Arbitrage Pricing Theory

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Partial equilibrium and APT

The *Arbitrage Pricing Theory*,[1] APT, is a partial equilibrium valuation approach developed under the assumptions:

1. A linear factor model exogenously specified provides a complete description of the uncertainty affecting investors decisions;
2. As the number of securities increases, the profits from arbitrage portfolios converge to zero.

(1) S.A. Ross (1976), The Arbitrage Theory of Capital Asset Pricing, *Journal of Economic Theory*, **13**, pp. 341–360.

APT assumes that stock return fluctuations admit a factor decomposition extending the one proposed for CAPM by Sharpe:

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i, \qquad i = 1, \ldots, n,$$

under the following statistical assumptions:

$$\begin{aligned} &\varepsilon_i \sim \mathrm{N}\left(0, \sigma^2_{\varepsilon_i}\right), \\ &\mathrm{Cov}\left(\varepsilon_i, \varepsilon_j\right) = \sigma_{\varepsilon_i, \varepsilon_j} = 0 \quad \text{per} \quad i \neq j, \\ &\mathrm{Cov}\left(\varepsilon_i, R_M\right) = \sigma_{\varepsilon_i, M} = 0. \end{aligned}$$

These properties and definitions imply:

$$\begin{cases} \mathrm{Cov}\left(R_i, R_j\right) = \beta_i \beta_j \sigma^2_M & \text{if} \quad i \neq j; \\ \mathrm{Var}\left(R_i\right) = \beta_i^2 \sigma^2_M + \sigma^2_{\varepsilon_i} & \text{if} \quad i = j. \end{cases}$$

The APT extends the same statistical hypotheses to a multifactor framework.

Consider a market with n risky securities and k factors. Each single return flucutuations are described by a linear factor model

$$R_i = \alpha_i + \mathbf{b}_i^\top \mathbf{F} + \varepsilon_i, \qquad i = 1, \ldots, n, \tag{1}$$

where $\mathbf{b}_i = (\beta_{i1}, \ldots, \beta_{ik})^\top$ and each β_{ih} $(i = 1, \ldots, n, \quad h = 1, \ldots, k)$ is the risk exposure of return i with respect to factor h, while $\mathbf{F} = (F_1, \ldots, F_k)^\top$ is the vector of the k risk factors.

In vector notation the linear factor model is

$$\mathbf{R} = \mathbf{a} + \mathbf{BF} + \epsilon,$$

where

$$\mathbf{R} =: \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}, \ \mathbf{a} =: \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \ \mathbf{B} =: \begin{bmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \ldots & \beta_{nk} \end{bmatrix}, \ \mathbf{F} =: \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_k \end{bmatrix}, \ \epsilon =: \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}.$$

and the following statistical properties are assumed:

$$\begin{array}{ll} \varepsilon_i \sim \mathrm{N}\left(0, \sigma^2_{\varepsilon_i}\right), & \mathrm{Cov}\left(\varepsilon_i, \varepsilon_j\right) = \sigma_{\varepsilon_i, \varepsilon_j} = \mathbb{E}\left[\varepsilon_i \varepsilon_j\right] = 0 \quad \text{per} \quad i \neq j, \\ \mathbb{E}\left(\varepsilon_i, F_h\right) = 0, & \mathrm{Cov}\left(F_h, F_l\right) = \sigma_{F_h, F_l} = 0 \quad \text{per} \quad h \neq l. \end{array}$$

Eq. (1) is a strucutural assumption about the uncertainty driving returns which has to be empirically validated

Notice that this assumption is particularly useful under the (testable) assumption that the number k of factors, which are necessary to explain the returns for a large number of stocks n, is small. This assumption is usually verified. Assume for simplicity and without loss of generality that the factors F_k, are independent. Then each element of the variance-covariance matrix has the representation:

$$\begin{cases} \mathrm{Cov}\left(R_i, R_j\right) = \sigma_{ij} = \sum_{h=1}^{k} \beta_{i,h}\beta_{j,h}\mathrm{Var}\left(F_h\right) = \sum_{h=1}^{k} \beta_{i,h}\beta_{j,h}\sigma^2_{F_h} \text{ per } i \neq j, \\ \mathrm{Var}(R_i) = \sigma_i^2 = \sum_{h=1}^{k} \beta^2_{i,h}\mathrm{Var}\left(F_h\right) + \sigma^2_{\varepsilon i} = \sum_{h=1}^{k} \beta^2_{i,h}\sigma^2_{F_h} + \sigma^2_{\varepsilon_i} \text{ per } i = j. \end{cases}$$

In matrix notation the above expressions of the variance covariance matrix are equivalent to:

$$\mathrm{Cov}(\mathbf{R}) = \mathbb{E}\left[\mathbf{R}\mathbf{R}^\top\right] - \mathbb{E}\left[\mathbf{R}\right]\mathbb{E}\left[\mathbf{R}^\top\right] = \mathbf{B}\mathbb{E}\left[\mathbf{F}\mathbf{F}^\top\right]\mathbf{B}^\top + \mathrm{Cov}(\epsilon),$$

where

$$\mathrm{Cov}(\epsilon) =: \begin{bmatrix} \sigma^2_{\varepsilon_1} & 0 & \dots & 0 \\ 0 & \sigma^2_{\varepsilon_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \dots & \sigma^2_{\varepsilon_n} \end{bmatrix},$$

since non systematic variance components $\sigma^2_{\varepsilon_i}$, for different securities are uncorrelated.

Under the assumption that factors are indepedent the matrix $\mathbb{E}\left[\mathbf{F}\mathbf{F}^\top\right]$ is a square diagonal $k-$dimensional matrix with diagonal elements given by $\sigma^2_{F_h}$, $h = 1, ..., k$.

The vector of expected returns is given by

$$\mathbb{E}[\mathbf{R}] = \alpha + \mathbf{B}\,\mathbb{E}[\mathbf{F}]$$

or with a different notation

$$\mu = \alpha + \mathbf{B}\mu_{\mathbf{F}}$$

where

$$\mathbb{E}[\mathbf{R}] = \mu =: \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \qquad e \qquad \mathbb{E}[\mathbf{F}] = \mu_{\mathbf{F}} =: \begin{bmatrix} \mu_{F_1} \\ \mu_{F_2} \\ \vdots \\ \mu_{F_k} \end{bmatrix}.$$

In applications it is often used a demeaned version of the factors $\mu_{\mathbf{F}} = \mathbf{0}$. In this special case $\mu = \alpha$. The use of demeaned factors does not introduce any loss of generality ; each factor F_k with $\mu_{F_k} \neq 0$ can be repalced by a new factordefined as $\mathbf{F}' = \mathbf{F} - \mu$. This change can be absorbed by shifting the constant term $\alpha' = \alpha + \mu$.

Arbitrage Portfolios

The definition of arbitrage portfolio is the key innovation of arbitrage valuation theory. A portfolio $P_{\mathcal{A}}$, is said an (asymptotic) arbitrage portfolio if the following conditions are verified. We assume a market with n risky securities and a riskless one.

Condition 1 (Zero Initial Investment)

The total investment in the n securities incldued in $P_{\mathcal{A}}$ is zero.

$$\sum_{i=1}^{n} w_i = 0 = \mathbf{w}^{\top}\mathbf{1}$$

where w_i is the investment in the risky security i.

Condition 2 (No Systematic Risk)

The total risk exposure of portfolio $P_{\mathcal{A}}$ with respect to any risk factor is zero [a]

$$\sum_{i=1}^{n} \beta_{ih} w_i = 0 = \mathbf{w}^\top \mathbf{b}_h \qquad \forall h, \qquad h = 1, \ldots, k.$$

[a] recall that the total risk exposure of a portfolio is simply given by the weighted combination of the single securities risk exposures. Weights are determined by the dollar portfolio allocations.

Condition 3 (Positive expected return)

Portfolio $P_{\mathcal{A}}$ has a positive return

$$\mathbb{E}\left[P_{\mathcal{A}}\right] = \mathbf{w}^{\top}\mu > 0,$$

where μ denotes the vector of expected returns.

Definition. A portfolio $P_{\mathcal{A}}$ that satisfies condition 1–3 (for $n \to +\infty$) is said arbitrage portfolio (asymptotic):

$$\mathbf{w}^{\top}\mathbf{1} = \mathbf{0}, \qquad \mathbf{w}^{\top}\mathbf{B} = \mathbf{0}, \qquad \mathbb{E}\left[P_{\mathcal{A}}\right] = \mathbf{w}^{\top}\mu > 0. \qquad \square$$

Within the APT model it is assumed that in a competitive market, in normal conditions, the market is free from arbitrage opportunities, i.e. no asymptotic arbitrage portfolio $P_{\mathcal{A}}$ can be traded by investors.

Valuation by no Arbitrage

The following result has been proved by Stephen A. Ross.

Theorem. *Assume the market is free from (asymptotic) arbitrage opportunities, then (for $n \to +\infty$) the expected μ_i of security i is a linear combination λ_0 plus a systematic component determined by the sum of the products between the risk exposures β_{ih} and the risk premia λ_h for factor h, hence*

$$\mu_i = \lambda_0 + \mathbf{b}_i^\top \lambda + \nu_i$$

or in scalar notation

$$\mu_i = \lambda_0 + \sum_{h=1}^{k} \beta_{ih} \lambda_h + \nu_i,$$

where $\lambda =: [\lambda_1, \lambda_2, \dots, \lambda_k]^\top$, ν_i satisfies [(2)]

$$\lim_{n \to +\infty} \frac{\|\nu\|^2}{n} = \lim_{n \to +\infty} \frac{\sum_{i=1}^{n} \nu_i^2}{n} = 0. \qquad \square$$

[(2)] ν_i is the residual of the cross sectional regression obtained by regressing the vector of expected returns μ on the matrix $\mathbf{B}$, thus the regression coefficients λ identify the best least square approximation of μ obtained by a linear combination of the columns of $\mathbf{B}$.

Observation. Assuming that the number of securities is large enough to remove the non systematic risk, one gets

$$\mu = \mathbf{1}\lambda_0 + \mathbf{B}\lambda.$$

In practice this is the relation which is used to compute expected returns and the residual risk premium components are neglected □

Proof.

1. Consider the regression of μ on the matrix of risk exposures $\mathbf{B}$ and on the constant $\mathbf{1}$

$$\mu = \mathbf{1}\lambda_0 + \mathbf{B}\lambda + \nu;$$

   which determines the regression coefficients $\lambda_0, \ldots, \lambda_h$, and the residual components ν_i.

2. Consider a portfolio i corresponding to the allocation $w_i = \frac{\nu_i}{\sqrt{n}\|\nu\|}$. By construction, the regression residuals ν satisfy conditions 1–2 as a consequence of the linear regression properties:

   (i) $\sum_{i=1}^{n} \nu_i = 0$,

   (ii) $\sum_{i=1}^{n} \beta_{ih}\nu_i = 0, \quad h = 1, \ldots, k$,

   hence also the portfolio allocation described by the vector $\mathbf{w}$ which is proportional to the vector of residues ν satisfies the conditions 1 and 2.

3. The portfolio with allocation $\mathbf{w}$ has an expected return

$$\begin{aligned}
\sum_{i=1}^{n} w_i \mu_i &= \sum_{i=1}^{n} w_i \left(\lambda_0 + \sum_{h=1}^{k} \beta_{ih} \lambda_h + \nu_i \right) \\
&= \sum_{i=1}^{n} \frac{\nu_i}{\sqrt{n} \, \|\nu\|} \left(\lambda_0 + \sum_{h=1}^{k} \beta_{ih} \lambda_h + \nu_i \right) \\
&= \frac{1}{\sqrt{n} \, \|\nu\|} \left\{ \underbrace{\left[\sum_{i=1}^{n} \nu_i \right]}_{=0} \lambda_0 + \sum_{h=1}^{k} \underbrace{\left[\sum_{i=1}^{n} (\nu_i \beta_{ih}) \right]}_{=0} \lambda_h + \sum_{i=1}^{n} \nu_i \nu_i \right\} \\
&= \frac{1}{\sqrt{n} \, \|\nu\|} \sum_{i=1}^{n} \nu_i^2 = \frac{\|\nu\|^2}{\sqrt{n} \, \|\nu\|} = \frac{\|\nu\|}{\sqrt{n}} .
\end{aligned}$$

4. We can finally prove that the absence of arbitrage opportunities forces the condition

$$\lim_{n\to+\infty} \frac{\|\nu\|}{\sqrt{n}} = 0.$$

On the other hand if the expected return of the portfolio with allocation **w** was positive

$$\lim_{n\to+\infty} \frac{\|\nu\|}{\sqrt{n}} > 0, \tag{2}$$

then it would be an asymptotic arbitrage contraddicting the assumption. In fact we proved that by construction the portfolio satisfies the defining properties 1 and 2 of an an arbitrage portfolio are verified. If (2) was true, then also condition 3 would be verified and the No Arbitrage assumption would be violated. Hence, by contraddiction, the theorem is proved and we must conclude that

$$\lim_{n\to+\infty} \frac{\|\nu\|}{\sqrt{n}} = 0$$

□

The model specification APT

From the definition of the linear factor model:

$$R_i = \alpha_i + \beta_{i1} F_1 + \beta_{i2} F_2 + \cdots + \beta_{ik} F_k + \varepsilon_i \tag{3}$$

and recalling that

$$\mathbb{E}\left[R_i\right] = \alpha_i + \beta_{i1} \mathbb{E}\left(F_1\right) + \beta_{i2} \mathbb{E}\left[F_2\right] + \cdots + \beta_{ik} \mathbb{E}\left[F_k\right].$$

that in the usual notation can be written as

$$\mu_i = \alpha_i + \beta_{i1} \mu_{F_1} + \beta_{i2} \mu_{F_2} + \cdots + \beta_{ik} \mu_{F_k}.$$

Observe that the previous equation applies in any market where return fluctuations are consistent with a linear factor model. This equation does not require the assumption of absence of arbitrage opportunities. On the contrary the main theorem of APT states that in the asymptotic limit the absence of arbitrage opportunities implies the relation

$$\mu_i = \lambda_0 + \beta_{i1} \lambda_1 + \beta_{i2} \lambda_2 + \cdots + \beta_{ik} \lambda_k.$$

APT when a risk free security is tradable

Denote with $i = 0$ a portfolio with zero risk exposure

$$\beta_{0h} = 0, \quad \forall h, \quad h = 1, \ldots, k.$$

Its expected return must be equal to λ_0.

Assume now that the expected return of the risk free security is given by μ_f. Then, under the (testable) hypothesis that the set of factors offers a complete description of all the systematic risks affecting the security returns, no arbitrage will force the property

$$\lambda_0 = \mu_f$$

APT APT when risk factors are tradable portfolios

Assume that risk factors are tracked by portfolios of tradable securities. Suppose for simplicity the situation with $k = 2$ APT and consider two securities A e B which are sufficient to replicate these factors.

We define the h-factor replicating portfolio as the portfolio whose risk exposure $\beta_{F_h h} = 1$, ed esposizione nulla rispetto a tutti gli altri $\beta_{F_h l} = 0$, $l \neq h$.

Ad esempio il portafogli di replicazione del fattore 1 si ottiene risolvendo il sistema lineare

$$\begin{cases} x_A^{F_1}\, \beta_{A1} + x_B^{F_1}\, \beta_{B1} = 1 \\ x_A^{F_1}\, \beta_{A2} + x_B^{F_1}\, \beta_{B2} = 0 \end{cases}$$

e analogamente possiamo determinare il portafogli di replicazione del fattore 2.

The Ross theorem shows that in the asymptotic limit, the APT equation

$$\mu_i = \mu_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2$$

describes the expected security returns and it is immediate to verify that the replication portfolios $\left(x_A^{F_1}, x_B^{F_1}\right)$ and $\left(x_A^{F_2}, x_B^{F_2}\right)$ satisfy the relations

$$\begin{cases} \mu_{P^{F_1}} = \lambda_0 + \beta_{F_1 1}\,\lambda_1 + \beta_{F_1 2}\,\lambda_2 = \mu_f + 1\cdot\lambda_1 + 0\cdot\lambda_2 = \mu_f + \lambda_1, \\ \mu_{P^{F_2}} = \lambda_0 + \beta_{F_2 1}\,\lambda_1 + \beta_{F_2 2}\,\lambda_2 = \mu_f + 0\cdot\lambda_1 + 1\cdot\lambda_2 = \mu_f + \lambda_2. \end{cases}$$

Hence, iIf the factors can be tracked by using a traded portfolio the λ_h will be the excess returns of the replicating portfolio of the factor F_h.