal{L}(x, \lambda^*)$.

*(iii) If (a) $\lambda^* \geq 0$ and (b) $\langle \lambda^*, g(x^*) \rangle = 0$, then $x^*$ is feasible for* (AP) *if and only if:*

$$\mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda) \quad \forall \lambda \in \ell^{\infty,\star} \text{ with } \lambda \geq 0.$$

*(iv) If Assumption 7 holds and $x^*$ solves* (AP)*, then $\mathcal{L}$ has a saddle point in $\ell^\infty \times \ell_+^{\infty,\star}$.*

*Proof of Theorem C.1.* (i) If $x$ is feasible for (AP), then all nonlinear constraints must hold with inequality and $\langle \lambda^*, g(x) \rangle \geq 0$. Thus, for any feasible choice $x$: $f(x^*) = f(x^*) + \langle \lambda^*, g(x^*) \rangle = \mathcal{L}(x^*, \lambda^*) \geq \mathcal{L}(x, \lambda^*) = f(x) + \langle \lambda^*, g(x) \rangle \geq f(x)$. (ii) This proof is standard. To save space, we do not report it here, but refer the interested reader to Luenberger (1969), Theorem 1, page 217-218. (iii) ($\Rightarrow$) If $x^*$ is feasible, then $g(x^*) \geq 0$. Hence, for all $\lambda \geq 0$, $\langle \lambda, g(x^*) \rangle \geq 0$ and 0 is the infimum of $\langle \lambda, g(x^*) \rangle$ over $\lambda \geq 0$. Hence, from (a) and (b), $f(x^*) + \langle \lambda^*, g(x^*) \rangle \leq f(x^*) + \langle \lambda, g(x^*) \rangle$, $\forall \lambda \in \ell_+^{\infty,\star}$. (iii) ($\Leftarrow$) Suppose for a given $x^*$ we have $\langle \lambda^*, g(x^*) \rangle = 0$, and $\lambda^* \in \arg\min_{\lambda \in \ell_+^{\infty,\star}} \mathcal{L}(x^*, \lambda)$. Suppose for some $r$, $g_r(x^*) < 0$. Let $\hat{\lambda} = \lambda^* + \chi$, with $\chi(v) = 1$ if $v = r$ and 0 otherwise. Then

[34] Define $\lambda \in \ell_+^{\infty,*}$ if $\lambda \in \ell^{\infty,*}$ and $\lambda \geq 0$, where $\lambda \geq 0 \iff \langle \lambda, y \rangle \geq 0\ \forall y \in \ell^\infty, y \geq 0$.

$\mathcal{L}(x^*, \hat{\lambda}) < \mathcal{L}(x^*, \lambda^*)$ contradicting the fact that $\lambda^*$ is a minimizer. Hence, it must be that $g(x^*) \geq 0$ (i.e., $x^*$ is feasible). (iv) It immediately follows from (ii) and (iii), that if Assumption 7 holds and $x^*$ solves (AP), then $\mathcal{L}$ has a saddle point in $\ell^\infty \times \ell_+^{\infty,\star}$. □

We now refine Theorem C.1 and give conditions such that the minimizing multiplier $\lambda^*$ in Theorem C.1(ii) lies in $\ell^1 \subset \ell^{\infty,\star}$. By Yosida and Hewitt (1952), Theorems 1.22 and 1.24, $\ell^{\infty,\star}$ admits the decomposition $\ell^{\infty,\star} = \ell^1 + \ell^s$ with $\ell^s$ the set of *pure finitely additive* components. Assumption 8 below ensures summability of the minimizing multiplier.[35] In the assumption and throughout the proof, for a pair $x$ and $y \in \ell^\infty$, and $T \in \mathbb{N}$, let $x^T(x,y) := x_r$ if $r \leq T$ and $y_r$, if $r > T$.

**Assumption 8.** *(C) Continuity:* $\lim_{T\to\infty} f(x^T(x,y)) = f(x)$.
*(AN) Asymptotically non-anticipatory:* $\forall t$, $\lim_{T\to\infty} g_r(x^T(x,y)) = g_r(x)$.
*(AI) Asymptotically insensitive: for all* $N$, $\lim_{r\to\infty}[g_r(x^N(x,y)) - g_r(y)] = 0$.
*(B) Uniform Boundedness:* $\exists M$ *s.t. for all* $T$, $\sup_r \|g_r(x^T(x,y))\|_E \leq M$.[36]

**Theorem C.2.** *Suppose* $f$ *and* $g$ *satisfy Assumption 7 (S) and Assumption 8. If* $(x^*, \lambda^*) \in \ell^\infty \times \ell_+^{\infty,\star}$ *is a saddle point of* $\mathcal{L}$*, then* $(x^*, \lambda^*) \in \ell^\infty \times \ell_+^1$.

*Proof of Theorem C.2.* The proof uses two key lemmas.

**Lemma C.1.** *Given Assumption 8 (AI),* $\forall\ \lambda^s \in \ell^s$ *and* $N \geq 1$, $\langle \lambda^s, g(x^N(x,y))\rangle = \langle \lambda^s, g(y)\rangle$.

*Proof.* If $\lambda^s \in \ell^s$, then for all $z = \{z_r\} \in \ell^\infty$ with $\lim_{r\to\infty} z_r = 0$, we have $\langle \lambda^s, z\rangle = 0$. By Assumption 8 (AI), $\forall N$, $\lim_{r\to\infty}[g_r(x^N(x,y)) - g_r(y)] = 0$ and so $\forall N$, $\langle \lambda^s, [g(x^N(x,y)) - g(y)]\rangle = 0$ as required. □

**Lemma C.2.** *Given Assumption 8 (ANA) and (B),* $\forall\ \lambda^1 \in \ell^1$, $\lim_{T\to\infty}\langle \lambda^1, g(x^T(x,y))\rangle = \langle \lambda^1, g(x)\rangle$.

*Proof.* For all $T, N \in \mathbb{N}$, $\|\langle \lambda^1, g(x^T(x,y))\rangle - \langle \lambda^1, g(x)\rangle\|_E \leq \sum_{r=0}^{N} \|\lambda_r^1\|_E\ \|g_r(x^T(x,y)) - g_r(x)\|_E + \sup_r \|g_r(x^T(x,y)) - g_r(x)\|_E \sum_{r=N+1}^{\infty} \|\lambda_t^1\|_E$. From Assumption 8 (B), there is an $M > 0$ such that $\forall T$, $\sup_r \|g_r(x^T(x,y)) - g_r(x)\|_E \leq \bar{M} := M + \sup_r \|g_r(x)\|_E$. Since $\lambda^1 \in \ell^1$ for each $\varepsilon > 0$ there is an $N_0$ such that $\sum_{r=N_0+1}^{\infty} \|\lambda_r^1\|_E < \varepsilon/2\bar{M}$ , and so, from Assumption 8 (ANA), there is a $\bar{T}_r$ such that $\forall T \geq \bar{T}_r$, $\|\lambda_t^1\|_E\ \|g_r(x^T(x,y)) - g_r(x)\|_E < \varepsilon/2N_0$ . Hence, combining conditions, for all $T > \max_{r\leq N_0} \{\bar{T}_r\}$, $\|\langle \lambda^1, g(x^T(x,y))\rangle - \langle \lambda^1, g(x)\rangle\|_E < N_0\, \varepsilon/2N_0 + \bar{M}\, \varepsilon/2\bar{M} = \varepsilon$. Since $\varepsilon > 0$ was arbitrary this proves the result. □

We now conclude the proof of Theorem C.2. Since $(x^*, \lambda^*)$ is a saddle in $\ell^\infty \times \ell_+^{\infty,\star}$, we have for all $x \in \ell^\infty$, $\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*)$, and $\langle \lambda^*, g(x^*)\rangle = 0$. Let $\lambda^* = \lambda^1 + \lambda^s$. Since $\lambda^* \geq 0$ we have both $\langle \lambda^1, g(x^*)\rangle = 0$ and $\langle \lambda^s, g(x^*)\rangle = 0$. Since $x^*$ maximizes $\mathcal{L}(\cdot, \lambda^*)$ over $\ell^\infty$ and $\langle \lambda^*, g(x^*)\rangle = 0$:

$$f(x^*) \geq f(x^T(x^*, \hat{x})) + \langle \lambda^1, g(x^T(x^*, \hat{x}))\rangle + \langle \lambda^s, g(x^T(x^*, \hat{x}))\rangle. \tag{C.1}$$

Recall that if $\hat{x}$ is chosen to satisfy the Slater condition then $\inf_r g_r(\hat{x}) > 0$. Lemma C.1 and Lemma C.2 together imply that for $T \to \infty$ we have both: $\langle \lambda^1, g(x^T(x^*, \hat{x}))\rangle \to \langle \lambda^1, g(x^*)\rangle$ and $\langle \lambda^s, g(x^T(x^*, \hat{x}))\rangle \to \langle \lambda^s, g(\hat{x})\rangle$. Moreover, from Assumption 8 (C) we have $f(x^T(x^*, \hat{x})) \to f(x^*)$. If $\lambda^s \neq 0$ since $\lambda^s \geq 0$ taking limits in (C.1) we have the following contradiction: $f(x^*) \geq f(x^*) + \langle \lambda^s, g(\hat{x})\rangle > f(x^*)$. Thus, $\lambda^s = 0$. □

[35] Dechert (1982) introduced the terminology used in Assumption 8 (AN) and (AI). He shows summability of multipliers under slightly different assumptions. The proofs of Lemma C.1 and Lemma C.2 follow Le Van and Saglam (2004) who focus on variations to the deterministic model of optimal growth.

[36] The number $\|g_r(x)\|_E$ represents the Euclidean norm of the vector $g_r(x)$.

**Modified Problem** Let

$$\mathbf{P}^R = \{\mathbf{p} = (\mathbf{a}, \mathbf{k}, \mathbf{v}) \mid \quad \forall t \geq 0, \quad a_t : \mathcal{S}^t \to \mathcal{A}, \quad k_t : \mathcal{S}^{t-1} \to \mathcal{K}, \quad v_t^c : \mathcal{S}^t \to \mathcal{V}^c \}$$

denote the set of (modified) primal plans that exclude the quasilinear state variables. In addition, letting $b_0^l = \mathbb{I}_{n_l}$ (with $\mathbb{I}_{n_l}$ the $n_l$-identity matrix) and for all $\forall t, s^t, s' \quad b_{t+1}^l(s^t, s'|s_0) = b_t^l(s^t|s_0) B^l(s_t, s')$, define:

$$V^l(s_t, \mathbf{a}|s^t) := (1-\delta) \sum_{n=0}^{\infty} \delta^n \sum_{\mathcal{S}^n} b_n^l(s^{t+n}|s_t) u^l(s_{t+n}, a_{t+n}^l(s^{t+n})) \pi^n(s^{t+n}|s_t), \tag{C.2}$$

where the previous expression is well defined since Assumption 5(iv) implies that all entries of the diagonal matrices $B^l$ are bounded between $-1$ and $+1$. Relax the non-linear laws of motion for state variables and replace the quasilinear state variables using (C.2) to obtain the modified problem:

$$MP_0^* := \sup F[s_0, v_0^c, V^l(s_0, \mathbf{a})] \tag{MP}$$

subject to $\mathbf{p}^R \in \mathbf{P}^R$, $k_0 \leq \bar{k}$ and $\forall t,\ s^t$,

$$k_{t+1}(s^t) \leq W^k[k_t(s^{t-1}), s_t, a_t(s^t)], \tag{C.3}$$

$$v_t^c(s^t) \leq W^c[s_t, a_t(s^t), M^c[s_t, v_{t+1}^c(s^t, \cdot)]], \tag{C.4}$$

$$\text{and} \qquad H[k_t(s^{t-1}), s_t, a_t(s^t), v_{t+1}^c(s^t, \cdot), V^l(\cdot, \mathbf{a}|s^t, \cdot)] \geq 0.$$

We relate the original problem (P) to the modified problem (MP).

**Lemma C.3.** *Let Assumptions 2, 4 and 5 (iv) hold and let $\bar{k} \in \mathcal{K}^*(s_0)$. Then $MP_0^* = P_0^*$ and for any solution $\mathbf{p}^* = (\mathbf{p}^{R*}, \mathbf{v}^{l*})$ to (P), $\mathbf{p}^{R*}$ is a solution to (MP).*

*Proof.* Let $\mathbf{p} = (\mathbf{p}^R, \mathbf{v}^l)$ denote a feasible plan for (P), then, given Assumption 5 (iv), it is readily shown via iteration on the law of motion for quasilinear states that $v_0 = V^l(s_0, \mathbf{a})$ and for all $t = 1, 2, \ldots$ and $s^t \in \mathcal{S}^t$, $v_t^l(s^t) = V^l(s_t, \mathbf{a}|s^t)$. Hence, since the nonlinear laws of motion are relaxed in (MP), $\mathbf{p}^R$ is feasible for (MP) and, so, $MP_0^* \geq P_0^*$. Since $\bar{k} \in \mathcal{K}^*(s_0)$ and the constraints of (P) are non-empty, the constraint set for (MP) is also non-empty. By Assumption 2 and a similar argument to Proposition 1, (MP) has a solution $\hat{\mathbf{p}}^R$. Let $\hat{\mathbf{v}}^l = \{\hat{v}_t^l\}$, with $\hat{v}_0 = V^l(s_0, \hat{\mathbf{a}})$ and for all $t = 1, 2, \ldots$ and $s^t \in \mathcal{S}^t$, $\hat{v}_t^l(s^t) = V^l(s_t, \hat{\mathbf{a}}|s^t)$. If $\hat{\mathbf{p}} = (\hat{\mathbf{p}}^R, \hat{\mathbf{v}}^l)$ is feasible for (P), then $MP_0^* = P_0^*$ and, hence, the $\mathbf{p}^R$ component of any solution to the original problem (P) solves (MP). Suppose that $\hat{\mathbf{p}}$ is not feasible for (P) and that $F[s_0, \hat{v}_0] > P_0^*$. Then $\hat{k}_0 \leq \bar{k}$,

$$\hat{k}_{t+1}(s^t) \leq W^k[\hat{k}_t(s^{t-1}), s_t, \hat{a}_t(s^t)], \tag{C.5}$$

$$\hat{v}_t^c(s^t) \leq W^c[s_t, \hat{a}_t(s^t), M^c[s_t, \hat{v}_{t+1}^c(s^t, \cdot)]], \tag{C.6}$$

$$\hat{v}_t^l(s^t) = W^l[s_t, \hat{a}_t(s^t), M^l[s_t, \hat{v}_{t+1}^l(s^t, \cdot)]],$$

$$\text{and} \qquad H[\hat{k}_t(s^{t-1}), s_t, \hat{a}_t(s^t), \hat{v}_{t+1}(s^t, \cdot)] \geq 0,$$

with at least one of the constraints $\hat{k}_0 \leq \bar{k}$, (C.5) or (C.6) a strict inequality. Consider first modifying $\hat{\mathbf{p}}$ by increasing $\hat{k}_0$ until it equals $\bar{k} \in \mathcal{K}^*(s_0) \subset \mathcal{K}$ and successively at each history raising $\hat{k}_{t+1}(s^t)$ until it equals $W^k[\hat{k}_t(s^{t-1}), s_t, \hat{a}_t(s^t)]$. By Assumption 4 (ii) and (iii), the modified plan satisfies (C.3) with equality at each $s^t$, the $H$ constraints at each $s^t$ and has each $k_t(s^t) \in \mathcal{K}$. If each (C.6) holds with equality at $\hat{\mathbf{p}}$ and, hence, at the modified plan, then the modified plan has a payoff $MP_0^*$ (since it did not alter $\hat{v}_0$) and is feasible for (P). Thus, $P_0^* \geq MP_0^*$. Suppose that at some $s^t$, $\hat{v}_t^c(s^t) < W^c[s_t, \hat{a}_t(s^t), M^c[s_t, \hat{v}_{t+1}^c(s^t, \cdot)]]$. Then further modify the plan by raising $\hat{v}_t^c(s^t)$ until equality is restored. Since we assume throughout that $W^v[s, a, M^v[s, \cdot]] : \mathcal{V}^{n_S} \to \mathcal{V}$ and, hence, $W^c[s, a, M^c[s, \cdot]] : (\mathcal{V}^c)^{n_S} \to \mathcal{V}^c$ and since $v_{t+1}^c(s^t, \cdot) \in (\mathcal{V}^c)^{n_S}$, the adjusted $v_c^t(s^t) \in \mathcal{V}^c$. Continuing in this way through successively shorter histories $s^\tau$, each $v_\tau^c(s^\tau)$ is increased (by the strict monotonicity of $W^c$ and $M^c$) and, in particular, $v_0^c$ is increased. Hence, by the increasingness of $F[s_0, \cdot]$, the modified plan raises the value of $F[s_0, \cdot]$ above $F[s_0, \hat{v}_0]$. Since the $\mathbf{p}^R$ component of this plan is feasible for (MP), the optimality of $\hat{\mathbf{p}}^R$ for (MP) is contradicted. Thus, the modified plan must satisfy the conditions (C.4) with equality. We conclude that $P_0^* \geq MP_0^*$. Combining inequalities $P_0^* = MP_0^*$ and the $\mathbf{p}^R$ component of any optimum for (P) is feasible for (MP), attains a payoff of $MP_0^*$ and, hence, is optimal for (MP). □

**Relating the modified problem to the abstract problem** For each (modified) primal plan $\mathbf{p}^R$, define the constraint values as follows: $\mathbf{z}(\mathbf{p}^R) = (\mathbf{z}^k(\mathbf{p}^R), \mathbf{z}^c(\mathbf{p}^R), \mathbf{z}^h(\mathbf{p}^R))$, with $\mathbf{z}^k(\mathbf{p}^R) = \{z_t^k(\mathbf{p}^R)\}_{t=0}^\infty$, where $z_0^k(\mathbf{p}^R) := \bar{k} - k_0$ and, for all $t = 1, 2, \ldots$, and $s^t \in \mathcal{S}^t$,

$$z_t^k(\mathbf{p}^R)(s^t) := W^k[k_{t-1}(s^{t-1}), s_{t-1}, a_{t-1}(s^{t-1})] - k_t(s^t);$$

$\mathbf{z}^c(\mathbf{p}^R) = \{z_t^c(\mathbf{p}^R)\}_{t=0}^\infty$, where for all $t = 0, 1, 2, \ldots$, $s^t \in \mathcal{S}^t$,

$$z_t^c(\mathbf{p}^R)(s^t) := W^v[s_t, a_t(s^t), M^c[s_t, v_{t+1}^c(s^t, \cdot)]] - v_t^c(s^t)$$

and $\mathbf{z}^h(\mathbf{p}^R) = \{z_t^h(\mathbf{p}^R)\}_{t=0}^\infty$, where:

$$z_t^h(\mathbf{p}^R)(s^t) := H[k_t(s^{t-1}), s_t, a_t(s^t), v_{t+1}^c(s^t, \cdot), V^l(s', \mathbf{a}|s^t, \cdot)].$$

Let $\mathbf{y}^k = \{y_t^k\}_{t=0}^\infty$, with $y_t^k : \mathcal{S}^t \to \mathbb{R}_+^{n_K}$, denote non-negative costates for the backward-looking law of motion and $\mathbf{y}^c = \{y_t^c\}_{t=0}^\infty$, with $y_t^c : \mathcal{S}^t \to \mathbb{R}_+^{n_c}$, non-negative costates for the forward-looking non-linear laws of motion. Let $\mathbf{m} = \{m_t\}_{t=0}^\infty$, with $m_t : \mathcal{S}^t \to \mathbb{R}_+^{n_H}$, denote multipliers for the $H$-constraints. Collect these various multipliers into a (modified) *dual plan* $\mathbf{q}^R = \{\mathbf{m}, \mathbf{y}^k, \mathbf{y}^c\}$ and define the set of such dual plans to be:

$$\mathbf{Q}^R = \left\{ \mathbf{q}^R \left| \sum_{t=0}^\infty \sum_{\mathcal{S}^t} \delta^t \{\|m_t(s^t)\|_E + \|y_t^k(s^t)\|_E + \|y_t^c(s^t)\|_E\} \pi^t(s^t|s_0) < \infty \right. \right\},$$

We associate the following Lagrangian $\mathscr{L} : \mathbf{P}^R \times \mathbf{Q}^R \to \mathbb{R}$ with (MP):

$$\mathscr{L}^R(\mathbf{p}^R, \mathbf{q}^R) = F[s_0, v_0^c, V^l(s_0, \mathbf{a}|s_0)] + \langle \mathbf{q}^R, \mathbf{z}(\mathbf{p}^R) \rangle,$$

with: $\langle \mathbf{q}^R, \mathbf{z}(\mathbf{p}^R) \rangle = \sum_{t=0}^\infty \sum_{\mathcal{S}^t} \delta^t \{m_t(s^t) \cdot z_t^h(\mathbf{p}^R)(s^t) + y_t^k(s^t) \cdot z_t^k(\mathbf{p}^R)(s^t) + y_t^c(s^t) \cdot z_t^c(\mathbf{p}^R)(s^t)\} \pi^t(s^t|s_0)$.

**Lemma C.4.** *If $\mathbf{p}^{R*}$ solves (MP) and Assumptions 5 and 6 hold, then there exists a $\mathbf{q}^R \in \mathbf{Q}^R$ such that $(\mathbf{p}^{R*}, \mathbf{q}^{R*})$ is a saddle point of $\mathscr{L}^R$ on $\mathbf{P}^R \times \mathbf{Q}^R$.*

*Proof.* We first re-express (MP) as an abstract problem of the form (AP). Next, we verify Assumptions 7 and 8. The result then follows from Theorems C.1 and C.2. In (MP) constraints are indexed by histories, in (AP) by the natural numbers. To convert one to the other, let $\mathscr{S} = \cup_{t=0}^\infty \mathcal{S}^t$ denote the countable set of all histories (and recall that $\mathcal{S} = \{1, \ldots, n_S\}$). Relabel histories according to $\tau : \mathscr{S} \to \mathbb{N}$ with $\tau(s^t) = (1 + \sum_{r=1}^{t-1} s_r \cdot n_S^{t-r} + s_t)$.[37] Thus, the relabeled

[37]For example, consider the history $(s_0, s_1, s_2) = (s_0, 2, 5)$. We have $\tau(s_0) = 1$, $\tau(s_0, 2) = 1 + 2 = 3$, and $\tau(s_0, 2, 5) = 1 + 2 \cdot n_S + 5 = 2 \cdot n_S + 6$.

history 1 is the initial (date 0) history, relabeled histories $2, \dots, n_S + 1$ are the period 1 histories occurring after each period 1 shock realization and so on. Choice variables may be grouped and re-indexed accordingly. Specifically, let $x = \{x_\tau\}_{\tau=1}^{\infty} \in \ell^\infty$, denote a regrouped and labeled primal plan, with

$$x_{\tau(s^t)} = (k_t(s^t), a_t(s^t), v_t^c(s^t)) \in \mathcal{X} = \mathcal{K} \times \mathcal{A} \times \mathcal{V}^{c,n_S},$$

where $\mathcal{V}^c$ denotes the bounded set of nonlinear states. Let $g$ denote a regrouped and relabeled constraint function with $g = \{g_\tau\}_{\tau=1}^{\infty}$ and for each $\tau(s^t)$:

$$g_{\tau(s^t)}(x) = (z_t^k(\mathbf{p}^R(x))(s^t), z_t^c(\mathbf{p}^R(x))(s^t), z_t^h(\mathbf{p}^R(x))(s^t)),$$

where $\mathbf{p}^R(x) = \{p_t^R(x)(s^t)\}_{t=0}^{\infty}$ is the primal plan associated with $x$. The boundedness of the constraint functions implies $g : \ell^\infty \to \ell^\infty$. Finally, let $f(x) = F[s_0, v_0^c(x), V^l(s_0, \mathbf{a}(x))]$, where $v_0^c(x)$ and $\mathbf{a}(x)$ denote the $v_0$ and $\mathbf{a}$ components of $x$. In this way (MP) is re-expressed in the form (AP) and, in particular, a Lagrangian of the form $\mathcal{L}(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle$ may be associated with (MP).

We next show that Assumptions 5 and 6 in the main text and the structure of $f$, and $g$ in our setting imply Assumptions 7 and 8. From Assumption 5, the concavity of $F$, $H$, $W^c$ and $M^c$ imply concavity of $f$ and $g$ as required by Assumption 7 (CV). Assumption 6 ensures Assumption 7 (S) (the Slater Condition). Specifically, if there is, as required by Assumption 6, a $\hat{\mathbf{p}} \in \mathbf{P}$ satisfying the law of motion constraints for $\mathbf{v}^c$ and $\mathbf{k}$ and the $H$ constraints with strict inequality and $z^l(\hat{\mathbf{p}}) = 0$, then the corresponding $\hat{\mathbf{p}}^R \in \mathbf{P}^R$ satisfies Assumption 7 (S).

Assumption 8 (C) holds if for all $x$ and $y$, $\lim_{T\to\infty} F[s_0, v_0^c(x^T(x,y)), V^l(s_0, \mathbf{a}(x^T(x,y)))] = F[s_0, v_0^c(x), V^l(s_0, \mathbf{a}(x))]$. Since $v_0^c(x^T(x,y)) = v_0^c(x)$ for all $T \geq 1$ and $F$ is linear and, so, continuous in its third argument, Assumption 8 (C) holds if $\lim_{T\to\infty} V^l(s_0, \mathbf{a}(x^T(x,y))) = V^l(s_0, \mathbf{a}(x))$. For any pair $x$ and $y$, there is a non-decreasing sequence of dates $R(T)$ with $\lim_{T\to\infty} R(T) = \infty$ such that for all $s^t$ with $t < R(T)$ all elements $p_t^R(x)(s^t) = p_t^R(x^T(x,y))(s^t)$. Thus,[38]

$$\begin{aligned}
&|V^l(s_0, \mathbf{a}(x^T(x,y))) - V^l(s_0, \mathbf{a}(x))| \\
&\qquad \leq \delta^{R(T)} \sum_{s^{R(T)}} |b^l_{R(T)}(s^{R(T)}|s_0)||V^l(s_{R(T)}, \mathbf{a}(x^T(x,y))|s^{R(T)}) - V^l(s_{R(T)}, \mathbf{a}(x)|s^{R(T)})|\pi^{R(T)}(s^{R(T)}|s_0)
\end{aligned}$$

Since $\delta^{R(T)}$ converges to 0 and the sequence of sums $\sum_{s^{R(T)}} |b^l_{R(T)}(s^{R(T)}|s_0)|\ |V^l(s^t, \mathbf{a}(x^T(x,y))|s^t) - V^l(s^t, \mathbf{a}(x)|s^t)|\pi^{R(T)}(s^{R(T)}|s_0)$ is uniformly bounded, $\lim_{T\to\infty} |V^l(s_0, \mathbf{a}(x^T(x,y))) - V^l(s_0, \mathbf{a}(x))| = 0$. Thus, Assumption 8 (C) is satisfied. Assumption 8 (ANA) holds if for each $j = k, c, h$ and $t$ and all $x$ and $y$, $\lim_{T\to\infty} z_t^j(\mathbf{p}^R(x^T(x,y)))(s^t) = z_t^j(\mathbf{p}^R(x))(s^t)$. This result follows from the fact that each $z_t^j(\mathbf{p}^R)(s^t)$, $j = k, c$, depends only upon variables measurable with respect to $s^t$ and $s^{t+1}$ and each $z_t^h(\mathbf{p}^R)(s^t)$ depends only upon these variables and upon $V^l(s_t, \mathbf{a}|s^t)$ continuously. Thus, defining $R(T)$ as before, once $R(T) > t+1$, $z_t^j(\mathbf{p}^R(x^T(x,y))(s^t) = z_t^j(\mathbf{p}^R(x))(s^t)$, $j = k, c$. Also, by a similar argument to that given above: $|V^l(s_t, \mathbf{a}(x^T(x,y))|s^t) - V^l(s_t, \mathbf{a}(x)|s^t)| \to 0$. Thus, $\lim_{T\to\infty} z_t^h(\mathbf{p}^R(x^T(x,y))(s^t) = z_t^h(\mathbf{p}^R(x))(s^t)$ and Assumption 8 (ANA) is verified.

Assumption 8 (AI) requires that for each $j = c, k, h$ and $N$, $\lim_{t\to\infty}[z_t^j(\mathbf{p}^R(x^N(x,y)) - z_t^j(\mathbf{p}^R(y))] = 0$. This is an immediate consequence of the fact that each $z_t^j$ does not depend on any variable that is measurable with respect to $s^{t-1}$. Thus, for any fixed $N$ there exists an $M(N)$ such that for each $j$, $t > M(N)$ and $r \geq 0$, $p^R_{t+r}(x^N(x,y))(s^{t+r}) = p^R_{t+r}(y)(s^{t+r})$ and so

[38] We use the following notation: for a generic matrix $R$, $|R|$ denotes the element-wise application of the absolute value operator to $R$.

$[z_t^j(\mathbf{p}^R(x^N(x,y)) - z_t^j(\mathbf{p}^R(y))] = 0$. This confirms Assumption 8 (AI). Finally, Assumption 8 (B) follows from the boundedness of the constraint functions.

Thus, Assumptions 7 and 8 are verified. Let $x^*$ denote the regrouped and relabeled primal plan corresponding to the solution $\mathbf{p}^{R*}$ of (MP). By Theorems C.1 and C.2 there exists a multiplier $\lambda^* \in \ell^1_+$ such that $(x^*, \lambda^*)$ is a saddle point of $\mathcal{L}(x,\lambda) = f(x) + \langle \lambda, g(x) \rangle$, where $f$ and $g$ are those implied by (MP) and defined above. Resetting the labeling it follows that $\mathbf{p}^{R*}$ and $\mathbf{q}^{R*}$ (where $\mathbf{q}^{R*}$ is obtained from $\lambda^*$ by resetting the labeling and normalizing multipliers at each $t$ and history $s^t$ by $\delta^t \pi^t(s^t|s_0)$) is a saddle point of $\mathscr{L}^R$ on $\mathbf{P}^R \times \mathbf{Q}^R$. □

The preceding result establishes that if $\mathbf{p}^{R*}$ solves (MP), then there is a $\mathbf{q}^{R*} \in \mathbf{Q}^R$ such that $(\mathbf{p}^{R*}, \mathbf{q}^{R*})$ is a saddle point of $\mathscr{L}^R$ on $\mathbf{P}^R \times \mathbf{Q}^R$. We now seek to show that if $\mathbf{p}^*$ solves (P), then there is a $\mathbf{q}^* \in \mathbf{Q}$ such that $(\mathbf{p}^*, \mathbf{q}^*)$ is a saddle point of $\mathscr{L}$ on $\mathbf{P} \times \mathbf{Q}$. Recall that the Lagrangian $\mathscr{L}$ associated with (P) incorporates quasi-linear state variables and the laws of motion for such variables. It also allows the costates associated with nonlinear laws of motion to belong to $\mathbb{R}$ rather than $\mathbb{R}_+$. We use a constructive argument to show that this Lagrangian also has a saddle point. Under Assumption 5, $F[s_0, v_0^c, v_0^l]$ is linear in its third argument. Below we use the notation:

$$F[s_0, v_0^c, v_0^l] = \hat{F}[s_0, v_0^c] + F_l(s_0) \cdot v_0^l.$$

In addition, $H[k, s, a, v^{c\prime}, v^{l\prime}]$ is linear in its final argument. Below we use the notation:

$$H[k, s, a, v^{c\prime}, v^{l\prime}] = \hat{H}[k, s, a, v^{c\prime}] + \delta \sum_{s' \in \mathcal{S}} N^l(s, s') \cdot v^l(s') \pi(s'|s).$$

**Lemma C.5.** *Let $\mathbf{p}^* = (\mathbf{p}^{R*}, \mathbf{v}^{l*})$, with $\mathbf{p}^{R*} = (\mathbf{a}^*, \mathbf{k}^*, \mathbf{v}^{c*})$, be a solution to (P). If $(\mathbf{p}^{R*}, \mathbf{q}^{R*})$, with $\mathbf{q}^{R*} = (\mathbf{y}^{k*}, \mathbf{y}^{c*}, \mathbf{m}^*)$, is a saddle point of $\mathscr{L}^R$ on $\mathbf{P}^R \times \mathbf{Q}^R$, then $(\mathbf{p}^{R*}, \mathbf{v}^{l*}, \mathbf{q}^{R*}, \mathbf{y}^{l*})$, where $\mathbf{y}^{l*}$ satisfies the recursion $y_0^{l*} = F^l(s_0)$ and for all $t = 1, 2, \ldots$ and $s^t \in \mathcal{S}^t$,*

$$y_{t+1}^{l*}(s^t, s') = y_t^{l*}(s^t) \cdot B^l(s_t, s') + m_t^*(s^t) \cdot N^l(s_t, s'), \tag{C.7}$$

*is a saddle point of $\mathscr{L}$ on $\mathbf{P}^R \times \mathbf{Q}^R$*

*Proof.* Let $\mathbf{Q}_+$ denote the subset of $\mathbf{Q}$ in which the costates $\mathbf{y}^k$ and $\mathbf{y}^c$ on backward-looking and non-linear forward-looking state variables are non-negative. For $(\mathbf{p}, \mathbf{q}) \in \mathbf{P} \times \mathbf{Q}_+$ and $(\mathbf{p}^R, \mathbf{q}^R) \in \mathbf{P}^R \times \mathbf{Q}^R$, define

$$\begin{aligned}
&\Delta(\mathbf{a}, \mathbf{v}^l, \mathbf{m}, \mathbf{y}^l) := \mathscr{L}^R(\mathbf{p}^R, \mathbf{q}^R) - \mathscr{L}(\mathbf{p}, \mathbf{q}) \qquad (C.8)\\
&\quad = F^l(s_0) \cdot V^l(s_0, \mathbf{a}) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t) \cdot \sum_{s_{t+1} \in \mathcal{S}} \delta N^l(s_t, s_{t+1}) V^l(s_{t+1}, \mathbf{a}|s^t, s_{t+1}) \pi(s_{t+1}|s_t) \pi^t(s^t|s_0)\\
&\qquad - F^l(s_0) \cdot v_0^l - \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t) \cdot \sum_{s_{t+1} \in \mathcal{S}} \delta N^l(s_t, s_{t+1}) v_{t+1}^l(s^t, s_{t+1}) \pi(s_{t+1}|s_t) \pi^t(s^t|s_0)\\
&\qquad - \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} y_t^l(s^t) \cdot \left[ u^l(s_t, a_t(s^t)) + \delta \sum_{s_{t+1}} B^l(s_t, s_{t+1}) v_{t+1}^l(s^t, s_{t+1}) \pi(s_{t+1}|s_t) - v_t^l(s^t) \right] \pi^t(s^t|s_0),
\end{aligned}$$

where the second equality follows from the definitions of the Lagrangians. From the recursion (C.7) defining the plan $\mathbf{y}^{l*}$ and the assumption $|B(s, s')| \leq \mathbb{I}_{n_l}$, $\mathbf{y}^{l*}$ is summable with respect to the $\delta^t \pi^t(s^t|s_0)$ normalization as long as $\mathbf{m}^*$ is, that is, we have $\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} \delta^t \|y_t^{l*}(s^t)\|_E \pi^t(s^t|s_0) < \infty$. Consequently, $(\mathbf{q}^{R*}, \mathbf{y}^{l*}) \in \mathbf{Q}_+$ and is feasible for the minimization defining the saddle point of $\mathscr{L}$. Hence, using the saddle point inequalities for $\mathscr{L}$ and $\mathscr{L}^R$ and the definition of $\Delta$,

$(\mathbf{p}^{R*}, \mathbf{v}^{l*}, \mathbf{q}^{R*}, \mathbf{y}^{l*})$ is a saddle point of $\mathscr{L}$ on $\mathbf{P} \times \mathbf{Q}_+$ if for all $(\mathbf{a}, \mathbf{v}^l) \in \mathbf{P}$ and $(\mathbf{m}, \mathbf{y}^l) \in \mathbf{Q}_+$,

$$\Delta(\mathbf{a}, \mathbf{v}^l, \mathbf{m}^*, \mathbf{y}^{l*}) \geq \Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}^*, \mathbf{y}^{l*}) \geq \Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}, \mathbf{y}^l).$$

Consider first $\Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}, \mathbf{y}^l)$. Since $(\mathbf{a}^*, \mathbf{v}^{l*})$ is part of an optimum and, hence, feasible for (P), it satisfies the law of motion for quasilinear states with equality. Consequently, the last line in (C.8) when evaluated at $(\mathbf{a}^*, \mathbf{v}^{l*})$ equals zero. Moreover, the law of motion for quasilinear states and Assumption 5 (iv) imply that for all $t, s^t$ we have $v_t^*(s^t) = V^l(s_t, \mathbf{a}^*|s^t)$. Thus, (C.8) implies that for for all $\mathbf{m}$ and $\mathbf{y}^l$ forming part of $\mathbf{q} \in \mathbf{Q}_+$,

$$\Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}, \mathbf{y}^l) = 0.$$

Consequently, the inequality $\Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}^*, \mathbf{y}^{l*}) \geq \Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}, \mathbf{y}^l)$ is trivially satisfied. Next consider $\Delta(\mathbf{a}, \mathbf{v}^l, \mathbf{m}^*, \mathbf{y}^{l*})$. Substituting the recursion defining $\mathbf{y}^{l*}$ into (C.8) eliminates all terms involving $\mathbf{v}^l$ from $\Delta(\mathbf{a}, \mathbf{v}^l, \mathbf{m}^*, \mathbf{y}^{l*})$. Moreover, the definition of $V^l$ implies that for all $\mathbf{a} \in \mathbf{A}$,

$$\begin{aligned}
&\sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} y_t^{l*}(s^t) \cdot u^l(s_t, a_t(s^t)) \\
&= F^l(s_0) \cdot V^l(s_0, \mathbf{a}) + \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t^*(s^t) \cdot \sum_{s_{t+1} \in \mathcal{S}} \delta N^l(s_t, s_{t+1}) V^l(s_{t+1}, \mathbf{a}|s^t, s_{t+1}) \pi(s_{t+1}|s_t) \pi^t(s^t|s_0).
\end{aligned}$$

Substituting this into (C.8), it follows that $\Delta(\cdot, \cdot, \mathbf{m}^*, \mathbf{y}^{l*})$ is zero and independent of $\mathbf{a}$ and $\mathbf{v}^l$. Hence,

$$\Delta(\mathbf{a}, \mathbf{v}^l, \mathbf{m}^*, \mathbf{y}^{l*}) \geq \Delta(\mathbf{a}^*, \mathbf{v}^{l*}, \mathbf{m}^*, \mathbf{y}^{l*}), \quad \forall\, \mathbf{a}, \mathbf{v}^l \in \mathbf{P}.$$

Thus, $(\mathbf{p}^*, \mathbf{q}^*) = (\mathbf{p}^{R*}, \mathbf{v}^{l*}, \mathbf{q}^{R*}, \mathbf{y}^{l*})$ is a saddle point of $\mathscr{L}$ on $\mathbf{P} \times \mathbf{Q}_+$. It remains only to show that $(\mathbf{p}^{R*}, \mathbf{v}^{l*}, \mathbf{q}^{R*}, \mathbf{y}^{l*})$ is a saddle point of $\mathscr{L}$ on $\mathbf{P} \times \mathbf{Q}$ (i.e. of the Lagrangian without the costates $\mathbf{y}^k$ and $\mathbf{y}^c$ restricted to be non-negative). However, since $\mathbf{p}^*$ is a solution to (P) it satisfies the laws of motion for backward-looking and non-linear forward-looking states with equality. Hence, $\mathscr{L}(\mathbf{p}^*, \cdot)$ is independent of the multipliers on these states implying that $\mathbf{q}^*$ minimizes $\mathscr{L}(\mathbf{p}^*, \cdot)$ on the set $\mathbf{Q}$. □

*Proof of Proposition 8.* The proof now follows from the preceding lemmas. By Assumption 2, the restriction on $\bar{k}$ and Proposition 1, (P) has a solution $\mathbf{p}^* = (\mathbf{p}^{R*}, \mathbf{v}^{l*})$. Given Assumptions 2, 4 and 5 (iv), by Lemma C.3, $\mathbf{p}^{R*}$ solves (MP). Given Assumptions 5 and 6, Lemma C.4 implies that the Lagrangian $\mathscr{L}^R$ has a saddle point $(\mathbf{p}^{R*}, \mathbf{q}^{R*})$. Finally, by Lemma C.5, the Lagrangian $\mathscr{L}$ has a saddle point. □

# D The Quasilinear Case

As noted in the main text (see Subsection 4.3), many problems have aggregators and constraint functions that are quasi-linear in $k$ or $v$ or both. In this appendix, we show how directly exploiting this structure prior to the formulation of the Lagrangian can lead to considerable simplification. In particular, the backward and forward primal states $k_t$ and $v_t$ can be substituted out along with the equality constraints describing their evolution.

**Example: Limited Commitment with linear capital accumulation technology**
To make the subsequent analysis concrete, we consider the limited commitment example

from Section 2, set $\rho = \sigma$ and extend it to include capital, a linear "AK-like" technology and a default option that is linear in capital. Specifically, defining $V^i(s_0, \mathbf{a})$ to be:

$$V^i(s_0, \mathbf{a}) = (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} u^i(a_t(s^t)) \pi^t(s^t | s_0),$$

with $\delta \in (0,1)$, we consider a decision maker solving:

$$\sup_{\mathbf{a},\mathbf{k}} \sum_{i \in \mathcal{I}} \lambda^i V^i(s_0, \mathbf{a}). \tag{O}$$

subject to $k_0 = \bar{k} \geq 0$ and for all $s^t = (s^{t-1}, s_t) \in \mathcal{S}^t$, $a^i_t(s^t) \geq 0$, $k_{t+1}(s^t) \geq 0$, the no default constraints:

$$V^i(s_t, \mathbf{a}|s^t) - w^i_0(s^t) - w^i_1(s_t) k_t(s^{t-1}) \geq 0, \qquad \text{for } i \in \mathcal{I}, \tag{D}$$

with $w^i_0 \geq 0$ and $w^i_1 \geq 0$, and the law of motion for capital:

$$\gamma_0(s_t) + \gamma_1(s_t) k_t(s^{t-1}) - \sum_{i \in \mathcal{I}} a^i_t(s^t) - k_{t+1}(s^t) = 0, \tag{R}$$

with $\gamma_0 \geq 0$ and $\gamma_1 \geq 0$. With the restriction $\gamma_1 < 1$, this problem is a special case of the quasi-linear in states environment we describe below (an even more special case occurs when $\gamma_1 = 0$ and $w_1 = 0$; then the model of Section 2 for $\sigma = \rho$ is obtained).[39] We will call this case the "limited commitment with capital example" and refer back to it throughout the remainder of this section.

## D.1 Quasilinear environments

We now turn to the general quasilinear environment. We proceed as in the main text by first defining (in this case quasilinear) laws of motion for backward and forward looking states and then associating them with a decision-maker's problem.

**Shocks and Actions** We keep the assumptions and notation for shocks the same as in earlier sections. The set of actions is denoted by $\mathcal{A} \subset \mathbb{R}^{n_A}$ and the set of action plans $\mathbf{A} = \mathcal{A}^\infty$.

**Backward-looking states** Let $\mathcal{K} \subset \mathbb{R}^{n_K}$ be a bounded set of backward-looking states. Some problems omit backwards-looking states altogether (for example the problems in Section 2). The analysis below is easily specialized to exclude such states and, hence, consider these simpler settings. Let $\mathbf{k} = \{k_t\}_{t=0}^{\infty}$, with $k_0 = \bar{k} \in \mathcal{K}$ and for $t \in \mathbb{N}$, $k_t : \mathcal{S}^{t-1} \to \mathcal{K}$, be a plan for backward-looking states with initial condition $\bar{k}$. The law of motion for such states is specialized to be a quasilinear function $W^k$:

$$W^k[k, s, a] := B^k(s) k - c(s, a), \tag{D.1}$$

where $B^k(s)$ is a diagonal matrix of dimension $n_K$ and with each element less than one in absolute value and $c : \mathcal{S} \times \mathcal{A} \to [\underline{C}, \overline{C}]^{n_K} \subset \mathbb{R}^{n_K}$. These assumptions imply that $W^k(\mathcal{K}, \mathcal{S}, \mathcal{A})$ is a

[39]The analysis can be extended to a class of unbounded quasilinear problems that can accommodate the $\gamma_1 \geq 1$ case. To ensure a well defined Lagrangian, a technical modification to our basic arguments is required. We defer this to later work.

bounded set. A pair of plans $\mathbf{a}$ and $\mathbf{k}$ satisfy the law of motion (D.1) if for all $t \geq 0$ and $s^t \in \mathcal{S}^t$,

$$k_{t+1}(s^t) = B^k(s_t)k_t(s^{t-1}) - c(s_t, a_t(s^t)).$$

In our formulation of the decision-maker's problem below, the quasilinearity of (D.1) in $k$ is used to substitute backward-looking states from the problem. Anticipating this, it is useful to have an explicit representation of these states in terms of the action plan. Given an initial backward-looking state $\bar{k} \in \mathcal{K}$ and plans $\mathbf{a}$ and $\mathbf{k}$ satisfying (D.1), we have:

$$k_{t+1}(s^t) = K_{t+1}(\bar{k}, s^t, \mathbf{a}) := -\sum_{\tau=0}^{t} \prod_{j=\tau+1}^{t} B^k(s_j)c(s_\tau, a_\tau(s^\tau)) + \prod_{j=0}^{t} B^k(s_j)\bar{k}. \tag{D.2}$$

Later we absorb the restriction that each $k_{t+1}(s^t) = K_{t+1}(\bar{k}, s^t, \mathbf{a}) \in \mathcal{K}$ directly into a Lagrangian that is a function of actions and multipliers only.

In the limited commitment with capital example, individual consumptions remain in $\mathbb{R}_+^{n_I}$ and $\mathcal{K} \subset \mathbb{R}_+$. Also, $B^k(s) = \gamma_1(s)$, $c(s,a) = -\gamma_0 + \sum_{i\in\mathcal{I}} a^i$ and, hence, $W^k[k,s,a] = \gamma_1(s)k + \gamma_0 - \sum_{i\in\mathcal{I}} a^i$. With $\gamma_1 < 1$, the accumulation technology implies that capital is bounded above by $k_{\max} = \max(\bar{k}, \frac{\bar{\gamma}_0}{1-\bar{\gamma}_1})$ where, for $r = 0,\ 1$, $\bar{\gamma}_r = \max_{s\in\mathcal{S}} \gamma_r(s)$. Then, the nonnegativity of capital implies that consumption must remain below $a_{\max} := \frac{k_{\max}+\bar{\gamma}_0}{1-\bar{\gamma}_1}$. Requiring that $a$ belongs to the interval $\mathcal{A} := [0, a_{\max}]^{n_I}$ ensures that $c(s,\cdot)$ is bounded for all $s \in \mathcal{S}$ without restricting the solution to the problem.

**Forward-looking states** Let $u : \mathcal{S} \times \mathcal{A} \to \mathcal{V} \subset \mathbb{R}^{n_V}$ where $\mathcal{V}$ is the bounded set of forward-looking states. For concreteness, it is useful to think of these states as lifetime payoffs of (possibly incentive-constrained) agents and $u$ as a function giving the current payoffs of agents, although other interpretations are possible. Let $\mathbf{v} = \{v_t\}_{t=0}^{\infty}$, with for $t \geq 0$, $v_t : \mathcal{S}^t \to \mathcal{V}$, be a plan for such states. The law of motion for forward-looking states is, as in the main text, a composition of two functions. The first, the time aggregator $W^v$, is:

$$W^v[s,a,m] = (1-\delta)u(s,a) + \delta \cdot m, \tag{D.3}$$

with $\delta \in (0,1)$. The second, the certainty equivalent operator $M^v$, is:

$$M^v[s,v'] = \sum_{s'\in\mathcal{S}} B^v(s,s')v'(s')\pi(s'|s), \tag{D.4}$$

where for each $(s,s') \in \mathcal{S}^2$, $B^v(s,s')$ is diagonal matrix of dimension $n_V$ with each entry bounded in absolute value by one. The composition of $W^v$ and $M^v$ is then quasilinear in current and future states $v$ and $v'$. A pair of plans for actions $\mathbf{a}$ and forward-looking states $\mathbf{v}$ satisfy this law of motion if for all $t \geq 0$,

$$v_t(s^t) = (1-\delta)u(s_t, a_t(s^t)) + \delta \sum_{s'\in\mathcal{S}} B^v(s_t,s')v_{t+1}(s^t,s')\pi(s'|s_t). \tag{D.5}$$

In addition, letting $b_0^v = \mathbb{I}_{n_V}$ (with $\mathbb{I}_{n_V}$ the $n_V$-identity matrix) and for all $\forall t, s^t, s'$ $b_{t+1}^v(s^t,s'|s_0) = b_t^v(s^t|s_0)B^v(s_t,s')$, then (D.5) implies that forward-looking states correspond to discounted expected sums:

$$v_t(s^t) = V(s_t, \mathbf{a}|s^t) := (1-\delta)\sum_{n=0}^{\infty} \delta^n \sum_{\mathcal{S}^n} b_n^v(s^{t+n}|s_t)u^l(s_{t+n}, a_{t+n}^l(s^{t+n}))\pi^n(s^{t+n}|s_t). \tag{D.6}$$

**$H$-constraint function** Analogous to the laws of motion, the constraint function $H$ is assumed to be quasi-linear in states:

$$H[k,s,a,v'] = N^k(s)k + h(s,a) + \delta \sum_{s'\in\mathcal{S}} N^v(s,s')v'(s')\pi(s'|s), \tag{D.7}$$

with $N^k(s)$ an $n_H$ vector indexed by $s$, $N^v(s,s')$ an $n_H \times n_V$ matrix indexed by $(s,s')$ and $h : \mathcal{S}\times\mathcal{A} \to \mathbb{R}^{n_H}$ a bounded function of current shocks and actions. As noted above, $H$ absorbs any restrictions that ensure backward-looking states remain in $\mathcal{K}$, i.e. if $(k,s,a,v') \in \mathcal{K}\times\mathcal{S}\times\mathcal{A}\times\mathcal{V}^{n_S}$ satisfies $H[k,s,a,v'] \geq 0$, then $W^k[k,s,a] = B^k(s)k - c(s,a) \in \mathcal{K}$. $H[k,s,a,v'] \geq 0$ also captures any incentive constraints on agents.

**Objective and Problem Statement** The objective $F$ is assumed to be linear in forward states $v$: $F[s,v] = \lambda \cdot v$. Typically, it will be a Pareto weighted aggregate of agent payoffs and in some problems will place all weight on one of the agents (who is often designated the principal). The problem may be stated, as in the main text, in terms of action plans and state plans. However, given the expressions for states in terms of actions: $k_t(s^{t-1}) = K_t(\bar{k}, s^{t-1}, \mathbf{a})$ and $v_t(s^t) = V(s_t, \mathbf{a}|s^t)$, the problem may be directly expressed in terms of action plans. Thus, the decision maker's problem can be expressed as:

$$\sup \lambda \cdot V(s_0, \mathbf{a}) \tag{PP$_{QL}$}$$

subject to $\mathbf{a} \in \mathbf{A}$ and for all $t$, $s^t$,

$$N^k(s_t)K_t(\bar{k}, s^{t-1}, \mathbf{a}) + h(s_t, a_t(s^t)) + \delta \sum_{s_{t+1}\in\mathcal{S}} N^v(s_t, s_{t+1})V(s_{t+1}, \mathbf{a}|s^t, s_{t+1})\pi(s_{t+1}|s_t) \geq 0. \tag{D.8}$$

It is easy to see that this formulation encompasses the limited commitment with capital example described above. In particular, the no default and non-negativity of future capital restrictions are obtained by setting:

$$N^k(s) = \begin{pmatrix} -w_1(s) \\ \gamma_1(s) \end{pmatrix}, \quad h(s,a) = \begin{pmatrix} u(a) - w_0(s) \\ \gamma_0(s) - \sum_{i\in\mathcal{I}} a^i \end{pmatrix}, \quad \text{and} \quad N^v = \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix},$$

where $u$ is the vector-valued function of current agent payoffs and $\mathbb{I}$ is an identity matrix (of dimension $n_I$).

## D.2 The Lagrangian and Dual Problem

A Lagrangian for the dual problem is constructed as follows. Since (PP$_{QL}$) incorporates only the $H$-function constraints, the constraint process is given by $\mathbf{z}^h(\bar{k}, \mathbf{a}) = \{z_t^h(\bar{k}, \mathbf{a})\}_{t=0}^{\infty}$, with

$$z_t^h(\bar{k}, \mathbf{a})(s^t) = N^k(s_t)K_t(\bar{k}, s^{t-1}, \mathbf{a}) + h(s_t, a_t(s^t)) + \delta \sum_{s_{t+1}\in\mathcal{S}} N^v(s_t, s_{t+1})V(s_{t+1}, \mathbf{a}|(s^t, s_{t+1}))\pi(s_{t+1}|s_t),$$

Since $K_t$, $V$ and $h$ are all bounded, constraint processes have values in $\ell^\infty$. Define the Lagrangian $\mathscr{L}(\bar{k}, s_0, \cdot) : \mathbf{A}\times\mathbf{M} \to \mathbb{R}$, as:

$$\mathscr{L}(\bar{k}, s_0, \mathbf{a}, \mathbf{m}) = \lambda \cdot V(s_0, \mathbf{a}) + \langle \mathbf{m}, \mathbf{z}^h(\bar{k}, \mathbf{a})\rangle,$$

with $\mathbf{M} := \{\mathbf{m} | m_t(s^t) \geq 0, \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t)\pi^t(s^t|s_0) < \infty\}$, and

$$\langle \mathbf{m}, \mathbf{z}^h(\bar{k}, \mathbf{a}) \rangle = \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t) \cdot z_t^h(\bar{k}, \mathbf{a})(s^t)\pi^t(s^t|s_0).$$

Associate the following dual problem with (PP$_{QL}$):

$$D_0^* = \inf_{\mathbf{m} \in \mathbf{M}} \sup_{\mathbf{a} \in \mathbf{A}} \mathscr{L}(\bar{k}, s_0, \mathbf{a}, \mathbf{m}). \tag{D.9}$$

## D.3 Continuation Dual Function and Recursive Dual Problem

The quasi-linearity of the aggregators ensures that this Lagrangian has the necessary separability for a recursive dual approach. As a first step, in this direction we recover the continuation dual problem from (D.9). The definition of $\mathbf{z}^h$ and (D.2) imply that:

$$\langle \mathbf{m}, \mathbf{z}^h(\bar{k}, \mathbf{a}) \rangle = \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t) \cdot N^k(s_t) b_t^k(s^t) \; \bar{k} \; \pi^t(s^t|s_0) + \langle \mathbf{m}, \mathbf{z}^h(0, \mathbf{a}) \rangle, \tag{D.10}$$

where, for a given history $s^t$, $b_t^k(s^t) = \prod_{j=0}^{t} B^k(s_j)$. Substituting the right hand side of (D.10) into the Lagrangian implies that the dual problem (D.9) can be rewritten as:

$$D_0^* = \inf_{\mathbf{m} \in \mathbf{M}} \sup_{\mathbf{a} \in \mathbf{A}} \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t) \cdot N^k(s_t) b_t^k(s^t) \; \bar{k} \; \pi^t(s^t|s_0) + \lambda \cdot V(s_0, \mathbf{a}) + \langle \mathbf{m}, \mathbf{z}^h(0, \mathbf{a}) \rangle. \tag{D.11}$$

The coefficient on $\bar{k}$ in (D.11) is a linear function of multipliers, $\mathcal{T} : \mathbf{M} \to \mathbb{R}^{n_K}$, where

$$\mathcal{T}(\mathbf{m}) := \sum_{t=0}^{\infty} \delta^t \sum_{\mathcal{S}^t} m_t(s^t) \cdot N^k(s_t) b_t^k(s^t) \; \pi^t(s^t|s_0).$$

Let $\mathbf{M}(y_0^k)$ denote the pre-image of $\mathcal{T}$ at $y_0^k$:

$$\mathbf{M}(y_0^k) = \{\mathbf{m} \in \mathbf{M} | y_0^k = \mathcal{T}(\mathbf{m})\}. \tag{D.12}$$

Given a value of the coefficient (the costate) on the initial capital $\bar{k}$, $\mathbf{M}(y_0^k)$ gives the set of multiplier sequences $\mathbf{m}$ consistent with this. Note that the correspondence $\mathbf{M}(y_0^k)$ may be empty-valued for some $y_0^k$. It is useful to identify the set on which is it is not. Let:

$$\mathcal{Y}^k := \{y^k \in \mathbb{R} | \mathbf{M}(y_0^k) \neq \emptyset\}.$$

In other words, $\mathcal{Y}^k$ is simply the range of $\mathcal{T}$. This set will be used to construct the state space of the dual value function below. While $\mathcal{Y}^k$ maybe a strict subset of $\mathbb{R}^{n_K}$ it is always a closed convex cone. In many settings it is easy to find. For example, in limited commitment with capital example $N^k(s) = (-w_1(s) \;\; \gamma_1(s))^T$, $\mathcal{Y}^k = \mathbb{R}$ and $\mathcal{Y}^v = \mathbb{R}^{n_V}$ once more.

Using this notation the infimum in (D.11) may be decomposed as:

$$D_0^* = \inf_{y_0^k \in \mathcal{Y}^k} \left\{ y_0^k \cdot \bar{k} + \inf_{\mathbf{m} \in \mathbf{M}(y_0^k)} \sup_{\mathbf{a} \in \mathbf{A}} \lambda \cdot V(s_0, \mathbf{a}) + \langle \mathbf{m}, \mathbf{z}^h(0, \mathbf{a}) \rangle \right\},$$

where the outer infimum is over values for the costate and the (constrained) inner infimum is over the set of feasible multiplier sequences. Thus, the dual can be expressed as:

$$D_0^* = \inf_{y_0^k \in \mathcal{Y}^k} \left\{ y_0^k \cdot \bar{k} + D^*(s_0, y_0^k, \lambda) \right\},$$

where $D^*$ is the continuation dual function, for each $y = (y^k, y^v) \in \mathcal{Y} := \mathcal{Y}^k \times \mathbb{R}^{n_V}$,

$$D^*(s, y) = \inf_{\mathbf{m} \in \mathbf{M}(y^k)} \sup_{\mathbf{a} \in \mathbf{A}} \; y^v \cdot V(s, \mathbf{a}) + \langle \mathbf{m}, \mathbf{z}^h(0, \mathbf{a}) \rangle. \tag{D.13}$$

We now turn to the recursive form of (D.13). Define the current dual correspondence $\mathcal{Q} : \mathcal{Y}^k \rightrightarrows \mathbb{R}_+^{n_H} \times (\mathcal{Y}^k)^{n_S}$,

$$\mathcal{Q}(y^k) = \left\{ (m, y^{k\prime}) \in \mathbb{R}_+^{n_H} \times (\mathcal{Y}^k)^{n_S} \middle| y^k = (m)^T \cdot N^k(s) + B^k(s) \cdot \sum_{\mathcal{S}} y^{k\prime}(s') \right\}, \tag{D.14}$$

Given a current costate $y^k$, $\mathcal{Q}(y^k)$ gives current $H$-constraint multipliers $m$ and continuation costates $y^{k\prime}$ that are consistent with (D.12). Next define the current dual objective $J : \mathcal{S} \times \mathbb{R}^{n_V} \times \mathbb{R}_+^{n_H} \times (\mathcal{Y}^{n_K})^{n_S} \times \mathcal{A} \to \mathbb{R}$,

$$J(s, y^v; m, y^{k\prime}, a) = y^v \cdot u(s, a) - \sum_{s' \in \mathcal{S}} y^{k\prime}(s') c(s, a) + m \cdot h(s, a). \tag{D.15}$$

Relative to the general setting in the main text, this omits terms involving $k$ and $v'$ and is simpler. It evaluates the shadow value of current increments to constraints inclusive of laws of motion. It is immediate to see that absent backward looking states (i.e., $\sum_{s' \in \mathcal{S}} y^{k\prime}(s') c(s, a) = 0$) we recover the $J$ function in Subsection 4.3. Define the law of motion for costates $y^v$:

$$\phi(s, y^v, m, s') = y^v \cdot B^v(s, s') + m \cdot N^v(s, s'). \tag{D.16}$$

The quasilinear recursive dual problem combines (D.14), (D.15) and (D.16) and is described in the following proposition.

**Proposition D.1. (Value Functions)**. *The optimal value $D_0^*$ and the continuation dual value function satisfy:*

$$D_0^* = \inf_{y^k \in \mathcal{Y}^k} D^*(s_0, y^k, \lambda) + y^k \cdot \bar{k} \tag{D.17}$$

*The continuation dual value function satisfies the recursion, for all $(s, y) \in \mathcal{S} \times \mathcal{Y}$,*

$$D^*(s, y) = \inf_{(m, y^{k\prime}) \in \mathcal{Q}(y^k)} \sup_{a \in \mathcal{A}} J(s, y^v; m, y^{k\prime}, a) + \delta \sum_{s' \in \mathcal{S}} D^*(s', y^{k\prime}(s'), \phi(s, y^v, m, s')) \pi(s'|s). \tag{D.18}$$

The proof of the previous proposition is the same as Proposition 3 and is omitted. In addition, it is easy to see that $D^*$ is a sub-linear function. Notice that in (D.17) the initial condition for the costate $y^k$ is picked, whilst that for $y^v$ is pinned down by the parameter $\lambda$. Thus, the costate $y^k$ for the backward-looking state $k$ is forward-looking and the the costate $y^v$ for the forward-looking state $v$ is backward-looking. It follows from (D.18) that the dual Bellman operator is:

$$\mathscr{B}(D)(s, y) = \inf_{(m, y^{k\prime}) \in \mathcal{Q}(y^k)} \sup_{a \in \mathcal{A}} J(s, y^v; m, y^{k\prime}, a) + \delta \sum_{s' \in \mathcal{S}} D(s', y^{k\prime}(s'), \phi(s, y^v, m, s')) \pi(s'|s). \tag{D.19}$$

Comparison of terms defining the dual Bellman operator in the main text in Definition 1 and in (D.19) (in particular, comparison of the terms defining the $J$ functions) reveals how exploitation of quasilinearity simplifies matters.

To make the preceding discussion concrete, consider again the limited commitment with capital example. Applying (D.18), the dual Bellman equation is:

$$D^*(s,y) = \inf_{(m,y^{k\prime})\mathcal{Q}(y^k)} \sup_{a\in\mathcal{A}} J(s,y^v;m,y^{k\prime},a) + \delta \sum_{s'\in\mathcal{S}} D^*(s',y^{k\prime}(s'),\phi(s,y^v,m,s'))\pi(s'|s),$$

with:

$$J(s,y^v;m,y^{k\prime},a) = \sum_{i\in\mathcal{I}}(y^{v,i}+m^i)u^i(a^i) - \Big(\sum_{s'\in\mathcal{S}} y^{k\prime}(s') + m^{n_I+1}\Big)\sum_{i\in\mathcal{I}} a^i + m^{n_I+1}\gamma_0,$$
$$\mathcal{Q}(y^k) = \Big\{(m,y^{k\prime})\Big| y^k = -\sum_{i\in\mathcal{I}} m^{h,i}w^i(s) + \Big(m^{n_I+1} + \sum_{\mathcal{S}} y^{k\prime}(s')\Big)\gamma_1(s)\Big\}$$

and $\phi(s,y^v,m,s') = y^v + \bar{m}$, where $\bar{m} = (m^1,\ldots,m^{n_I})^T$. Thus, the weight $y^{v,i}$ on agent $i$'s current utility $u^i(a^i)$ is augmented by the multiplier on her incentive constraint $m^i$. The constraint set $\mathcal{Q}(y^k)$ reveals the evolution of the costate $y^k$, the shadow value of capital. This value is depressed to the extent that capital tightens incentive constraints $-\sum_{i\in\mathcal{I}} m^i w^i(s)$, but enhanced to the extent that capital relaxes the current resource constraint or augments the future capital stock $(m^{n_I+1} + \sum_{\mathcal{S}} y^{k\prime}(s'))\gamma_1(s)$.

## D.4 Contraction

Consider the *dual Bellman operator* defined in (D.19). As in the main text, it is immediate from (D.18) and (D.19) that $D^* = \mathscr{B}(D^*)$. Moreover, since a sub-linear function on a convex cone is fully determined on $\mathcal{C} := \{y \in \mathcal{Y} \mid \|y\|_E = 1\}$, it is sufficient to consider the restriction of candidate (sub-linear) value functions to the domain $\mathcal{S}\times\mathcal{C}$. Define the set $\mathscr{G}$ as in the main text and recall that, when endowed with the (modified) Thomson metric $d$, $(\mathscr{G},d)$ is a complete metric space. The following result mimics Theorem 2 in the main text (and admits the same proof).

**Theorem D.1. (Contraction).** *Given $\underline{\underline{D}}$, $\underline{D}$ and $\overline{D}$ satisfying Assumption 3 in the main text, the operator $\mathscr{B}$ defined in (D.19) is a contraction on the metric space $(\mathscr{G},d)$ of sub-linear functions bounded below by $\underline{D}$ and above by $\overline{D}$ and endowed with the metric $d$. In addition, $\mathscr{B}$ admits a unique fixed point $\hat{D}$ and if $\underline{D} \le D^* \le \overline{D}$, then $\hat{D} = D^*$.*

# E Further Examples and Variations

Our framework accommodates many problems from the literature. In Section 2, we considered limited commitment economies; in Appendix D we described a (quasilinear) limited commitment problem with physical capital. Below we briefly describe other examples that highlight the scope of our method. In each case, we state the corresponding recursive dual problem and comment on the assumptions needed to apply results from the main text. We start in Appendix E.1 with an optimal monetary policy problem similar to those considered in Woodford (2003). This problem features a non-linear law of motion for the forward-looking state variables. Next, in Appendix E.2 we describe a hidden information insurance problem from Atkeson and Lucas (1992) and an extension that incorporates persistent private information. Finally, in Appendix E.3 we briefly comment on how to extend our formulation to include problems with (hidden action) moral hazard.

## E.1 Optimal monetary policy

In this section we consider a deterministic version of an optimal monetary policy problem as in Woodford (2003). It is straightforward, although notationally heavier, to extend this to a stochastic setting. The government's social objective over sequences of output $\mathbf{a} = \{a_t\}_{t=0}^{\infty}$ and inflation $\{\Delta p_t\}_{t=0}^{\infty}$ is given by $\sum_{t=0}^{\infty} \delta^t L(a_t, \Delta p_t)$ with $L : \mathbb{R}^2 \to \mathbb{R}$ strictly concave and continuous.[40] Output sequences are restricted to $\mathbf{A} := \mathcal{A}^{\infty}$, with $\mathcal{A} = [\underline{a}, \overline{a}]$ a bounded interval. Inflation evolves according to a simple New Keynesian Philips Curve,

$$\Delta p_t = \kappa a_t + \delta \Delta p_{t+1},$$

with the terminal condition $\lim_{t\to\infty} \sum_{t=0}^{\infty} \delta^t \Delta p_t = 0$. Consequently, given an output plan $\mathbf{a}$, inflation at $t$ is $\Delta p_t = V^2(\mathbf{a}|t) := \kappa \sum_{\tau=0}^{\infty} \delta^{\tau} a_{t+\tau}$ and the government's continuation payoff:

$$V^1(\mathbf{a}|t) := \sum_{\tau=0}^{\infty} \delta^t L(a_{t+\tau}, V^2(\mathbf{a}|t+\tau)).$$

The government's payoff and inflation serve as forward-looking state variables; there are no backward-looking state variables in this problem. Adopting our previous notation and letting $v = (v^1, v^2)$ and $v' = (v'^{,1}, v'^{,2})$ denote, respectively, current and future pairs of payoff and inflation, the aggregator $W^v$ is given by:

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = W^v\left[a, \begin{pmatrix} v'^{,1} \\ v'^{,2} \end{pmatrix}\right] = \begin{pmatrix} L(a, \kappa a + \delta v'^{,2}) + \delta v'^{,1} \\ \kappa a + \delta v'^{,2} \end{pmatrix}.$$

It is nonlinear in inflation. There is no separate $H$ function in this case and the social objective is simply $F[V(\mathbf{a})] = V^1(\mathbf{a})$. Proceeding as in the main text, the period 0 dual period value is given by:

$$D_0^* = \inf_{y_0 \in \mathbb{R}^2} \sup_{v_0 \in \mathcal{V}} \quad v_0^1 - y_0 \cdot v_0 + D^*(y_0),$$

where $v_0 = (v_0^1, v_0^2)$ is the period 0 government payoff and inflation, $y_0 = (y_0^1, y_0^2)$ is the period 0 costate and $D^*$ the continuation dual value function. The current dual function specializes to:

$$J(y; q, p) = y^1 \left\{ L(a, \kappa a + \delta v'^{,1}) + \delta v'^{,1} \right\} + y^2 \left\{ \kappa a + \delta v'^{,2} \right\} - \delta y' \cdot v', \tag{E.1}$$

where the first term is the current shadow value of delivering payoff to the government, the second term is the current shadow value of inflation, while the final term is the shadow cost of future payoff and inflation promises. The dual Bellman equation for this problem is:

$$D^*(y) = \inf_{y' \in \mathbb{R}^2} \sup_{(a, v') \in \mathcal{A} \times \mathcal{V}} \quad J(y; q, p) + \delta D^*(y') \tag{E.2}$$

[40] In most of Woodford (2003)'s examples, $L$ is a (concave) quadratic approximation to an underlying objective over primitives. For now we place no such restrictions on $L$.

with $J$ as in (E.1). Application of Theorem 2 and Proposition 10 require the existence of bounding functions satisfying Assumption 3. We now turn to this.

**Bounding Functions** Let $\mathcal{V} = \prod_{i=1,2}[\underline{v}^i, \overline{v}^i]$ denote a set of possible government payoffs and inflation rates. Assume an $\tilde{a} \in \mathcal{A} = [\underline{a}, \overline{a}]$ and $\xi > 0$ such that:

$$\begin{pmatrix} \overline{v}^1 - \xi \\ \overline{v}^2 - \xi \end{pmatrix} \geq \begin{pmatrix} L(\tilde{a}, \kappa\tilde{a} + \delta\overline{v}^2) \\ \kappa\tilde{a} \end{pmatrix} + \delta \begin{pmatrix} \overline{v}^1 \\ \overline{v}^2 \end{pmatrix}$$
$$\geq \begin{pmatrix} L(\tilde{a}, \kappa\tilde{a} + \delta\underline{v}^2) \\ \kappa\tilde{a} \end{pmatrix} + \delta \begin{pmatrix} \underline{v}^1 \\ \underline{v}^2 \end{pmatrix} > \begin{pmatrix} \underline{v}^1 + \xi \\ \underline{v}^2 + \xi \end{pmatrix}. \tag{E.3}$$

It is then easily verified that:

$$\overline{D}(y) = \sum_{i=1}^{2} y^i \varphi^i(y^i), \qquad \varphi^i(y^i) = \begin{cases} \overline{v}^i & y^i \geq 0 \\ \underline{v}^i & y^i < 0, \end{cases}$$
$$\underline{\underline{D}}(y) = \sum_{i=1}^{2} \{y^i \psi^i(y^i) + |y^i|\xi\}, \qquad \psi^i(y^i) = \begin{cases} \underline{v}^i & y^i \geq 0 \\ \overline{v}^i & y^i < 0. \end{cases}$$

and $\underline{D} = \mathbf{B}(\underline{\underline{D}})$ satisfy all desired conditions.[41]

**Partial quasi-linearity of the problem** The recursive dual problem derived in (E.2) is obtained via the general approach of Section 4. Appendix D describes how quasilinear structure in aggregators may be exploited to simplify the analysis. The optimal monetary policy problem is quasilinear in government payoffs, but not inflation. We now use this example to show how the analysis of Appendix D extends to problems in which aggregators are quasi-linear in a subset of state variables.

Since the forward-looking state describing the government's future payoff $v'^{,1}$ enters $W^v$ in a quasi-linear way it can be substituted out of the problem. In contrast, the forward-looking state describing inflation $v'^{,2}$ enters non-linearly and cannot be so removed. After substitution of $v^1$, the problem becomes:

$$\sup \sum_{t=0}^{\infty} \delta^t L(a_t, \kappa a_t + \delta v^2_{t+1})$$

subject to, for all $t$, $v^2_t = \kappa a_t + \delta v^2_{t+1}$. This leads to the dual problem:

$$D^*_0 = \inf_{\mathbf{Q}} \sup_{\mathbf{P}} \sum_{t=0}^{\infty} \delta^t L(a_t, \kappa a_t + \delta v^2_{t+1}) + \sum_{t=0}^{\infty} \delta^t y^2_t \{\kappa a_t + \delta v^2_{t+1} - v^2_t\}, \tag{E.4}$$

where $\mathbf{Q}$ is the set of inflation costate sequences $\{y^2_t\}$ and $\mathbf{P}$ the set of inflation-output sequences $\{v^2_t, a_t\}_{t=0}^{\infty}$. Notice that in (E.4) the costate on the government's payoff $y^1$ is initialized to and remains at 1. Using arguments similar to before the initial problem specializes to:

$$D^*_0 = \inf_{y^2_0 \in \mathbb{R}} \sup_{v^2_0 \in \mathcal{V}^2} -y^2_0 v^2_0 + D^*(1, y^2_0),$$

[41]The verification is similar to that given for the limited commitment case in Appendix E.

where $\mathcal{V}^2 = \frac{\kappa}{1-\delta}[\underline{a}, \overline{a}]$ is the set of possible inflations, while the dual Bellman equation becomes:

$$D^*(1, y^2) = \inf_{y'^{,2} \in \mathbb{R}} \sup_{(a, v'^{,2}) \in \mathcal{A} \times \mathcal{V}^2} L(a, \kappa a + \delta v'^{,2}) + y^2(\kappa a + \delta v'^{,2}) - y'^{,2} v'^{,2} + \delta D^*(1, y'^{,2}).$$

In the latter the inner supremum is over current output-inflation pairs $(a, v'^{,2})$, while the infimum operation is over the future inflation costate $y'^{,2}$.

## E.2 Hidden Information

In hidden information problems agents must be motivated to reveal the outcomes of privately observe shock processes. Our framework accommodates such problems. To illustrate this we describe the application of the dual recursive method to a dynamic risk sharing problem with privately observed preference shocks similar to Atkeson and Lucas (1992). In this problem a planner seeks to maximize a weighted sum of an agent's lifetime utility and a lifetime resource cost. Incentive compatibility requires that the agent is induced to reveal her privately observed shocks. Thus, within a period incentive constraints "run across shock contingent allocations" (since the agent must compare the utility from the allocation associated with her truthfully revealed shock to that from lying). This is in contrast to, say, limited commitment problems in which constraints do not run across shock outcomes (the agent must only consider whether the allocation associated with her current shock is better than her outside option). To capture this structure within our framework requires an appropriate definition of actions. Specifically, assume that in each period $t \geq 0$, conditional on history $s^t$ the planner chooses a vector of consumptions $a_t(s^t) \in \mathcal{A} = \mathbb{R}_+^{n_S}$, with $\mathcal{A}$ compact. The elements of $a_t(s^t)$ are consumption amounts to be consumed in period $t+1$ after each history $(s^t, s)$, $s \in \mathcal{S}$ by the agent. Let $\gamma$ denote a constant (possibly zero) and publicly observable endowment received by the agent in each period; let $u : \mathcal{S} \times \mathbb{R}_+ \to \mathbb{R}$ denote a shock contingent utility for the agent that is increasing, continuous and concave in its second argument. Assuming expected utility preferences on the part of the agent, the resource cost and the agent's payoff from a consumption vector $a$ is:

$$U^1(a) := \sum_{s' \in \mathcal{S}} \pi(s')(\gamma - a(s'))$$

and

$$U^2(a) := \sum_{s' \in \mathcal{S}} \pi(s') u(s', a(s')).$$

Laws of motion for the lifetime resource amount and the agent's promised utility are given by for $i = 1, 2$:

$$W^{v,i}[a, M(v'^{,i})] = (1-\delta)U^i(a) + \delta M(v'^{,i}) = (1-\delta)U^i(a) + \delta \sum_{s' \in \mathcal{S}} \pi(s') v'^{,i}(s').$$

In general, the hidden information model requires $n_S \times (n_S - 1)$ incentive constraints to ensure truth telling at each history. In most applications, a *single crossing property* is assumed, which, together with a monotonicity condition on optimal payments, ensures that checking only for sub-optimality of "local downward deviations" suffices for incentive compatibility. The local deviation approach reduces the number of incentive constraints from $n_S \times (n_S - 1)$ to $n_S - 1$.[42] Thus, $H$ is given by:

[42] In our dynamic context, the planner has two instruments $a(s')$ and $v'(s')$. *Single Crossing* is guaranteed by the following condition on marginal rate of substitution: $-\frac{1}{\delta}\frac{\partial u(s', a)}{\partial a}$ to increase with $s'$,

$$
H[a,v'] = \begin{bmatrix} (1-\delta)u\left(n_S, a(n_S)\right) + \delta v'(n_S) & -[(1-\delta)u(n_S, a(n_S-1)) + \delta v'(n_S-1)] \\ & \vdots \\ (1-\delta)u\left(2, a(2)\right) + \delta v'(2) & -[(1-\delta)u(2, a(1)) + \delta v'(1)] \end{bmatrix}.
$$

Since we have no backwards looking variables, $y = y^v$, and the period 0 dual value satisfies:

$$
D_0^* = \inf_{y_0 \in \mathbb{R}^2} \sup_{v_0 \in \mathcal{V}} \sum_{i=1}^{2} \lambda^i v_0^i - y_0 \cdot v_0 + D^*(y_0),
$$

where $v_0 = (v_0^1, v_0^2)$ is the period 0 lifetime resource amount and agent payoff, and $\lambda^i$, $i = 1, 2$, is, respectively, the initial weight on resources and the agent's payoff. The current dual function specialises to:

$$
J(y; q, p) = y \cdot W^v[a, M(v')] + m \cdot H[a, v'^{,2}] - \delta \sum_{s' \in \mathcal{S}} y'(s') \cdot v'(s')\pi(s'), \tag{E.5}
$$

where $y \cdot W^v[a, M(v')] = \sum_{i=1}^{2} y^i W^{v,i}[a, M(v'^{,i})]$, and

$$
m \cdot H[a, v'^{,2}] = \sum_{s'=2}^{n_S} m(s') \Big\{ (1-\delta)u(s', a(s')) + \delta v'(s') - [(1-\delta)u(s', a(s'-1)) + \delta v'(s'-1)] \Big\} \pi(s').
$$

The dual Bellman equation then takes the form:

$$
D^*(y) = \inf_{(m,y') \in \mathbb{R}^{n_S - 1} \times \mathbb{R}^{2 \times n_S}} \sup_{(a,v') \in \mathcal{A} \times \mathcal{V}} J(y; q, p) + \delta \sum_{s' \in \mathcal{S}} D^*(y'(s'))\pi(s'),
$$

with $J$ defined as in (E.5). Theorem 2 and Proposition 10 apply after the construction of appropriate bounding functions.

**Quasilinearity** Note that the above problem fits within the quasilinear class (and can be reformulated accordingly). Similar to the class of problems considered in Subsection 4.3, both forward-looking states that describe the future payoff $v'$ enter $W^v$ and $H$ in a linear way and can be substituted out. Notice, moreover, that the costate associated with the principal's payoff remains constant over time and histories. Hence, it is sufficient to keep track of the costate for the agent. After eliminating the forward states $v'(s')$ and normalizing $y^1 = 1$, the dual Bellman equation becomes

$$
D^*(1, y^2) = \inf_{(m, y'^{,2}) \in \mathbb{R}_+^{n_S}} \sup_{a \in \mathcal{A}} J(y^2; m, a) + \delta \sum_{s' \in \mathcal{S}} D^*\left(1, y'^{,2}(s')\right) \pi(s'), \tag{E.6}
$$

where now:

$$
J(y^2; m, a) = (1-\delta) \sum_{s' \in \mathcal{S}} \pi(s') \left[\gamma - a(s') + y^2 u(s', a(s'))\right] + m \cdot h(a),
$$

with

$$
\begin{aligned}
m \cdot h(a) &= \sum_{s'=2}^{n_S} m(s') \Big\{ (1-\delta)u(s', a(s')) - [(1-\delta)u(s', a(s'-1))] \Big\} \pi(s') \\
&= (1-\delta) \sum_{s' \in \mathcal{S}} \left[ m(s') - \frac{\pi(s'+1)}{\pi(s')} m(s'+1) \right] u(s', a(s'))\pi(s'),
\end{aligned}
$$

---

while *monotonicity* requires $a(\cdot)$ to increase with $s'$.

and $m(1) = m(n_S + 1) = 0$. The law of motion for the costate variable is

$$y'^{,2}(s') = y^2 + m(s') - \frac{\pi(s'+1)}{\pi(s')} m(s'+1).$$

In this example, costates $y^2$ can take both positive and negative values. This reflects the fact that in the original formulations $H$ is not monotone in $a$ or $v'^{,2}$.

**Hidden Information with persistent shocks** The case of hidden information with persistent shocks can be handled similarly. Now, however, in addition to the expansion in the dimension of the current action set (to cover a menu of consumptions across current shocks in $\mathbb{R}_+^{n_S}$), the dimension of the promise set must be similarly expanded (to cover a menu of promises for future shocks). The law of motion for lifetime resources and promises to the agent are, for $i = 1, 2$,

$$W^{v,i}[a, M(v'^{,i})] = \left\{ (1-\delta) u^i(s, a(s)) + \delta \sum_{s' \in \mathcal{S}} \pi(s'|s) v'^{,i}(s') \right\}_{s \in \mathcal{S}} \in \mathbb{R}^{n_S},$$

with $u^1(s, a(s)) = \gamma - a(s)$ and $u^2(s, a(s)) = u(s, a(s))$. The incentive constraints are collected into:

$$H[a, v'] = \left[ (1-\delta) u(s, a(s)) + \delta \sum_{\mathcal{S}} \pi(s'|s) v'^{,2}(s, s') - (1-\delta) u(s, a(s-1)) - \delta \sum_{\mathcal{S}} \pi(s'|s) v'^{,2}(s-1, s') \right]_{s=2}^{n_S}.$$

With these modifications, the analysis proceeds similarly to the proceeding section.

## E.3 Action Moral Hazard

For notational simplicity we have assumed that the stochastic aggregator $M$ does not depend on $a$. It is straightforward to allow for such dependence. This variation of our framework immediately accommodates dynamic (hidden action) moral hazard problems with general recursive preferences and the timing assumed in Hopenhayn and Nicolini (1997).[43]

# F Recursive Primal: Set Approximation Procedures

We briefly review alternative recursive approaches in our general setting. Define a forward-looking state $v$ to be feasible at $(\bar{k}, s)$ if there is a primal plan $\mathbf{p}$ such that $v = v_0$, $k_0 = \bar{k}$ and for all $t$ and $s^t$, conditions (15), (16), and (17) hold. Let $\mathcal{X}^* : \mathcal{K} \times \mathcal{S} \rightrightarrows \mathcal{V}$ be a correspondence mapping the current backward-looking state and shock to the set of feasible forward-looking states. The decision-maker's problem may then be expressed as:

$$P_0^* = \sup_{v \in \mathcal{X}^*(\bar{k}, s_0)} F[s_0, v]. \tag{F.1}$$

[43] Under this timing the public signal (a job or unemployment) of a hidden action (job search) is realized in the period after the action is taken.

Let $\Gamma^*(k,s,v) := \Gamma(k,s,v;\mathcal{X}^*)$, where:

$$\Gamma(k,s,v;\mathcal{X}) := \left\{ (a,k',v') \in \mathcal{A} \times \mathcal{K} \times \mathcal{V}^{n_S} \;\middle|\; \begin{array}{l} v = W^v[s,a,M^v[s,v']], \quad k' = W^k[k,s,a], \\ \\ H[k,s,a,v'] \geq 0 \text{ and } \forall s' \in \mathcal{S},\; v'(s') \in \mathcal{X}(k',s') \end{array} \right\}.$$

Given $k_0^* = \bar{k}$ and a $v_0^*$ solving (F.1), an optimal plan may be obtained from the iteration:

$$(a_t^*(s^t), k_{t+1}^*, v_{t+1}^*(s^t,\cdot)) \in \Gamma^*(k_t^*(s^{t-1}), s_t, v_t^*(s^t)). \tag{F.2}$$

Solution of (F.1) and implementation of the iteration (F.2) requires prior recovery of $\mathcal{X}^*$. Let the space $\Xi := \{\mathcal{X} : \mathcal{K} \times \mathcal{S} \rightrightarrows \mathcal{V}\}$ be equipped with the set inclusion ordering: $\mathcal{X}_1 \geq \mathcal{X}_2$ if for all $(k,s) \in \mathcal{K} \times \mathcal{S}$, $\mathcal{X}_1(k,s) \supseteq \mathcal{X}_2(k,s)$. Extending arguments of Abreu et al. (1990) and Atkeson (1991), $\mathcal{X}^*$ is the largest correspondence in $\Xi$ (with respect to this ordering) satisfying the fixed point equation $\mathcal{X} = \Psi(\mathcal{X})$, where $\Psi : \Xi \to \Xi$ is given by:

$$\Psi(\mathcal{X})(k,s) = \{v \,|\, \Gamma(k,s,v;\mathcal{X}) \neq \emptyset \}. \tag{F.3}$$

The continuity properties of $\Psi$ are not well understood. In particular, $\Psi$ has not been shown to be contractive. Thus, there is no guarantee that an iteration of $\Psi$ from an arbitrary $\mathcal{X}_0$ in $\Xi$ converges to $\mathcal{X}^*$. However, extending arguments of Abreu et al. (1990) and Atkeson (1991), $\Psi$ is monotone in the set inclusion ordering described above and iteration of $\Psi$ from a suitable $\mathcal{X}_0 \geq \Psi(X_0) = \mathcal{X}_1 \geq \mathcal{X}^*$ generates a sequence of correspondences that converge monotonically to $\mathcal{X}^*$:

$$\mathcal{X}_0 \geq \ldots \geq \mathcal{X}_n \geq \Psi(\mathcal{X}_n) = \mathcal{X}_{n+1} \geq \ldots \geq \lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}^*. \tag{F.4}$$

On the other hand, a $\Psi$-iteration from $\mathcal{X}_0 \leq \Psi(\mathcal{X}_0) = \mathcal{X}_1 \leq \mathcal{X}^*$ generates a sequence of correspondences that converge monotonically and are bounded above by $\mathcal{X}^*$:

$$\mathcal{X}_0 \leq \ldots \leq \mathcal{X}_n \leq \Psi(\mathcal{X}_n) = \mathcal{X}_{n+1} \leq \ldots \leq \mathcal{X}^*. \tag{F.5}$$

Note that in this case convergence to $\mathcal{X}^*$ is not ensured. Instead, the sequence converges to an "inner approximation": i.e. $\lim_{n\to\infty} \mathcal{X}_n \leq \mathcal{X}^*$.

A basic challenge in implementing these algorithms is selecting a low dimensional approximation to the correspondences $\mathcal{X}_n$ that respects the monotonicity of $\Psi$. In a setting without backward-looking states, Chang (1998) forms discrete approximations to the iterates $\mathcal{X}_n$ (and to the set $\mathcal{A} \times \mathcal{V}^{n_S}$).[44] Under his approach, to assess whether a particular $v$ is in an updated set $\mathcal{X}_{n+1}(s) = \Psi(\mathcal{X}_n)(s)$ requires an exhaustive search over all points in the grid associated with $\mathcal{A} \times \mathcal{V}^{n_S}$ for a combination $(a,v')$ that satisfies (or approximately satisfies) the restrictions defining $\Psi(\mathcal{X}_n)(s)$. However, since $\mathcal{X}^*$ is not typically discrete, this procedure is not usually consistent with the selection of an initial $\mathcal{X}_0 \geq \mathcal{X}^*$ and implementation of a sequence (F.4). At best it produces an inner approximation to $\mathcal{X}^*$ and the approximation may be rather poor. Moreover, it suffers from a severe curse of dimensionality especially as $n_I$ or $n_S$ increase.[45]

In the context of repeated games without backward-looking states or shocks and with quasilinear laws of motion for forward-looking states (i.e. for payoffs) and a convex value set $\mathcal{X}^*$, Judd et al. (2003) suggest an "inner approximation" implementation of the sequence

[44] Abraham and Pavoni (2008) apply a similar approach to calculate the domain of feasible promises in a principal-agent setting.

[45] This may force the use of rather coarse grids leading to poor approximation. Moreover, as noted, little is known about the continuity properties of $\Psi$. Approximation errors introduced by grid approximation may propagate as $\Psi$ is repeatedly applied in an iteration.

(F.5). This procedure represents candidate sets $\mathcal{X}$ with finite tuples of extreme points. At each step of the iteration, the convex hull of the extreme points is computed. This is used to generate a feasible set of continuation promises. A new set of extreme points is then obtained via a collection of updating optimizations. The generation of the convex hull and the possible high dimensionality of the optimizations involved make this approach computationally costly to implement. Moreover, since (F.5) is not guaranteed to converge to $\mathcal{X}^*$, even if implemented with high accuracy, this approach may converge to a limiting set contained within and far from $\mathcal{X}^*$. Finally, no error bounds or convergence criteria are available. Judd et al. (2003) suggest combining this procedure with an "outer approximation" implementation of (F.4). The latter uses the intersection of a finite number of supporting hyperplanes to represent value sets. Such intersections contain the set to be approximated. At each step of the iteration, an intersection of hyperplanes is used to generate a feasible set of continuation promises. A new set of hyperplane coefficients is then obtained via a collection of updating optimizations. The possible high dimensionality of the optimizations involved make this approach computationally costly to implement. It converges to $\mathcal{X}^*$ only if this set is convex and it generates optimal policies only with a further round of optimization. Finally, it suffers from the usual drawbacks of monotone iterations: a bounding start point possibly far from the limit is needed and error bounds are only available if the procedure is used with the inner approximation approach. Sleet and Yeltekin (2016) supply an extension of this procedure to settings with backward-looking states.

**A Special Case** In the limited commitment problem considered in Section 2, $F$ is increasing in forward-looking states and both the law of motion (i.e. the composition of $W^v$ and $M^v$) and the constraint function $H$ are increasing in the continuation values for these states. The alternative primal recursive approach procedures described in Section 2 is applicable to other problems with a similar structure, i.e. in which $F$ is non-decreasing in (at least) one element of the forward-looking state (call it $v^i$) and both the composition of $W^v$ and $M^v$ and $H$ are non-decreasing in the corresponding continuation values $v'^{,i}$. For these problems only the upper surface of the correspondence $\mathcal{X}^*(s,\cdot)$ in the direction of $v^i$ and the projection of $\mathcal{X}^*(s,\cdot)$ onto the other forward-looking states is needed to calculate an optimal value and a solution to (F.1).

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